Some Results on Fuzzy SubKS-semigroups

بعض النتائج حول شبه الزمرة الجزئية الضبابية KS

Sundus Najah Assp.sajda kadhum Mohammed
College of education
Kufa university

Abstracts

In this paper we study a fuzzy subKS-semigroups, Cartesian product of fuzzy sets in KS-semigroups and strong fuzzy relation in KS-semigroups , and prove some results about this .

المستخلص

في هذا البحث درسنا شبه الزمرة الضبابية KS والضرب الكارتيزي والعلاقة الضبابية القوية على شبه الزمرة KS وبرهنا العديد من النتائج المتعلقة بهذا الموضوع .

1.Introduction

The notation of BCK-algebra was proposed by Iami and K-Iseki [1] in 1966 in the same year, K-Iseki [2] introduced the notion of BCI- algebra which is a generalization of a BCK-algebra in 1965 L.A .Zadeh[3] introduced the notion of fuzzy set , in 1971 A. Rosenfeld [4] introduced the notion of fuzzy group. and in 1991 O.G. Xi[5] introduced the notion of fuzzy BCK-algebra .The new class of algebraic structure introduced in 2006 by K.H. Kim[6] called KS-semigroups , which is the combination of BCK-algebra and semigroups , in 2007 D.R Prince Williams , Shamshad Husain [7] introduced the notion of fuzzy KS-semigroup .in this paper , we prove some results in a subKS-semigroup ,Cartesian product of fuzzy sets and strong fuzzy relation on KS-semigroups .

2.Preliminary

In this section , we introduce the fundamental definitions that will be used in the sequel .

Definition 2.1 A **BCK algebra** is a non empty set X with a binary operation "*" and a constant 0 satisfying the following axiom :

1. $((x * y) * (x * z)) * (z * y) = 0$,

2. $((x * (x * y)) * y = 0$,

3. $x * x = 0$,

4. $0 * x = 0$

5. $x * y = 0$ *and* $y * x = 0 \Rightarrow x = y$, *for all* $x, y, z \in X$, [8] .

Remarks 2.2

• A partial ordering " $\leq$ " on X can be defined by $x \leq y$ if and only if $x * y = 0$,[7] .

Definition **2.3** A ***semigroup*** is an ordered pair (S,*), where S is a nonempty set and "*" is an associative binary operation on S .[9].

Definition **2.4** *Let X be a non-empty set.A* ***fuzzy subset*** *of X is a function* $\mu : X \to [0, 1]$.[3].

Definition **2.5** *A* ***KS-semigroup*** *is a non-empty set X with two binary operation* $*$ *and . ,and a constant 0 satisfies the following axioms:*

1. $(X, *, 0)$ *is a BCK-algebra,*

2. $(X, .)$ *is a semigroup,*

3. $x.(y * z) = (x.y) * (x.z)$ *and* $(x * y).z = (x.z) * (y.z)$,*for all* $x, y, z \in X$.[6],[7],[9].

Example 2.6 The set X = {0, a, b, c, d} with two binary operations "*" and "." defined by the following tables:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	0	0

.	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	0	b
c	0	0	0	a	b
d	0	a	b	c	d

X is a **KS-semigroup**,[7] .

Definition 2.7 A non empty subset S of a KS-semigroup X with two binary operation "*" and "." is called a **sub ks-semigroup** if it satisfies the following conditions :

1. $x * y \in S$,

2. $xy \in S \quad \forall x, y \in S$. **[**7],[6],[9] **.**

Example 2.8 Let X = {0, e,f} be a KS-semigroup with the following tables:

*	0	e	f
0	0	0	0
e	e	0	e
f	f	f	0

.	0	e	f
0	0	0	0
e	0	e	0
f	0	0	f

The subset S={0,e} of X is a subKS-semigroup of X ,[7] .

Definition **2.9** A fuzzy set μ of X is called a fuzzy subKS-semigroup of X if

1. $\mu(x_1 * x_2) \geq \min\{\mu(x_1), \mu(x_2)\}$ 2. $\mu(x_1 x_2) \geq \min\{\mu(x_1), \mu(x_2)\}$.[7]

Example 2.10 Let X = {0, e,f} be a KS-semigroup with the following tables:

*	0	e	f
0	0	0	0
e	e	0	e
f	f	f	0

.	0	e	f
0	0	0	0
e	0	e	0
f	0	0	f

The fuzzy subset μ of X which is defined by $\mu(0) = 0.6, \mu(x) = 0.3 \quad \forall x \neq 0$ is a fuzzy subKS-semigroup of X ,[7] .

Definition 2.11 Let X be a KS-semigroup and μ be a fuzzy subset of X.for a fixed $0 \leq t \leq 1$,the set $\mu_t = \{x \in X | \mu(x) \geq t\}$ is called **an upper level** set of μ.[7].

Definition 2.12 The Cartesian product of two fuzzy sets μ and ν in X is defined by :

$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}$ *for all* $x, y \in X$.[3] ,[7] .

Definition 2.13 Let v be a fuzzy set in X the strong fuzzy relation on X is a fuzzy set $\rho_v : X \times X \to [0,1]$ defined by $\rho_v(x, y) = \min\{v(x), v(y)\} \quad \forall x, y \in X$.[3] ,[7] .

Definition 2.14 Let A and B be a fuzzy sets on X ,define the fuzzy set $A \cap B$ as follows : $\forall x \in X$
$(A \cap B)(x) = \min\{A(x), B(x)\}$, [3] ,[10] .

Proposition 2.15 If μ, v be two fuzzy set of X and $a \le b$ such that $a, b \in [0,1]$, then $\mu_b \subseteq \mu_a$, [3] ,[10] .

3.Main Results

In this section , we find some results about subKS-semigroup of X and fuzzy subKS-semigroup and Cartesian product of fuzzy sets and strong fuzzy relation on KS-semigroups .

Proposition 3.1 Let A and B be a subKS-semigroup of X then $A \cap B$ is a subks-semigroup .

Proof: Let A and B are subKS-semigroup and let $x, y \in A \cap B.$ Then

$x, y \in A \quad and \quad x, y \in B \quad since \quad A, B \quad are\ subKS-semigroup$

$so\ x * y \in A \quad and \quad x * y \in B \quad then \quad x * y \in A \cap B \ . \quad Now,$

$so\ xy \in A \quad and \quad xy \in B \quad therefore\ xy \in A \cap B$

$Hence \quad A \cap B\ is\ a\ subKS-semigroup.$

Proposition 3.2 Let A and B are fuzzy subKS-semigroup of X. Then $A \cap B$ is a fuzzy subKS-semigroup .

Proof: Let A and B are the fuzzy subKS-semigroups and let $x, y \in A \cap B$ then

$$\begin{aligned}(1)\ (A \cap B)(x * y) &= \min\{A(x * y), B((x * y)\} \\ &\ge \min\{\min\{A(x), A(y)\}, \min\{B(x), B(y)\}\} \\ &= \min\{\min\{A(x), B(x)\}, \min\{A(y), B(y)\}\} \\ &= \min\{A \cap B(x), A \cap B(y)\}.\end{aligned}$$

$$\begin{aligned}(2)\ (A \cap B)(xy) &= \min\{A(xy), B((xy)\} \\ &\ge \min\{\min\{A(x), A(y)\}, \min\{B(x), B(y)\}\} \\ &= \min\{\min\{A(x), B(x)\}, \min\{A(y), B(y)\}\} \\ &= \min\{A \cap B(x), A \cap B(y)\}.\end{aligned}$$

Hence $A \cap B$ is a fuzzy subKS-semigroup .

Lemma 3.3 If μ is a fuzzy subKS-semigroup of X , then $\mu(0) \ge \mu(x) \quad \forall x \in X$ **.**

Proof : Let μ be a fuzzy subKS-semigroup and let $x \in X$ so x*x=0 for all $x \in X$,therefore

$\mu(0) = \mu(x * x)) \ge \min\{\mu(x), \mu(x)\} = \mu(x)$ that is mean $\mu(0) \ge \mu(x) \quad \forall x \in X$ **.**

Which completes the proof **.**

Proposition 3.4 Let μ be a fuzzy subKS-semigroup of X with finite Im(μ) .if $\mu_s = \mu_t$ for some $s, t \in \mathrm{Im}(\mu)\ then\ s = t$.

Proof: Let $s, t \in \mathrm{Im}(\mu), where\ \mathrm{Im}(\mu)$ is finite ,let $x, y \in X\ s.t\ \ \mu(x) = s\ and\ \ \mu(y) = t$ then $x \in \mu_s\ \ and\ \ y \in \mu_t$, since $\mu_t = \mu_s \rightarrow x \in \mu_s = \mu_t\ \ then\ x \in \mu_t\ \ so\ \ s = \mu(x) \ge t ....(1)$,

also $y \in \mu_t = \mu_s \rightarrow y \in \mu_s \rightarrow t = \mu(y) \ge s ....(2)$ frome (1) and (2) $\rightarrow s = t$.

Proposition 3.5 Let μ and ρ be two fuzzy subKS-semigroup of X with the same family of levels.if Im(μ) $= \{t_1, ..., t_m\}$ and Im(ρ) $= \{s_1, ..., s_p\}$ where $t_1 > t_2 > ..... > t_m \quad and \quad s_1 > s_2 > ...... > s_p$ then

1- $m = p$

2- $\mu_{t_i} = \rho_{s_i} \quad for\ \ i = 1, ..., m.$

3- If $\mu(x) = t_i \quad then\ \ \rho(x) = s_i \quad \forall\ i = 1, ..., m$.

Proof: (1) Let μ and ρ be two fuzzy subKS-semigroup of X, since μ and ρ we have the same family of levels then $m = p$.

(2) let Im(μ) $=\{t_1,........,t_m\}$ and Im(ρ) $=\{s_1,....,s_p\}$ where $t_1 > t_2 > ..... > t_m$ *and* $s_1 > s_2 > ...... > s_p$, since μ and ρ have the same family of levels, so first if we take i=m then since $\mu_{t_m} \supset .... \supset \mu_{t_1}$, $\rho_{s_p} \supset .... \supset \rho_{s_1}$ and by (1) we have $\mu_{t_m} = \rho_{s_p}$ now let $i = m-1$ then we have $\mu_{t_{m-1}} = \rho_{s_{p-1}}$ and so on $\mu_{t_i} = \rho_{s_i}$ $\forall i = 1,....,m$.

to proof (3) let $x \in G$ *s.t* $\mu(x) = t_i$ *and* $\rho(x) = s_j$ from (2) and $\mu(x) = t_i$ we have $x \in \rho_{s_i}$ *thus* $\rho(x) \geq s_i$ *and* $s_j \geq s_i$ *that is mean* $\rho_{s_j} \subseteq \rho_{s_i}$

since $x \in \rho_{s_j} = \mu_{t_j}$ *we have* $t_i = \mu(x) \geq t_j$ *, so*

$\mu_{t_i} \subseteq \mu_{t_j}$ *and in the consequence (by 2)* $\rho_{s_i} = \mu_{t_i} \subseteq \mu_{t_j} = \rho_{s_j}$ *thus* $\rho_{s_i} = \rho_{s_j}$

but, by corollary 3.4 $s_i = s_j$ *therefore* $\rho(x) = s_i$.

Corollary 3.6 If a fuzzy sub KS-semigroup μ and ρ defined on X have the same finite family of levels then $\mu = \rho$ if and only if Im(μ)=Im(ρ) .

Proof: Let $\mu = \rho \rightarrow \mu(x) = \rho(x) \quad \forall x \in X \rightarrow \text{Im}(\mu) = \text{Im}(\rho)$.

Conversely let $\text{Im}(\mu) = \text{Im}(\rho) = \{t_1,....,t_n\}$

Let $x \in X$ *and* $\mu(x) = t_i \ni i = 1,.....,n$ since μ and ρ have the same finite family of levels so by corollary 3.5 $\mu_{t_i} = \rho_{s_i}$ *for* $i = 1,...,n$ and

$\mu(x) = t_i$ *then* $\rho(x) = t_i \rightarrow \mu(x) = t_i = \rho(x) \quad \forall x \in X \qquad \therefore \mu = \rho$.

Theorem 3.7 A fuzzy set μ of X is a fuzzy subKS-semigroup if and only if for every $t \in [0,1]$, μ_t is either empty or a sub KS-semigroup of X ,[7] .

Lemma 3.8 Let X be a KS-semigroup and let μ , ν be a fuzzy subKS-semigroup then $\mu \times \nu$ is a fuzzy subKS-semigroup .

Proof: Let μ , ν be a fuzzy subKS-semigroups $\ni (x_1, y_1),(x_2, y_2) \in X \times X$ then

$$
\begin{aligned}
(\mu \times \nu)((x_1, y_1) * (x_2, y_2)) &= \mu \times \nu\,((x_1 * x_2, y_1 * y_2)) \\
&= \min\{\mu(x_1 * x_2), \nu(y_1 * y_2)\} \\
&\geq \min\{\min\{\mu(x_1), \mu(x_2)\}, \min\{\nu(y_1), \nu(y_2)\}\} \\
&= \min\{\min\{\mu(x_1), \nu(y_1)\}, \min\{\mu(x_2), \nu(y_2)\}\} \\
&= \min\{\mu \times \nu(x_1, y_1), \mu \times \nu(x_2, y_2)\}
\end{aligned}
$$

$$
\begin{aligned}
(\mu \times \nu)((x_1, y_1).(x_2, y_2)) &= \mu \times \nu\,((x_1..x_2, y_1.y_2)) \\
&= \min\{\mu(x_1..x_2), \nu(y_1.y_2)\} \\
&\geq \min\{\min\{\mu(x_1), \mu(x_2)\}, \min\{\nu(y_1), \nu(y_2)\}\} \\
&= \min\{\min\{\mu(x_1), \nu(y_1)\}, \min\{\mu(x_2), \nu(y_2)\}\} \\
&= \min\{\mu \times \nu(x_1, y_1), \mu \times \nu(x_2, y_2)\}
\end{aligned}
$$

Hence $\mu \times \nu$ is a subKS-semigroup .

Remark 3.9 It is important to note that the converse of above lemma is not true since if we take two fuzzy sets μ and ν such that $\mu(x) \leq \nu(x) \quad \forall x \in X$.

the Cartesian product of μ and ν depends only on μ .

Theorem 3.10 Let X be a KS-semigroup and μ , λ be two fuzzy sets in X such that $\mu \times \nu$ is a fuzzy subKS-semigroup of $X \times X$ then :

1. either $\mu(x) \le \mu(0)$ *or* $\lambda(x) \le \lambda(0)$ *for all* $x \in X$.
2. if $\mu(x) \le \mu(0)$ *for all* $x \in X$ then either $\mu(x) \le \lambda(0)$ *or* $\lambda(x) \le \lambda(0)$.
3. if $\lambda(x) \le \lambda(0)$ *for all* $x \in X$ then either $\mu(x) \le \mu(0)$ *or* $\lambda(x) \le \mu(0)$.
4. either μ or λ is a fuzzy subKS-semigroup of X .

Proof: (1) Suppose that $\mu(x) > \mu(0)$ *and* $\lambda(y) > \lambda(0)$ *for some* $x, y \in X$ .then
$(\mu \times \lambda)(x, y) = \min\{\mu(x), \lambda(y)\} > \min\{\mu(0), \lambda(0)\} = (\mu \times \lambda)(0,0)$
which is contradiction .so either $\mu(x) \le \mu(0)$ *or* $\lambda(x) \le \lambda(0)$ $\forall$ $x \in X$.

(2) Let $\mu(x) > \lambda(0)$ *and* $\lambda(y) > \lambda(0)$ *for some* $x, y \in X$ then $(\mu \times \lambda)(0,0)$ =min $\{\mu(0), \lambda(0)\}$= $\lambda(0)$ so
$(\mu \times \lambda)(x, y) = \min\{\mu(x), \lambda(y)\} > \lambda(0) = (\mu \times \lambda)(0,0)$ This is a contradiction .
Therefore (2) holds .

We can prove (3) in a similar way of (2) .

(4) Since by (1) either $\mu(x) \le \mu(0)$ *or* $\lambda(x) \le \lambda(0)$ *for all* $x \in X$
without loss of generality we may assume that $\lambda(x) \le \lambda(0)$
it follows from (3) that either $\mu(x) \le \mu(0)$ *or* $\lambda(x) \le \mu(0)$ if $\lambda(x) \le \mu(0)$ $\forall x \in X$ then

$$
\begin{aligned}
\lambda(x_1 * x_2) = \min\{\mu(0), \lambda(x_1 * x_2)\} &= (\mu \times \lambda)(0, x_1 * x_2) \\
&= (\mu \times \lambda)(0 * 0, x_1 * x_2) \\
&= (\mu \times \lambda)((0, x_1) * (0, x_2)) \\
&\ge \min\{(\mu \times \lambda)(0, x_1), (\mu \times \lambda)(0, x_2)\} \\
&= \min\{\min\{\mu(0), \lambda(x_1)\}, \min\{\mu(0), \lambda(x_2)\}\} \\
&= \min\{\lambda(x_1), \lambda(x_2)\}.
\end{aligned}
$$

$$
\begin{aligned}
\lambda(x_1 . x_2) = \min\{\mu(0), \lambda(x_1 . x_2)\} &= (\mu \times \lambda)(0, x_1 . x_2) \\
&= (\mu \times \lambda)(0.0, x_1 . x_2) \\
&= (\mu \times \lambda)((0, x_1).(0, x_2)) \\
&\ge \min\{(\mu \times \lambda)(0, x_1), (\mu \times \lambda)(0, x_2)\} \\
&= \min\{\min\{\mu(0), \lambda(x_1)\}, \min\{\mu(0), \lambda(x_2)\}\} \\
&= \min\{\lambda(x_1), \lambda(x_2)\}
\end{aligned}
$$

So λ is a a fuzzy subKS-semigroup in X .

If $\lambda(x) \le \mu(0)$ is not satisfied then $\lambda(y) > \mu(0)$ *for some* $y \in X$
and by the assumption , $\mu(x) \le \mu(0)$ *for all* $x \in X$ we have

$$\lambda(0) \ge \lambda(y) > \mu(0) \ge \mu(x) \ \ i.e \ \lambda(0) \ge \mu(x) \ \ \forall x \in X .$$

Therefore $(\mu \times \lambda)(x,0) = \min\{\mu(x), \lambda(0)\} = \mu(x)$ and , in the consequence

$$\begin{aligned}\mu(x_1 * x_2) &= (\mu\times\lambda)(x_1 * x_2, 0)\\ &= (\mu\times\lambda)(x_1 * x_2)(0 * 0)\\ &= (\mu\times\lambda)((x_1,0)*(x_2,0))\\ &\geq \min\{(\mu\times\lambda)(x_1,0),(\mu\times\lambda)(x_2,0)\}\\ &= \min\{\mu(x_1),\mu(x_2)\}\ .\end{aligned}$$

$$\begin{aligned}\mu(x_1 . x_2) &= (\mu\times\lambda)(x_1 . x_2, 0)\\ &= (\mu\times\lambda)(x_1 . x_2)(0.0)\\ &= (\mu\times\lambda)((x_1,0).(x_2,0))\\ &\geq \min\{(\mu\times\lambda)(x_1,0),(\mu\times\lambda)(x_2,0)\}\\ &= \min\{\mu(x_1),\mu(x_2)\}\ .\end{aligned}$$

Which proves that μ is a fuzzy subKS-semigroup in X .this completes the proof .

Theorem 3.11 Let X be a KS-semigroup v be fuzzy set Then ρ_v is fuzzy subKS-semigroup if and only if v is fuzzy subKS-semigroup .

Proof: Let v be fuzzy set on X and let $x_1, x_2, y_1, y_2 \in X$

1. $$\begin{aligned}\rho_v((x_1,y_1)*(x_2,y_2)) &= \rho_v((x_1 * x_2, y_1 * y_2))\\ &= \min\{v(x_1 * x_2), v(y_1 * y_2)\}\\ &\geq \min\{\min\{v(x_1),v(x_2)\},\min\{v(y_1),v(y_2)\}\}\\ &= \min\{\min\{v(x_1),v(y_1)\},\min\{v(x_2),v(y_2)\}\}\\ &= \min\{\rho_v(x_1,y_1),\rho_v(x_2,y_2)\}\end{aligned}$$

2. $$\begin{aligned}\rho_v((x_1,y_1).(x_2,y_2)) &= \rho_v(x_1x_2, y_1y_2)\\ &= \min\{v(x_1x_2), v(y_1y_2)\}\\ &\geq \min\{\min\{v(x_1),v(x_2)\},\min\{v(y_1),v(y_2)\}\}\\ &= \min\{\min\{v(x_1),v(y_1)\},\min\{v(x_2),v(y_2)\}\}\\ &= \min\{\rho_v(x_1,y_1),\rho_v(x_2,y_2)\}\end{aligned}$$

$\therefore \rho_v$ is fuzzy a subKS-semigroup .

Conversely,

Let ρ_v be a fuzzy subKS-semigroup and $(x,y) \in X \times X$ then

$(x,y)*(x,y) = (x*x, y*y) = (0,0)$,so

$\rho_v(0,0) \geq \rho_v(x,y) \quad \forall (x,y) \in X \times X$ by lemma 3.3 , now

$v(0) = \min\{v(0),v(0)\} = \rho_v(0,0) \geq \rho_v(x,x) = \min\{v(x),v(x)\} = v(x)$

$\therefore v(0) \geq v(x) \quad \forall x \in G$

Let $x_1, x_2 \in X$ So

1. $v(x_1 * x_2) = \min\{v(x_1 * x_2), v(0)\}$

$$= \rho_v(x_1 * x_2, 0 * 0)$$
$$= \rho_v((x_1,0),(x_2,0))$$
$$\geq \min\{\rho_v(x_1,0), \rho_v(x_2,0)\}$$
$$= \min\{\min\{v(x_1), v(0)\}, \min\{v(x_2), v(0)\}\}$$
$$= \min\{v(x_1), v(x_2)\}.$$

2. $v(x_1 x_2) = \min\{v(x_1 x_2), v(0)\}$

$$= \rho_v\{(x_1 x_2),(0.0)\}$$
$$= \rho_v\{(x_1.0),(x_2.0)\}$$
$$\geq \min\{\rho_v(x_1,0), \rho_v(x_2,0)\}$$
$$= \min\{\min\{v(x_1), v(0)\}, \min\{v(x_2), v(0)\}\}$$
$$= \min\{v(x_1), v(x_2)\}$$

$\therefore$ *v is a fuzzy subKS – semigroup.*

References

[1] Y.Imai and K.Iseki, "On axiom systems of propositional calculiXIV ", Proc.Japan Acad,42 (1966),19-22 .
[2] K.Iseki , " An algebra related with a propositional calculus" , Japan Acad . ,42 1966 .
[3] L.A Zadeh ,"Fuzzy sets" ,Information Control ,8 (1965),338-353 .
[4] A.Rosenfeld ,"Fuzzy groups" ,J.Math .Anal .and Appl,35 (1971),512-517 .
[5] O.G,Xi ," Fuzzy BCK- algebras" , Math .Japon,36(1991),935-942 .
[6] K.H Kim ,"On structure of KS-semigroups" ,Int . Math. Forum ,1(2006),67-76 .
[7] Williams, D.R, Prince and Husain Shamshad , "On Fuzzy KS-semigroup" . International Mathematical Forum ,2,2007, no.32 ,1577-1588 .
[8] Celestin Lele and Congxin Wu Harbin "Generalization of Some Fuzzy ideal in BCK- algebras" Institute of Technology ,Department of Mathematics, Harbin 150001,China .
[9] Joncelyn S.Paradero-Vilela and Mila Cawi " On KS-Semigroup Homomorphism " International Mathematical Forum ,4, 2009, no. 23, 1129-1138 .
[10] Mordeson , J.N. ,"Lecture Notes in Fuzzy Mathematics and computer Science ", Creighton University ,Omaha ,Newaska 68178 USA , 1996 .
[11] K.Iseki and S.Tanaka ," An introduction to the theory of BCK-algebras ", Math . Japon 23, (1969),1-26 .
[12] Hungerford, Thomas W., "Algebra" Springer-Verlag New York Inc, 1974 .