Composition operator induced by φ(z) = sz + t for which |s|≤1, |t|<1 and |s|+|t|≤1

*Samira Naji Kadhim**

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Abstract:
We study in this paper the composition operator that is induced by φ(z) = sz + t. We give a characterization of the adjoint of composiotion operators generated by self-maps of the unit ball of form φ(z) = sz + t for which |s|≤1, |t|<1 and |s|+|t|≤1. In fact we prove that the adjoint is a product of toeplitz operators and composition operator. Also, we have studied the compactness of C_φ and give some other partial results.

Key words: composition operator, toeplitz operator, compact operator

Introduction:

Let U denote the unit ball in the complex plan, the Hardy space H^2 is the collection of holomorphic (analytic) function f(z) = $\sum_{n=0}^{\infty} \hat{f}(n) z^n$ with $\hat{f}(n)$ denoting the n-th Taylor coefficient , for which $\sum_{n=0}^{\infty} |\hat{f}(n)|^2<\infty$. The norm is defined by $\|f\|^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2$ ($f \in H^2$). The particular importance of H^2 is duo to the fact that it is a Hilbert space. Let φ be a holomorphic function that take the unit ball U into itself (which is called homomorphic self-map of U). The composition operator C_φ induced by φ is defined on H^2 by the equation $C_\varphi f = f \circ \varphi$ ($f \in H^2$) [1].

We state very loosely some basic facts on composition operator on H^2.

Theorem 1:- Every composition operator C_φ is bounded.

Theorem 2:- C_φ is normal if and only if φ(z) = λz, |λ|≤1.

Theorem 3:- $C_\varphi C_\psi = C_{\psi\circ\varphi}$.

Furthermore an important special family of function in H^2, namely $\{K_\alpha\}_{\alpha \in U}$. For each $\alpha \in U$, Shapiro in [1], defined $K_\alpha = \frac{1}{1-\bar{\alpha}z} = \sum_{n=0}^{\infty} \bar{\alpha}^n z^n$.

It is clear for each $f \in H^2$, $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ that $< f, K_\alpha > = \sum_{n=0}^{\infty} \hat{f}(n) \alpha^n = f(\alpha)$. Shapiro in [1] gives the adjoint of a composition operator on $\{K_\alpha\}_{\alpha \in U}$ in the following theorem.

Theorem 4:- Let φ be a holomorphic self-map of U, then for all $\alpha \in U$, $C_\varphi^* K_\alpha = K_{\varphi(\alpha)}$.

Finally, Bourdon in [2] gives an exact value of the H^2-norm of composition operators induced by φ(z)

$$\|C_\varphi\| = \sqrt{\frac{2}{1+|s|^2-|t|^2+\sqrt{(1-|s|^2+|t|^2)^2-4|t|^2}}}$$

The adjoint of composition operator C_φ

Let H^∞ denote the collection of bounded holomorphic functions on U. The norm on H^∞ is defined by $\|f\|_\infty = \mathrm{Sup}_{z\in U}|f(z)|$ [1].

Recall that for $g \in H^\infty$, the toeplitz operator T_g is the operator on H^2 given by $(T_g f)(z) = g(z)f(z)$, $f \in H^2$, $z \in U$ [3].

In this section we will try to calculate the adjoint of composition operator C_φ

*Department of mathematics/ College of Science for women / University of Baghdad

induced by $\varphi(z) = sz + t$ for which $|s|\le 1$, $|t|<1$ and $|s|+|t|\le 1$.

Theorem 5:- $C_\varphi^* = T_g C_\delta$, where $g(z) = (1 - \bar{t}z)$, $\delta(z) = \frac{sz}{1-\bar{t}z}$.

Proof:- Since $|s|+|t|\le 1$, then $|1-\bar{t}z| > |1-|t|| \ge |s|$ ($|z|<1$ and $|t|<1$). Hence $|\delta(z)|<1$ ($z \in U$). Thus clearly δ maps U into itself. Moreover, $\|g\|_\infty = \mathrm{Sup}_{z\in U}|1-\bar{t}z| < \infty$ (since $|t|<1$). Thus $g \in H^\infty$. This means that the formula makes sense . Now, for each $\alpha \in U$, we have by theorem 4 $C_\varphi^* K_\alpha(z) = K_{\varphi(\alpha)}(z) = \frac{1}{1-(\overline{s\alpha+t})z} = \frac{1}{1-(\bar{s}\bar{\alpha}+\bar{t})z} = \frac{1}{1-\bar{t}z-\bar{s}\bar{\alpha}z} = (1-\bar{t}z)\frac{1}{1-\bar{\alpha}(\frac{\bar{s}z}{1-\bar{t}z})}$

Let $g(z) = (1-\bar{t}z)$, $\delta(z) = \frac{\bar{s}z}{1-\bar{t}z}$. Thus $C_\varphi^* K_\alpha(z) = T_g C_\delta K_\alpha(z)$. Therefore, $C_\varphi^* K_\alpha(z) = T_g C_\delta K_\alpha(z)$. ($z \in U$). Since $\{K_\alpha\}_{\alpha\in U}$ span a dense subset of H^2, the desired equality holds.

Proposition 6:- $C_\delta^* = T_{\tilde{g}}^* C_\varphi$ where $\tilde{g} = 1-\bar{t}z$.

Proof:- By theorem (1.) , $C_\delta^* K_\alpha(z) = K_{\delta(\alpha)}(z) = \frac{1}{1-\overline{\delta(\alpha)}z} = \frac{1}{1-(\frac{\bar{s}\bar{\alpha}}{1-t\bar{\alpha}})z} = \frac{1}{\frac{1-t\bar{\alpha}-\bar{s}\bar{\alpha}z}{1-t\bar{\alpha}}} = \frac{1-t\bar{\alpha}}{1-t\bar{\alpha}-\bar{s}\bar{\alpha}z} = (1-t\bar{\alpha})\frac{1}{1-\bar{\alpha}(sz+t)} = \frac{}{(1-t\bar{\alpha})} = \frac{1}{1-\overline{\alpha\vartheta(z)}}$

$= T_{\bar{\tilde{g}}} K_\alpha(\varphi(z))$

$= T_{\tilde{g}}^* C_\varphi K_\alpha(z)$ (since $T_h^* f = T_{\bar{h}} f$, by [2]).

Since $\{K_\alpha\}_{\alpha\in U}$ span a dense subset of H^2, the desired equality holds.

The compactness of composition operator C_φ induced by $\varphi(z)=sz+t$, for which $|s|\le 1$, $|t|<1$ and $|s|+|t|\le 1$, on Hardy space H^2.

Recall that an operator T on Hilbert space H is compact if it maps every bounded set into a relativity compact one (one whose cloure in H is compact set) [1]. We start this section by the following result which is proved in [1] by Shapiro.

Theorem 7:- Let ψ be a liner fractional self-map of U, that is $\psi(z) = \frac{az+b}{cz+d}$ where a,b,c and d are complex numbers. Then C_ψ is not compact if φ maps a point of the unit circle ∂U to a point of ∂U.

Now, we give the sufficient and necessary condition for compactness of C_φ.

Proposition 8:- C_φ is not compact if and only if $|s|+|t|=1$.

Proof:- Assume that C_φ is not compact, then by theorem 7 there exist $z_1, z_2 \in \partial U$ such that $\varphi(z_1) = z_2$. Hence $1 = |\varphi(z_1)| = |sz_1 + t| \le |s|\,|z_1| + |t| = |s|+|t|\le 1$. Therefore $|s|+|t|=1$.

Conversely assume that $|s|+|t|=1$. Since $|\varphi(z)| = |sz+t| \le |s|\,|z| + |t| = |s|+|t|\le 1$, then by Maximum principle of analytic function [4]. We have for each $z \in \partial U$, then there exists $z_1 \in \partial U$ such that $|\varphi(z_1)| = 1$. Hence by theorem 7 C_φ is not compact.

Notation :- We use the notation $\varphi_n = \varphi \circ \varphi \circ \cdots \circ \varphi$ (n times). To denote the n-th iterate of φ for n a positive integer.

Remark 9:- By theorem 3 we can conclude that $C_\varphi^n = C\varphi_n$ for each positive integer.

Now, we study the compactness of n-th power of C_φ.

Theorem 10:- C_φ^n is compact operator for every positive integer n if and only if $|s| + |t| < 1$ where $|s| < 1$ and $|t| < 1$.

Proof:- By using mathematical induction of $\varphi(z)$ we get $\varphi_n(z) = s^n z + (\sum_n^\alpha = 0\, s^n)t$. Since the geometric series $\sum_n^\alpha = 0\ S^n$ is convergent if $|s| < 1$. Then φ_n is a linear-fractional self map of U where $|s| < 1$. First suppose that $|s| + |t| < 1$, then by proposition 8 C_φ is compact, so C_φ^n is compact for every positive integer n.

Conversely, assume that C_φ^n is compact for every positive integer n. To show that $|s| + |t| < 1$, assume the converse that $|s| + |t| \geq 1$. This implies by proposition 8 C_φ is not compact which is a contradiction. Thus $|s| + |t| < 1$.

Now, we give the following results.

Proposition 11:- Suppose that Φ is a linear-fractional self-map of U. Then $C_\Phi C_\delta$ is compact if and only if $C_\Phi C_\varphi^*$ is compact, where $C_\varphi^* = T_g C_\delta$.

Proof:- Suppose $C_\Phi C_\delta$ is compact. Note that, $C_\Phi C_\varphi^* = C_\Phi T_g C_\delta = T_{g\circ\Phi} C_\varphi C_\delta$ (by theorem 5 and $C_\varphi T_g = T_{g\circ\Phi} C_\varphi$).

Since $C_\Phi C_\delta$ is compact operator. Moreover, $T_{g\circ\Phi}$ is bounded, then $C_\Phi C_\varphi^*$ is compact. Conversely, if $C_\Phi C_\varphi^*$ is compact.

Note that $C_\Phi C_\delta = C_\Phi (C_\delta^*)^*$

$= C_\Phi (T_{\bar{g}}^* C_\varphi)^*$ (by proposition 6).

$= C_\Phi C_\varphi^* T_{\bar{g}}^*$

$= C_\Phi C_\varphi^* T_{\bar{g}}$ (since $T_{\bar{g}}^* = T_{\bar{g}}$).

Since $C_\Phi C_\varphi^*$ is compact and $T_{\bar{g}}$ is bounded then $C_\Phi C_\delta$ is compact.

Proposition 12:- Suppose that Φ is a linear-fractional self-map of U. Then $C_\Phi C_\delta^*$ is compact if and only if $C_\Phi C_\varphi$ is compact.

Proof:- Suppose that $C_\Phi C_\delta^*$ is compact. Then

$C_\Phi C_\varphi = C_\Phi (C_\varphi^*)^* = C_\Phi (T_g C_\delta)^*$ (by theorem 5)

$= C_\Phi C_\delta^* T_g^*$

$= C_\Phi C_\delta^* T_{\bar{g}}$ $(T_g^* = T_{\bar{g}})$

Since $C_\Phi C_\delta^*$ is compact and $T_{\bar{g}}$ is bounded it follows that $C_\Phi C_\varphi$ is compact.

Conversely, if $C_\Phi C_\varphi$ is compact, $C_\Phi C_\delta^* = C_\Phi T_{\bar{g}}^* C_\varphi$ (by proposition 6)

$= C_\Phi T_{\bar{g}} C_\varphi$ (since $T_{\bar{g}}^* = T_{\bar{g}}$) [3]

$= T_{\bar{g}\circ\varphi} C_\Phi C_\varphi$ $(C_\Phi T_{\bar{g}} = T_{\bar{g}\circ\varphi} C_\varphi)$[3]

Since $C_\Phi C_\varphi$ is compact and $C_\Phi C_\varphi$ is bounded, then $C_\Phi C_\delta^*$ is compact.

Proposition 13:- Let Φ be a linear fractional self-map of U. Then $C_\delta C_\Phi$ is compact, if and only if $C_\varphi^* C_\Phi$ is compact, where $C_\varphi^* = T_g C_\delta$.

Proof:- suppose that $C_\delta C_\Phi$ is compact. Then $C_\varphi^* C_\Phi = T_g C_\delta C_\Phi$ (by theorem 5). Since $C_\delta C_\Phi$ is compact, then $C_\varphi^* C_\Phi$ is compact.

Conversely, assume that $C_\varphi^* C_\Phi$ is compact. Since the family $\{K_\alpha\}_{\alpha \in U}$ span a dense subset in H^2, then it is enough to prove the compactness on this family. Hence for each $\alpha \in U$,

$C_\delta C_\Phi K_\alpha(z) = (C_\delta^*)^* C_\Phi K_\alpha(z)$

$= (T_{\bar{g}} C_\varphi)^* C_\Phi K_\alpha(z)$ $(C_\delta^* = T_{\bar{g}}^* C_\varphi)$

$= C_\varphi^* T_{\bar{g}}^* C_\Phi K_\alpha(z)$

$= C_\varphi^* T_{\bar{g}} C_\Phi K_\alpha(z)$ (since $T_{\bar{g}}^* = T_{\bar{g}}$) [3]

$= C_\varphi^* T_{\bar{g}} K_\alpha(\Phi(z))$

$= C_\varphi^* \overline{g(\alpha)} K_\alpha(\varphi(z))(T_{\bar{g}}^* K_\alpha = \overline{g(\alpha)} K_\alpha)$

$= \overline{g(\alpha)} C_\varphi^* C_\Phi K_\alpha(z)$ (C_φ^* is linear)

Since $C_\varphi^* C_\Phi$ is compact, moreover, $g \in H^\infty$, then $C_\delta C_\Phi$ is compact on $\{K_\alpha\}_{\alpha \in U}$. But $\{K_\alpha\}_{\alpha \in U}$ span a dense subset in H^2. Hence $C_\delta C_\Phi$ is compact on H^2.

Similarly to the proof of the previous proposition we can get the following result.

Proposition 14:- Let Φ be a linear-fractional self-map of U. Then $C_\delta^* C_\Phi$ is compact, if and only if $C_\varphi C_\Phi$ is compact, where $C_\varphi^* = T_g C_\delta$.

Corollary 15:- Suppose that Φ is a linear fractional self-map of U such that $C_\Phi C_\delta^*$ is not compact, then there exist $w_1,w_2 \in \partial U$ such that $\varphi \circ \Phi (w_1) = w_2$.

Proof:- By proposition 12, if $C_\Phi C_\delta^*$ is not compact, then C_Φ C_φ is not compact. But each of Φ and φ are linear- functional self-map of U, then also $\varphi \circ \Phi$. Then by theorem 7 $C_{\varphi\circ\Phi} = C_\varphi \circ C_\Phi$ is not compact, if and only if $\varphi \circ \Phi$ maps a point of the unit circle onto the unite circle. So, there exist $w_1,w_2 \in \partial U$ such that $\varphi \circ \Phi (w_1) = w_2$.

Similarly to the proof of corollary 15. We have by proposition 14 and theorem 4 the next result.

Corollary 16:- Suppose that Φ is a linear-fractional self-map of U such that $C_\delta^* C_\Phi$ is not compact, then there exist $w_1,w_2 \in \partial U$ such that $\Phi \circ \varphi (w_1) = w_2$.

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المؤثر التركيبي المتولد بالدالة $\varphi(z) = sz + t$ لكل $|t|<1$, $|s|\leq 1$ و $|s|+|t|\leq 1$

*سميرة ناجي كاظم**

*قسم الرياضيات / كلية العلوم للبنات / جامعة بغداد

الخلاصة:

في هذا البحث أعطي وصف للمؤثر مرافق للمؤثر C_φ المتولد بواسطة الدالة $\varphi(z) = sz + t$ بحيث ان $|s|\leq 1$, و$|t|<1$ $|s|+|t|\leq 1$. بالحقيقة برهن انه المرافق هو عبارة عن ضرب المؤثرات نتوبلتز مع مؤثر تركيبي . و كذلك درسنا تراص المؤثر التركيبي C_φ مع بعض النتائج التي هي حسب علمنا جديدة.