Semi d-idealin d-algebra
شبه مثالي الـ(d) في جبر الـ(d)

Habeeb k. Abdullah
Department of Mathematics
Faculty of Education for Girls
Kufa University, Najaf, Iraq
e-mail: habeeb_973@yahoo.com

Ali khaled Hassan
Department of Mathematics
Faculty of Education for Girls
Kufa University, Najaf, Iraq
e-mail: almosaoy.ali@gmail.com

Abstract :
in this paper we introduce the notion of semi d-ideal of d-algebra, and investigate many theorems and examples for this notion, Furthermore we investigate many relations and theorem between semi d-ideal and each of d-ideal and BCK-ideal.

الخلاصة :
قدمنا في هذا البحث مفهوم جديد لشبه مثالي الـ(d) في جبر الـ(d) وناقشنا العديد من المبرهنات والخصائص حول هذا المفهوم بالإضافة إلى بعض الأمثلة، ثم درسنا بعض العلاقات والمبرهنات بين مفهوم شبه مثالي الـ(d) وبين مثالي الـd ومثالي الـ BCK في جبر الـ d.

1. Introduction

Y. Iami and K. Iseki introduced two classes of abstract algebra BCK-algebra and BCI-algebra ([1] , [2]) .It is known that the class of BCK-algebra is proper subclass of the class of BCI-algebra . J. Neggers and H. S. Kim introduced the notion of d-algebra , which is another useful generalization of BCK-algebra [3], and in [4] they introduced the notation of d-ideal on d-algebra . We introduce the notation of semi d-ideal on d-algebra and investigate relations among each of semid-ideal, d-ideal and BCK-ideal.

2. Preliminaries

In this section we recall the basic definition and information which are needed in our work.

Definition 2.1[3]: A d-algebra is a non-empty set X with a constant 0 and a binaryoperation $*$ satisfying the following axioms:

I. $x*x=0$
II. $0*x=0$
III. $x*y=0$ and $y*x=0$ imply that $x=y$ for all x,y in X .

A BCK-algebra is a d-algebra $(X;*,0)$ satisfying the following additional axioms :

IV. $((x*y)*(x*z))*(z*y)=0$
V. $(x*(x*y))*y=0$ for all x,y,z in X .

Definition 2.2[4] Let $(X;*,0)$ be a d-algebra and $\phi \neq I \subseteq X$. I is calleda d-subalgebra of X if $x*y \in I$ whenever $x \in I$ and $y \in I$. I is called aBCK-ideal of X if it satisfies:

(D_0) $0 \in I$

(D_1) $x*y \in I$ and $y \in I$ imply $x \in I$.

I is called a d-ideal of X if it satisfies (D_1) and

(D_2) $x \in I$ and $y \in X$ imply $x*y \in I$ i.e., $I*X \subseteq I$.

Definition 2.2[3]Let $(X;*,0)$ be a d-algebra and $x \in X$. Define $x*X=\{x*a, a \in X\}$. X is said to be edge if for any x in X , $x*X=\{x,0\}$.

Definition 2.3: Let $(X;*,0)$ and $(Y;*,0)$ be a d-algebra. A mapping $f: X \to Y$ is called a

1- d-morphism if $f(x*y) = f(x)*f(y)$ for any $x, y \in X$. Note that $f(0)_X = 0_Y$ **[3]**.

2- d-isomorphism If f is a bijective and d-morphism function

Definition 2.4 [4] : A d-algebra X is called a d^*-algebra if it satisfies the identity $(x*y)*x = 0$ for all $x, y \in X$.

Theorem 2.5 [4] : In a d^*-algebra, every BCK-ideal is a d-ideal .

Corollary 2.6 [4] : In a d^*-algebra, every BCK-ideal is a d-subalgebra.

3. Semi d- ideal

In this section we introduce the notation of semi d-ideal on d-algebra and investigate relations among semi d-ideal, d-ideal and BCK-ideal.

Definition 3.1: A semi d-ideal of a d-algebra X is a non empty subset F of X satisfies

i) $x, y \in F$ imply $x*y \in F$,

ii) $x*y \in F$ and $y \in F$ imply $x \in F$, for all $x, y \in X$.

Note that X and $\{0\}$ are semi d-ideal for any d-algebra X .and If X is a d-algebra then every semi d-ideal of X is a d-algebra with the same binary operation on X and the constant 0 .

Examples 3.2 :

1) consider the following d-algebra X [3] where $X = \{0,1,2,3\}$ with the following table

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

It is clear that F = {0,1} and M={0,1,2} are a semi d-ideal in X since (i) and (ii) are hold in both .

2)Let $X = \{0, a, b, c, d\}$ with the following table be a d-algebra [5]

*	0	a	b	c	d
0	0	0	0	0	0
a	A	0	0	a	0
b	B	b	0	0	b
c	C	c	c	0	c
d	D	d	d	d	0

Then $I = \{0, a\}$ is semi d-ideal and also $J = \{0, a, b\}$, $L = \{0, a, d\}$, $K = \{0, a, b, c\}$, $M = \{0, a, b, d\}$ all are a semi d-ideal in X .

3)Let $\Re$ be the set of all real numbers and define $x*y = x.(x-y)$, $x, y \in \Re$, where $\cdot$ and – are the ordinary product and subtraction of real numbers. Then $(\Re;*,0)$ is a d-algebra [3]. Let $Q \subset \Re$ be the set of rational number then Q is an infinite semi d-ideal in $\Re$ since :

i) Let $a, b \in Q$, $a*b = a.(a-b) = a^2 - ab$, since $a, b \in Q$ then $a^2, ab \in Q$ so $a^2 - ab \in Q$ thus $a*b \in Q$.

ii) Let $a*b \in Q$, $b \in Q$, $a*b = a^2 - ab \in Q$, $b \in Q$ so it is clear that $a^2 - ab \in Q$, since $a.b \in Q$ and $.b \in Q$ then $a \in Q$.

Theorem 3.3: The intersection of a family of semi d-ideal in a d-algebra X is a semi d-ideal .
Proof : Let $A_i, i \in I$ is a semi d-ideal of d-algebra X
let $x, y \in \bigcap_{i\in I} A_i$, then $x, y \in A_i$ for all i in I,
so $x * y \in A_i$ (since A_i is a semi d-ideal for all i in I), so $x * y \in \bigcap_{i\in I} A_i$.
Now let $x * y \in \bigcap_{i\in I} A_i$ and $y \in \bigcap_{i\in I} A_i$, so $x * y \in A_i$ and $y \in A_i$, for all i in I ,
since A_i is a semi d-ideal in X for all i in I , then $x \in A_i$, for all i in I thus $x \in \bigcap_{i\in I} A_i$, and this complete the proof . ■

Remark 3.4: The union of two semi d-ideal of d-algebra X not necessary to be semi d-ideal in X as a following example

Example 3.5: Let X = { 0,a,b,c } with the following table

*	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	b
c	c	c	c	0

X is a d-algebra [6] , and it is clear that I = {0,a}and J = {0,c} both are semi d-ideal of X . but $I \cup J$ = {0,a,c}is not semi d-ideal of X , since $a * c = b$ that mean the condition (i) not hold .

The condition that make the union of two semi d-ideal is semi d-ideal when X is edge d-algebra and the following theorem showing that , but before the theorem we will take an important lemma .

Lemma 3.6 : If F is a semi d-ideal in d-algebra X then $0 \in F$.
Proof : since $\phi \neq F$, then there exist $x \in F$, hence $x * x = 0 \in F$ by (i). ■

Theorem 3.7 : Let I and J be a semi d-ideal in edge d-algebra X then $I \cup J$ is a semi d-ideal .
Proof : Let $x, y \in I \cup J$.
If $x, y \in I$ or $x, y \in J$, is clear that $x * y \in I \cup J$, now if $x \in I$ and $y \in J$, since X is edge d-algebra $x * y = 0$ or $x * y = x$, if $x * y = 0$ and since $0 \in J$ (by lemma 3.6) then $x * y \in J$ thus $x * y \in I \cup J$, if $x * y = x$ then $x * y \in I$, thus $x * y \in I \cup J$
NowLet $x * y \in I \cup J$ and $y \in I \cup J$, it is clear that if $x * y \in I$ and $y \in I$ or $x * y \in J$ and $y \in J$ then $x \in I \cup J$, now if $x * y \in I$ and $y \in J$ and $y \notin I$, since X is edge d-algebra then either $x * y = 0$ or $x * y = x$ if $x * y = 0$ then $x * y \in J$ (since $0 \in J$ (by lemma 3.6)) and thus $x \in J$ so $x \in I \cup J$, if $x * y = x$ is clear that $x \in I \cup J$. Similarly if $x * y \in J$ and $y \in I$ and $y \notin J$ we can prove that $x \in I \cup J$. Thus $I \cup J$ is a semi d-ideal of X . ■

Remark3.8 [3] : If $(X, \leq)$ is an ordered set (poset) , then the operation $*$ on X given by $x * y = 0$ if and only if $x \leq y$.

Proposition 3.9 : in a semi d-ideal I if $x \in I$ and $y \leq x$ then $y \in I$
Proof : it is clear that if $y \leq x$ then $y * x = 0$ thus $y \in I$ (by lemma 3.6, and ii in Definition 3.1) ■

Remark 3.10: Every semi d-ideal is a d-subalgebra . but the converse need not be true as showing in following example

Example 3.11 : Let $(X; *, 0)$ in (Example 3.2 ,2) is a d-algebra so is clear that S = {0,c} is a d-subalgebra , butisn't a semi d-ideal since $b * c = 0$ but $b \notin S$, so the condition (ii) not hold.

Theorem 3.12 : Every d- ideal is a semi d-ideal in d-algebra X .
Proof : let F be a d-ideal of X is clear that every d-ideal is a d-subalgebra , so the condition (i) is hold and the condition (ii) is same condition (D_1) in Definition 2.1, thus f is semi d-ideal . ■

The converse of this theorem need not be true in general and the following example showing that .

Example 3.13 : Let X = {0,a,b,c} and a binary operation $*$ is define as following table

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	c
c	c	c	c	0

It is clear that (X;*,0) is a d-algebra , and A = {o,b} is a semi d-ideal in X since (i) and (ii) are hold. But it is not d-ideal since $b \in A,\ c \in X$ and $b * c = c \notin A$ i.e. $F * X \not\subset F$.

The condition that make every semi d-ideal is a d-ideal , when X is edge d-algebra, and we will prove that in the following theorem .

Theorem 3.14 : If X is edge d-algebra then every semi d-ideal in X is a d-ideal.
Proof : Let F be a semi d-ideal in edge d-algebra X, then $0 \in F$ (by lemma 3.6)
let $a * b \in F$, $b \in F$ then $a \in F$ for all $a, b \in X$
now let $x \in F$, $y \in X$, since X is edge d-algebra then $x * X = \{0, x\}$ for all $x \in F$ there for $F * X \not\subset F$, thus F is d-ideal in X . ■

Theorem 3.15 : every semi d-ideal in d-algebra X is a BCK- ideal.
Proof : let F be a semi d-ideal in X then $0 \in F$ (by lemma 3.6) so (D_1) is hold and it is clear that (D_2) is hold since it is same condition (ii) in Definition 3.1 thus F is a BCK-ideal . ■

The converse of this theorem need not be true in general , i.e. a BCK-ideal need not be a semi d-ideal as showing in the following example .

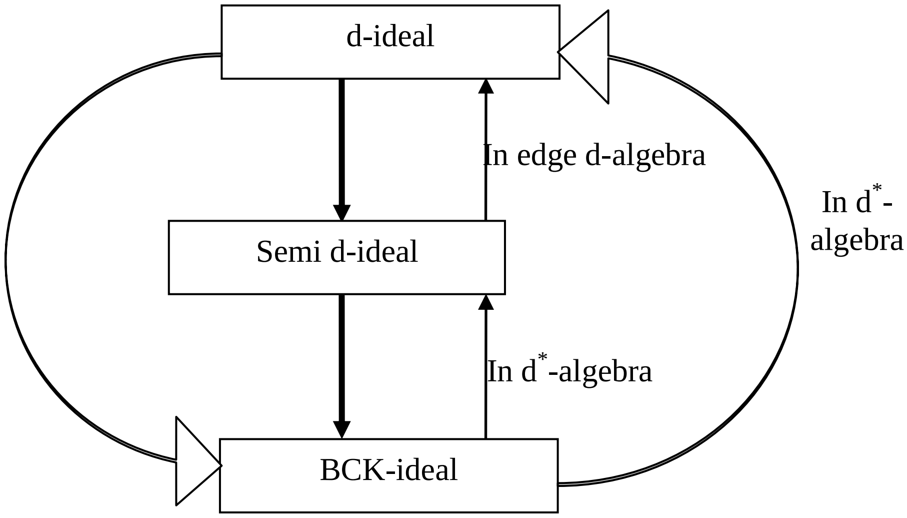
Example 3.16 : Let $X = \{0, a, b, c\}$ be a d-algebra with the following table

*	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	c	0	c
c	c	c	c	0

It is clear that $I = \{0, a, b\}$ is a BCK-ideal [5] , which is not a semi d-ideal since $b * a = c$ that mean the condition (i) is not hold.
The condition that make every BCK-ideal is a semi d-ideal, that when X is d^*-algebra and we will prove that in the following theorem.

Theorem 3.17 : In d^*-algebra, every BCK-ideal is a semi d-ideal.
Proof : it is clear by corollary 2.6 and definition 2.2 ■

The following diagram showing the relation between semi d-ideal, d-ideal and BCK-ideal.



Proposition 3.18 : The d-isomorphism image of semi d-ideal is a semi d-ideal.

Proof : Let $f : X \to Y$ be a function from d-algebra X to d-algebra Y, and let A is a semi d-ideal in X ,
Let $x, y \in f(A)$ then there exist tow element a, b in A such that $f(a) = x, f(b) = y$, so $x * y = f(a) * f(b) = f(a * b) \in f(A)$, (since A is a semi d-ideal in X).
Nowlet $x * y \in f(A)$ and $y \in f(A)$, then there exist tow element z, w in A such that $f(z) = x * y$, $f(w) = y$, so $f(z * w) = f(z) * f(w) \in f(A)$, since f is onto, then there exist an element a in A such that $f(a) = x$ and $f(z) = f(a) * f(w)$ then $[f(a) * f(w)] * f(w) \in f(A)$ there for $f[(a * w) * w] \in f(A)$, since f is one to one then $(a * w) * w \in A$ then $a \in A$ so $x \in f(A)$.
Thus $f(A)$ is a semi d-ideal in Y. ■

Proposition 3.19 : The inverse d-isomorphism image of semi d-ideal is a semi d-ideal..
Proof :: Let $f : X \to Y$ be a function from d-algebra X to d-algebra Y, and let B is a semi d-ideal in Y,
let $x, y \in f^{-1}(B)$ then $f(x), f(y) \in B$ so $f(x) * f(y) \in B$, then $x * y \in f^{-1}(B)$.
Now let $x * y \in f^{-1}(B)$ and $y \in f^{-1}(B)$ so $f(x * y) \in B$ and $f(y) \in B$ then $f(x) \in B$ so $x \in f^{-1}(B)$. This complete the proof . ■

Proposition 3.20 : Let f be a d-morphism function from d-algebra X to d-algebra Y, then ker(f) is a semi d-ideal.
Proof :
let $x, y \in \ker(f)$ so $f(x) = f(y) = 0$ and $f(x * y) = 0$, then $x * y \in \ker(f)$.
Now let $x * y \in \ker(f)$ and $y \in \ker(f)$ so $f(x * y) = 0$ and $f(y) = 0$, then $f(x * y) = f(x) * f(y) = 0$, but $f(y) * f(x) = 0 * f(x) = 0$ then $f(x) = f(y) = 0$ so $x \in \ker(f)$.
Thus $\ker(f)$ is a semi d-ideal. ■

References

1. Y. Iami and K. Iseki " on axiom system of propositional calculi XIV " Proc. Japan Acad, 42 (1966) 19-20.
2. K. Iseki " An algebra relation with propositional calculus " Proc. Japan Acad, 42 (1966) 26-29.
3. J. Neggers and H. S. Kim, " on d-algebra " , Math. Slovaco . 49(1999) No. 1, 19-26.
4. J. Neggers, Y. B. Jun and H. S. Kim, " on d-ideals in d-algebra " , Math. Slovaco . 49(1999) No. 3, 243-251.
5. S. S. Ahn and G. H. Han, Intuitionistic fuzzy quick ideals in d-algebras, Honam Math. J. 31 (2009), no. 3, 351–368.
6. S. S. Ahn and G. H. Han " Rouch fuzzy quick ideal in d-algebra " commun. Korean Math Soc. 25(2010) N0. 4, 511-522.