The Convergent and Discrete Maximum Principle of Finite Element Methods for solving the Convection-Diffusion-Reaction problem.

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Abstract

In this paper, we describe several finite element methods for solving the Convection-Diffusion-Reaction problem(CDR)and study two important properties for the approximate solution, the convergent and the discrete maximum principle.

For convergence we considered two cases, semi-discrete and discrete methods, for semi-discrete, we prove that all scheme are converge with $O(h^r)$, where h refers to the discretization parameter, $1 \le r \le s+1$ and s is a degree of polynomials of finite element space and for discrete we proved that all schemes converge with $O(h^r+\tau)$. Finally, the discrete maximum principle and L^∞- stable are proved.

1. Introduction

Diffusion, convection and reaction are fundamental processes in physical, biological, chemical fluid dynamics, heat and mass transfer and so on. This is the reason that the numerical solution of these kinds of problems attracts a number of specialists, engineers as well as mathematician [2, 3, 5, and 7]. The objective of this parer is to compare several finite element methods for solving the linear diffusion – convection- reaction equation from the point of view of the formulation of the methods. The methods that will be described are the following [1]

- New scheme- Gale kin (NS1-G),
- Modified problem -new schem1- Galer kin (MPs-NS1-G)
- Modified problem -new schem2- Galerkin (MPs-NS2-G)
- Modified problem -new schem3- Galerkin (MPs-NS3-G)

Essentially, all these methods consist in the addition of a stabilizing term to the original Galerkin formulation of the problem.

We have organized this paper as follows. The statement of the problem is present in the next section. In section, three presented the semi and full discrete cases. The error estimate and the stability are presented in section 4. In section 5,

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we prove the discrete maximum principle. The numerical results are shown in section 6

2. Time-dependent modeling problem

Consider the linear time-dependent CDR problem] (2], [3])

$$u_t - a\Delta u + b.\nabla u + cu = f \quad in \quad \Omega\times(0,T] \quad , \tag{1}$$
$$u(x,t) = 0 \quad on \quad \Gamma\times(0,T],$$
$$u(x,0) = u^0 \quad on \quad \overline{\Omega}$$

Where $\Omega\subset R^2$ with boundary Γ, $\overline{\Omega}=\Omega\bigcup\Gamma, and \ (a>0, c\geq 0)\in L^\infty(\Omega)$ are the diffusion coefficient and the reaction coefficient respectively. $b=(b_1,b_2):\Omega\times(0,T]\to R^2$ Is a convection coefficient.

The weak form of equation (1) is: Find $u\in V$ satisfying:

$$(u_t,v)+(a\nabla u,\nabla v)+(b.\nabla u,v)+(c\,u,v)=(f,v) \qquad \forall v\in V=H_0^1(\Omega) \tag{2}$$

$$(u(x,0),v)=(u^0,v)\,,$$

where, $(u,v)=\int uvdx \ and \ L^2(\Omega)=\{u: u \ is \ define \ on \ \Omega \ and \ \int_\Omega u^2<\infty\}$

$H^1(\Omega)=\{u: u \ and \ \nabla u \in L^2(\Omega)\}$

$H_0^1(\Omega)=\{u\in H^1(\Omega): u=0 \ on \ boundary \ of \ \Omega\}$

$L^\infty(\Omega)=max_{u\in\Omega}|u|$

3. The semi and full discrete approximation

The semi- discrete approximation for equation (2) reads: Find an approximate solution $u_h\in V_h\subset V$ such that:

$$(u_{h,t},\varphi)+(a\nabla u_h,\nabla\varphi)+(b.\nabla u_h,\varphi)+(c\,u_h,\varphi)=(f,\varphi),\ \forall\varphi\in V_h \tag{3}$$
$$u_h(0)=u_h^0$$

Where φ is the basis functions of V_h. And the fully- discrete Approximation are

(a) Forward-Galerkin Method [6]

Letting τ be the time step and $u_h\in V_h$ be the approximation of $u(x,t)$ at $t=t_n=n\tau$.

$$\frac{1}{\tau}(u_h^{n+1}-u_h^n,\varphi)+(a\nabla u_h^n,\nabla\varphi)+(b.\nabla u_h^n,\varphi)+(cu_h^n,\varphi)=(f,\varphi) \qquad \forall\varphi\in V_h \qquad n=1(1)N \tag{4}$$

(b) Forward-Galerkin-New Schemes (Galerkin-NSs) Method [1]

In this scheme, we note that the diffusivity term multiplied by another term a_1, which is called effective diffusivity.

$$a_1=\begin{cases} Pe\coth(Pe), & New\ Scheme1\equiv NS1\\ 1/(1+Pe), & New\ Scheme2\equiv NS2\\ 1/(1-(Pe)^2), & New\ Scheme3\equiv NS3\end{cases} \tag{5}$$

$$\frac{1}{\tau}(u_h^{n+1}-u_h^n,\varphi)+(a_p\nabla u_h^n,\nabla\varphi)+(b.\nabla u_h^n,\varphi)+(cu_h^n,\varphi)=(f,\varphi)$$

where $Pe=(\text{b}\ h)/(2\,a)$ is mesh Peclet number and $a_p=aa_1$

(c) Forward-Galerkin-Modified Parameters New Schemes (Galerkin-MPs-NSs) method [1]:

In Galerkin method (4) if the parameters a, b and c are modified as follows [8]:

$$\tilde{a} = a_p + a_1 \eta b^2,$$
$$\tilde{b} = b - (\xi + 1)\eta c b,$$
$$\tilde{c} = c - \xi \eta c^2, \tag{6}$$

then

$$\frac{1}{\tau}(u_h^{n+1} - u_h^n, \varphi) + ((a_p + a_1 \eta b^2.)\nabla u_h^n, \nabla\varphi) + ((1 - (\xi+1)\eta c)b.\nabla u_h^n, \varphi) + ((c - \xi \eta c^2)u_h^n, \varphi) = (f, \varphi)$$
$$\forall \varphi \in V_h \quad , n = 1(1)N$$

where $\xi \in [-1,1]$ and the stability term η are weights.

Now to get the weak form for all schemes, define the space $V = H_0^1(\Omega)$. for $f \in L^2(\Omega), u^0 \in L^2(\Omega)$ and $b \in (L^\infty(\Omega))^2$, equation (1) can be written in the form

$$u_t - \hat{a}\,\Delta u + \hat{b}.\nabla u + \hat{c}u = f \quad in \quad \Omega \times (0,T] \tag{٧}$$
$$u(x,t) = 0 \quad on \quad \Gamma \times (0,T]$$
$$u(x,0) = u^0 \quad on \quad \overline{\Omega}$$

where

$$\hat{a} = \begin{cases} a & in \;\; the \;\; standerd \;\; Galerkin \;\; method \\ a_p & in \;\; the \;\; G-NSs \; (NSs-G) \;\; methods \\ a_p + a_1 \eta b^2 & in \;\; the \;\; G-MPs-NSs \; (MPs-NSs-G) \;\; methods \end{cases}$$

$$\hat{b} = \begin{cases} b & in \;\; the \;\; standerd \;\; Galerkin \;\; and \;\; in \;\; G-NSs \; (NSs-G) \\ b - (\xi+1)\eta c b & in \;\; G-MPs-NSs \end{cases}$$

$$\hat{c} = \begin{cases} c & in \;\; the \;\; standerd \;\; Galerkin \;\; and \;\; in \;\; G-NSs \; (NSs-G) \\ (c - \xi \eta c^2) & in \;\; the \;\; G-MPs-NSs \; (MPs-NSs-G) \end{cases}$$

The weak form of equation (٧) is

$$(u_t, v) + (\hat{a}\,\nabla u, \nabla v) + (\hat{b}.\nabla u, v) + (\hat{c}\,u, v) = (f, v) \qquad \forall v \in V \tag{٨}$$
$$(u(x,0), v) = (u^0, v)$$

5. The error estimate and stability

5.1 . The Semi-Discrete Approximation

The finite element methods for equation (٨) read:

Find an approximate solution $u_h \in V_h \subset V$ such that:

$$(u_{h,t}, \varphi) + (\hat{a}\nabla u_h, \nabla\varphi) + (\hat{b}.\nabla u_h, \varphi) + (\hat{c}u_h, \varphi) = (f, \varphi) \qquad \forall \varphi \in V_h . \tag{٩}$$

Lemma (1) [6].

Suppose that u_h is the discrete solution of equation (٩). Then, there exists a constant $C > 0$ independent of h such that:

$$\| u_h(T) \|^2 \le e^{-CT} \| u_h^0 \|^2 + \frac{1}{C} \| f \|^2_{(C:(0,T))} .$$

Theorem 1. Suppose that u and u_h are solutions of (٨) and (11)respectively, satisfying $u \in L^\infty(0,T;H^2(\Omega)) \cap L^\infty(0,T;W^1_\infty(\Omega))$, u_t , $u_{tt} \in L^\infty(0.T;L^\infty(\Omega))$. Then there exists a constant C independent of h such that

$$\| u - u_h \|_{L^\infty(L^2)} \le C\, h^r$$

Proof:

Let $I\,u$ be the interpolate of u then we can write

$$u - u_h = (u - I\,u) - (u_h - I\,u) ,$$

let

$$\rho = u - I\,u \qquad , \qquad \theta = u_h - I\,u ,$$

by triangle inequality, we have

$$\| u - u_h \|_{L^\infty(L^2)} \le \| \rho \|_{L^\infty(L^2)} + \| \theta \|_{L^\infty(L^2)} ,$$

we have

$$\| \rho \|_{L^\infty(L^2)} = \max_{0 \le t \le T} \| u - I\,u \| ,$$
$$\le Ch^r \| u \|_{L^\infty(H^r)} .$$

(١٠)

To estimate θ, subtracting equation (٩) from (٨), since

$$\hat{a}\,(u - I\,u, \varphi) = 0 ,$$

then

$$(\theta_t, \varphi) + (\hat{a} \nabla \theta, \nabla \varphi) + (\hat{b}.\nabla \theta, \varphi) + (\hat{c}\,\theta, \varphi) = (\rho_t, \varphi) ,$$

(١١)

choosing $\varphi = \theta$ gives,

$$(\theta_t, \theta) + (\hat{a} \nabla \theta, \nabla \theta) + (\hat{b}.\nabla \theta, \theta) + (\hat{c}\,\theta, \theta) = (\rho_t, \theta) ,$$

since

$$(\theta_t, \theta) = \int_\Omega \theta_t \theta\, dt = \frac{1}{2}\frac{d}{dt} \int_\Omega \theta^2 dx = \frac{1}{2}\frac{d}{dt} \| \theta \|^2 ,$$

we have

$$\frac{1}{2}\frac{d}{dt} \| \theta \|^2 + \alpha \| \nabla \theta \|^2 + \beta \| \theta \|^2 \le (\rho_t, \theta)$$

by Cauchy-Schwartz inequality and ε -inquality, we have

$$\| \rho_t \| \| \theta \| \le \frac{1}{4\beta} \| \rho_t \|^2 + \beta \| \theta \|^2 .$$

then

$$\frac{d}{dt} \| \theta \|^2 + 2\alpha \| \nabla \theta \|^2 \le \frac{1}{2\beta} \| \rho_t \|^2 ,$$

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since the second term is non-negative, we have

$$\frac{d}{dt}\|\theta\|^2 \le \frac{1}{2\beta}\|\rho_t\|^2 ,$$

(1٢)

so, there exists $0 \le t^* \le T$ such that

$$\left\|\theta(t^*)\right\| = \max_{0\le t\le T}\|\theta\| = \|\theta\|_{l^\infty(L^2)} .$$

Integrating (1٢) from $t=0$ to $t=t^*$ gives

$$\left\|\theta(t^*)\right\|^2 \le \|\theta(0)\|^2 + \frac{1}{2\beta}\int_0^{t^*}\|\rho_t\|^2 dt \le \|\theta(0)\|^2 + \frac{1}{2\beta}\int_0^{T}\|\rho_t\|^2 dt$$

then

$$\|\theta\|_{L^\infty(L^2)} \le \|\theta(0)\| + \left(\frac{1}{2\beta}\int_0^T\|\rho_t\|^2 dt\right)^{\frac{1}{2}} ,$$

the first term on the right hand side gives,

$$\|\theta(0)\| = \|u_h(0) - Iu(0)\| \le \|u_h(0)-u(0)\| + \|u(0)-Iu(0)\| \le \left\|u_h^0 - u^0\right\| + Ch\left\|u^0\right\| \qquad (1٣)$$

for the second term, we have

$$\left(\frac{1}{2\beta}\int_0^T\|\rho_t\|^2 dt\right)^{\frac{1}{2}} = \left(\frac{1}{2\beta}\int_0^T\|u_t - Iu_t\|^2 dt\right)^{\frac{1}{2}} \le \left(\int_0^T Ch^{2r}\|u_t\|_r^2 dt\right)^{\frac{1}{2}}$$

$$= Ch^r\left(\int_0^T\|u_t\|_r^2 dt\right)^{\frac{1}{2}} = Ch^r\|u_t\|_{L^2(H^r)}$$

then (1٤)

$$\|\theta\|_{L^\infty(L^2)} \le \left\|u_h^0 - u^0\right\| + Ch^r\left\{\|u_t\|_{L^2(H^r)} + \left\|u^0\right\|_r\right\},$$

hence

$$\|\theta\|_{L^\infty(L^2)} \le Ch^r .$$

The proof is complete. □

5.2 The Fully Discrete Approximation

Now we describe all schemes in which the fully discrete equation are defined by replacing the time derivative in (٩) with the forward difference quotient to give

$$\frac{1}{\tau}(u_h^{n+1} - u_h^n, \varphi) + (\hat{a}\nabla u_h^n, \nabla\varphi) + (\hat{b}.\nabla u_h^n, \varphi) + (\hat{c}u_h^n, \varphi) = (f, \varphi) \qquad \varphi \in V_h, n = 1(1)N \qquad (1٥)$$

Lemma 2. [6] Suppose that u_h^n is the discrete solution of equation (1٥), then for any $N > 0$

$$\left\|u_h^N\right\| \le \left\|u_h^0\right\| + \tau\sum_{n=0}^{N}\left\|f^n\right\| .$$

Theorem 2 : Suppose that u^n and u_h^n are solutions of equations (٨) and (1٥) respectively satisfy the requirements

$u \in L^\infty(0,T;H^2(\Omega)) \cap L^\infty(0,T;W_\infty^1(\Omega))$, $u_t, u_{tt} \in L^\infty(0,T;L^\infty(\Omega))$. Then there exists a constant C independent of h such that,

$$\left\|u^n - u_h^n\right\|_{L^\infty(L^2)} \le C\left\{h^r + \tau\right\} .$$

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Proof:

$$u^n - u_h^n = (u^n - Iu^n) - (u_h^n - Iu^n) \quad = \rho^n - \theta^n$$

by triangle inequality, we have

$$\left\| u^n - u_h^n \right\|_{L^\infty(L^2)} \le \left\| \rho^n \right\|_{L^\infty(L^2)} + \left\| \theta^n \right\|_{L^\infty(L^2)},$$

we have

$$\left\| \rho^n \right\|_{L^\infty(L^2)} \le C h^r \left\| u^n \right\|_{L^\infty(H^r)}.$$

To find a bound on θ, note that

$$\frac{1}{\tau}(\theta^{n+1} - \theta^n, \varphi) + (\hat{a}\nabla\theta^n, \nabla\varphi) + (\hat{b}.\nabla\theta^n, \varphi) + (\hat{c}\theta^n, \varphi)$$

$$= \frac{1}{\tau}(u_h^{n+1} - u_h^n) + (\hat{a}\nabla u_h^n, \nabla\varphi) + (\hat{b}.\nabla u_h^n, \varphi) + (\hat{c}u_h^n, \varphi) - \frac{1}{\tau}(Iu^{n+1} - Iu^n, \varphi) - (\hat{a}\nabla Iu^n, \nabla\varphi)$$

$$-(\hat{b}.\nabla Iu^n, \varphi) - (\hat{c}Iu^n, \varphi).$$

scince

$$\hat{a}\,(u^n - Iu^n, \varphi) = 0,$$

and

$$\frac{1}{\tau}(u_h^{n+1} - u_h^n, \varphi) + (\hat{a}\nabla u_h^n, \nabla\varphi) + (\hat{b}.\nabla u_h^n, \varphi) + (\hat{c}u_h^n, \varphi) = (f^n, \varphi),$$

then,

$$\frac{1}{\tau}(\theta^{n+1} - \theta^n, \varphi) + (\hat{a}\nabla\theta^n, \nabla\varphi) + (\hat{b}.\nabla\theta^n, \varphi) + (\hat{c}\theta^n, \varphi)$$

$$= (f^n, \varphi) - (\hat{a}\nabla u^n, \nabla\varphi) - (\hat{b}.\nabla u^n, \varphi) - (\hat{c}u^n, \varphi) - \frac{1}{\tau}(Iu^{n+1} - Iu^n, \varphi),$$

$$= (u^n{}_t, \varphi) - \frac{1}{\tau}(Iu^{n+1} - Iu^n, \varphi).$$

Adding and subtracting $\frac{1}{\tau}(u^{n+1} - u^n, \varphi)$ we get,

$$\frac{1}{\tau}(\theta^{n+1} - \theta^n, \varphi) + (\hat{a}\nabla\theta^n, \varphi) + (\hat{b}.\nabla\theta^n, \varphi) + (\hat{c}\theta^n, \varphi)$$

$$= \frac{1}{\tau}(u^{n+1} - u^n, \varphi) - \frac{1}{\tau}(Iu^{n+1} - Iu^n, \varphi) + (u_t^n, \varphi) - \frac{1}{\tau}(u^{n+1} - u^n, \varphi),$$

$$= \frac{1}{\tau}(\rho^{n+1} - \rho^n, \varphi) + (\xi^n, \varphi),$$

where

$$\xi^n = u_t^n - \frac{1}{\tau}(u^{n+1} - u^n).$$

Choosing $\varphi = \theta^n$ and multiplying by τ by using Cauchy-Schwartz inequality Young's inequality to the right hand side and multiplying by 2, we get

$$\left\|\theta^{n+1}\right\|^2 + 2\tau\alpha\left\|\nabla\theta^n\right\|^2 \le \left\|\theta^n\right\|^2 + C\left[\frac{1}{\tau}\left\|\rho^{n+1} - \rho^n\right\|^2 + \tau\left\|\xi^n\right\|^2\right],$$

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since the second term in the left hand side is non-negative, summing both sides from $n = 0$ to $n = N$ we have,

$$\left\| \theta^{N+1} \right\|^2 \le \left\| \theta^0 \right\|^2 + C\left[\frac{1}{\tau} \sum_{n=0}^{N} \left\| \rho^{n+1} - \rho^n \right\|^2 + \tau \sum_{n=0}^{N} \left\| \xi^n \right\|^2 \right].$$

As we have done in the previous theorems, given
$0 \le n^* \le N$ such that

$$\left\| \theta^{n^*+1} \right\| = \left\| \theta^{n+1} \right\|_{L^\infty(L^2)},$$

then

$$\left\| \theta^{n+1} \right\|^2_{L^\infty(L^2)} \le \left\| \theta^0 \right\| + C\left[\frac{1}{\tau} \sum_{n=0}^{N} \left\| \rho^{n+1} - \rho^n \right\|^2 + \tau \sum_{n=0}^{N} \left\| \xi^n \right\|^2 \right]. \qquad (١٦)$$

for the second term note that

$$\rho^{n+1} - \rho^n = \int_{t_n}^{t_{n+1}} \rho_t dt ,$$

this implies

$$\left\| \rho^{n+1} - \rho^n \right\| \le \int_{t_n}^{t_{n+1}} \left\| \rho_t \right\| dt ,$$

thus

$$\left\| \rho^{n+1} - \rho^n \right\|^2 \le \left(\int_{t_n}^{t_{n+1}} \left\| \rho_t \right\| dt \right)^2 ,$$

$$= \tau^2 \left(\int_{t_n}^{t_{n+1}} \left\| \rho_t \right\| \frac{dt}{\tau} \right)^2 ,$$

applying Jensen's inequality (see [5]) to the right hand side this implies

$$\frac{1}{\tau} \sum_{n=0}^{N} \left\| \rho^{n+1} - \rho^n \right\|^2 \le \int_0^T \left\| \rho_t \right\|^2 dt \quad = C h^{2r} \left\| u_t \right\|_{L^2(H^r)}$$

To bound the third term of (18), note that

$$\xi^n = u_t^n - \frac{1}{\tau}(u^{n+1} - u^n),$$

then,

$$\tau \xi^n = \tau u_t^n - \int_{t_n}^{t_{n+1}} u_t \, dt = (t_{n+1} - t_n)\, u_t^n - \int_{t_n}^{t_{n+1}} u_t dt ,$$

adding and subtracting $t_{n+1} u_t^{n+1}$ gives

$$\tau \xi^n = t_{n+1} u_t^{n+1} - t_n u_t^n - \int_{t_n}^{t_{n+1}} u_t dt - (t_{n+1} u_t^{n+1} - t_{n+1} u_t^n) ,$$

$$= \int_{t_n}^{t_{n+1}} (t - t_{n+1}) u_{tt} dt,$$

then, we have by Jensen's inequality (see [5])

$$\|\xi^n\|^2 \le \left(\int_{t_n}^{t_{n+1}} \|u_{tt}\|dt\right)^2 = \tau^2\left(\int_{t_n}^{t_{n+1}} \|u_{tt}\|\frac{dt}{\tau}\right)^2 ,$$

then

$$\tau\sum_{n=0}^{N} \|\xi^n\|^2 \le \tau^2 \int_0^T \|u_{tt}\|^2 dt = \tau^2\|u_{tt}\|^2_{L^2(L^2)} .$$

Applying these results to (1٦) gives,

$$\|\theta^{n+1}\|_{L^\infty(L^2)} \le \|u_h^0 - u^0\| + C\left[h^r\{\|u^0\|_r + \|u_t\|_{L^2(H^r)}\} + \tau\|u_{tt}\|_{L^2(L^2)}\right],$$

hence

$$\|\theta^{n+1}\|_{L^\infty(L^2)} \le C(h^r + \tau).$$

The proof is complete.

5- The Discrete Maximum Principle

Now in order to get a fully discrete numerical scheme, we choose a time-step τ and u_h^n the approximate solution in V_h of $u(x,t)$ at $t = t_n = n\tau$. We apply the θ-method ($\theta \in [0,1]$ is a given parameter) and obtain a system of linear algebraic equations [4]:

$$\left(\frac{u_h^{n+1} - u_h^n}{\tau}, \varphi_h\right) + (\hat{a}\nabla u_h^{n+\theta}, \nabla\varphi_h) + (\hat{b}.\nabla u_h^{n+\theta}, \varphi_h) + (\hat{c}\, u_h^{n+\theta}, \varphi_h) = (f^{n+\theta}, \varphi_h),$$

(1٧)

$$(u_h^{n+1}, \varphi_h) - (u_h^n, \varphi_h) + \tau(\hat{a}\nabla u_h^{n+\theta}, \nabla\varphi_h) + \tau(\hat{b}.\nabla u_h^{n+\theta}, \varphi_h) + \tau(\hat{c} u_h^{n+\theta}, \varphi_h) = \tau(f^{n+\theta}, \varphi_h) .$$

In terms of linear basis functions $\{\varphi_j\}_1^N$ of the space V_h, where N is the dimension of V_h can represent the approximate solution of the fully discrete scheme (1٧) reads:

$$u_h(x,t) = \sum_{j=1}^{N} v(t)\varphi_j(x), \qquad \varphi_h = \varphi_i, \qquad i = 1,\ldots,N, j = 1,\ldots,N$$

such that

$$\sum_{j=1}^{N} v^{n+1}(\varphi_j, \varphi_i) - \sum_{j=1}^{N} v^n(\varphi_j, \varphi_i) + \tau\sum_{j=1}^{N} v^{n+\theta}(\hat{a}\,\nabla\varphi_j, \nabla\varphi_i) + \tau\sum_{j=1}^{N} v^{n+\theta}(\hat{b}.\nabla\varphi_j, \varphi_i) + \tau\sum_{j=1}^{N} v^{n+\theta}(\hat{c}\,\varphi_j, \varphi_i)$$

$$= \tau\sum_{j=1}^{N}(f^{n+\theta}, \varphi_i) .$$

In matrix form, this may be expressed as:

$M v^{n+1} - M v^n + \hat{a}\tau K v^{n+\theta} + \tau G v^{n+\theta} + \tau\hat{c} M v^{n+\theta} = \tau F^{n+\theta}$. Then we have

$$[M + \tau\theta(\hat{a} K + G + \hat{c} M)]v^{n+1} = [M - \tau(1-\theta)(\hat{a} K + G + \hat{c} M)]\, v^n + \tau F^{n+\theta},$$

(١٨)

where M denotes the $N \times N$ mass matrix , K denotes the $N \times N$ stiffness matrix and G denotes the $N \times N$ convection matrix are defined by:

$M=[m_{ij}]_{N\times N}$, $m_{ij}=\int_\Omega \varphi_j\varphi_i dx$, $K=[k_{ij}]_{N\times N}$, $k_{ij}=\int_\Omega \nabla\varphi_j\nabla\varphi_i dx$

$G=[g_{ij}]_{N\times N}$, $g_{ij}=\int_\Omega \hat{b}.\nabla\varphi_j\varphi_i dx$

Here $F^{n+\theta}=(f_1^{n+\theta},\ldots,f_N^{n+\theta})^t$, (t indicates the transpose).

$f_h^{n+\theta}=(1-\theta)f_h^n+\theta f_h^{n+1}$, $((f_h^n=I_h f(n\tau))$, and let

$$D=(d_{ij}) \quad , \quad D=[M+\tau\theta(\hat{a}K+G+\hat{c}M)], \tag{١٩}$$

$$E=(e_{ij}) \quad , \quad E=\left[M-\tau(1-\theta)(\hat{a}K+G+\hat{c}M)\right]. \tag{2٠}$$

Lemma 3. [7] Let $D=(d_{ij})$, $E=(e_{ij})$, be $N\times Z$ matrices satisfying the conditions:

(i) $\sum_{j=1}^{Z} d_{ij}\ge\sum_{j=1}^{Z} e_{ij}>0$, $1\le i\le N$,

(ii) $e_{ij}\ge 0$, $1\le i\le N$, $1\le j\le Z$,

(iii) $d_{ij}\le 0$, $1\le i\le N$, $1\le j\le Z$, $j\ne i$,

and assume that $D\vec{u}=E\vec{w}+\tau\vec{g}$. (2١)

Then each component $u_i (1\le i\le N)$ is estimated by

$$\max_{1\le j\le N}|u_j|\le\max\left\{\max_{1\le j\le Z}|w_j|+\tau\max_{1\le j\le Z}|g_j|,\ \max_{N+1\le j\le Z}|u_j|\right\}.$$

Theorem 3.. Let d_{ij} be defined in equation (١٩), and e_{ij} be defined in equation (2٠), if $m_{ij}>0$, $i=j$ and $m_{ij}=0$, $i\ne j$ then all schemes satisfy the DMP.

Proof: We have

$$m_{ii}>0 \quad (i=j) \quad and \quad m_{ij}=0. \quad (i\ne j)$$

Now matrix M is nonnegative because the basis functions are nonnegative [4]. Thus $M\ge 0$,

we get

$$\hat{c}M=\sum_{j=1}^{Z}\hat{c}m_{ij}=\hat{c}\sum_{j=1}^{Z} m_{ii}\ge 0 \text{ and } \hat{c}m_{ij}=0 \quad , \quad (i\ne j)$$

and for the ith coordinate of the vector K we have $(K)_i=\sum_{j=1}^{Z}k_{ij}=\sum_{j=1}^{Z}\hat{a}\,(\phi_j,\phi_i)=\hat{a}\,(\sum_{j=1}^{Z}\phi_j,\phi_i)=\hat{a}\,(1,\phi_i)=\hat{a}\int_\Omega grad1\ grad\phi\ dx=0$,

also

$$k_{ij}\le 0 \quad , \quad (i\ne j)\text{, we get}$$

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$$\sum_{j=1}^{Z} g_{ij} = \sum_{j=1}^{Z} (\hat{b}.\nabla\phi_j,\phi_i) = (\hat{b} \sum_{j=1}^{Z} \nabla\phi_j,\phi_i) \quad , \; 1 \le i \le N$$

since (see [4])

$$\phi_j \ge 0 \quad \text{and} \quad \sum_{j=1}^{Z} \phi_j \equiv 1 \quad \text{in} \quad \overline{\Omega} \text{ و } \quad (\hat{b}.\nabla \sum_{j=1}^{Z} \phi_j,\phi_i) = (\hat{b}.\ \nabla 1,\phi_i\ ) = 0,$$

then

It is easy to see that

$$\sum_{j=1}^{Z} D = \sum_{j=1}^{Z} [M + \theta\tau(\hat{a}K + G + \hat{c}M)] \ge$$

$$\sum_{j=1}^{Z} E = \sum_{j=1}^{Z}[M - (1-\theta)\tau(\hat{a}K + G + \hat{c}M)] = \sum_{j=1}^{Z} m_{ij} > 0,$$

hence, the condition (i) in Lemma 3 holds. Now we prove the condition (i i),

$$E = (e_{ij}) = [M - (1-\theta)\tau(\hat{a}K + G + \hat{c}M)] \ge 0 \qquad 1 \le i \le N \quad ,1 \le j \le Z$$

also

$$D = (d_{ij}) = [M + \theta\tau(\hat{a}K + G + \hat{c}M)] \le 0 \quad , \qquad i \ne j$$

because

$m_{ii} > 0$, $m_{ij} = 0$ $(i \ne j)$, $k_{ii} > 0$, $k_{ij} \le 0$ $(i \ne j)$, we have

$$G = (\hat{b}.\nabla u, v) = (\hat{b}.\nabla \sum_{j=1}^{Z} \varphi_j \phi_j, \phi_i) = 0 \,,$$

then the conditions (i), (i i) and (iii) in Lemma 3 are satisfied. Hence

$$\max_{1\le j\le N} \left| u_j \right| \le \max \left\{ \max_{1\le j\le Z} \left| w_j \right| + \tau \max_{1\le j\le Z} \left| g_j \right|, \max_{N+1\le j\le Z} \left| u_j \right| \right\}.$$

6 Numerical Example

This chapter presents a simple test case to see the behaviour of the Galerkin-NSs (NSs-Galerkin) and Galerkin-MPs-NSs (MPs-NSs-Galerkin) methods. The problem (1) was run with the data as follows: the domain Ω where the problem is to be solved in the unit square, $\overline{\Omega} = [0,1]\times[0,1]$, discretized using a uniform mesh of 21×21 bilinear elements (yielding 441 nodal points) and the diffusion coefficient $a = 10^{-4}$,the absorption coefficient is $c = 10^{-4}$. The velocity vector has been taken as $\mathrm{b} = |b|(\cos(\pi/3), \sin(\pi/3))$ with $|b| = 1$, and $\tau = 0.5 * h^2$ [9].

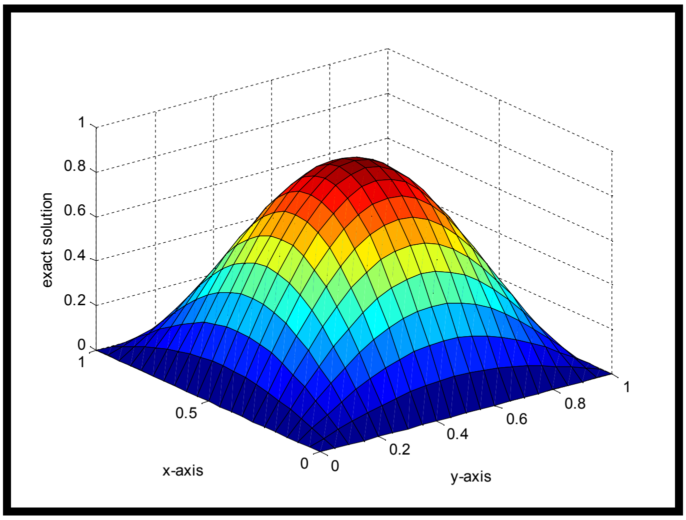
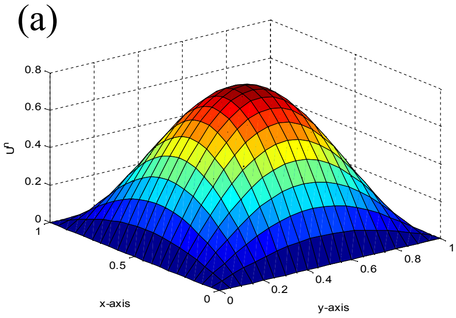
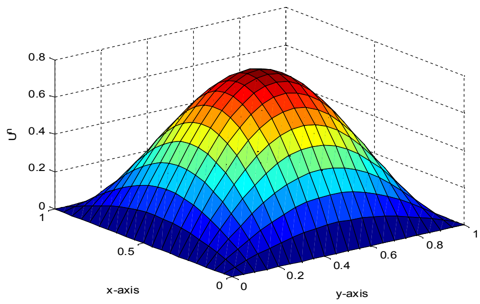


Figure 4.1: exact solution t=0.1

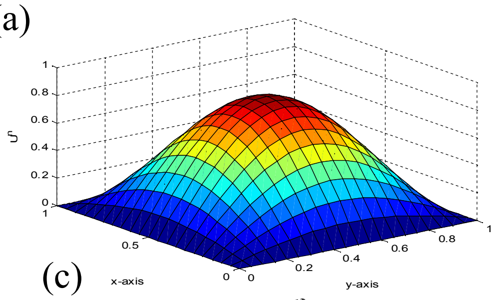
(a)

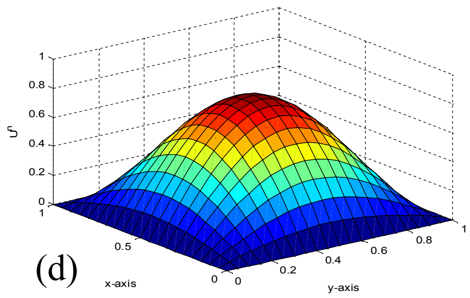




(a)

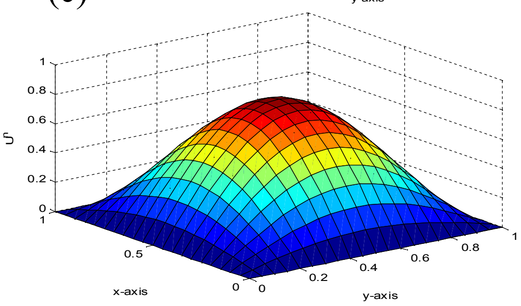
(b)

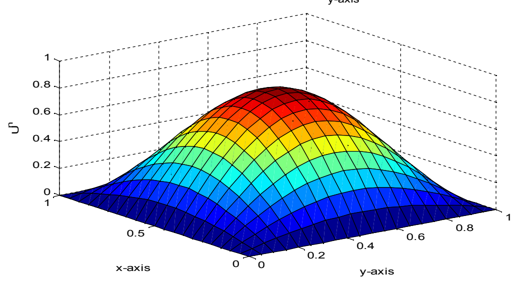




(c)

(d)





(e)

(f)

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Figure 4.2: solution of Galrkin-NS1,Galerkin-Mps-NSs, NS1-Galerkin, MPs-NS1-Galeki, MPs-NS2-Galerkin and MPs-NS3-Galerkin methods at t=0.1 from (a) to (f).

Conclution:

After theoretical and practical study to the convection –diffusion problem, we may make a number of remarks to the theoretical analysis and numerical result in this work.

1-The stability coefficient on the diffusion term removed the oscillation on standard Galerkin method

2- The numerical results consistent with the exact solution see fig.4.1 and 4.2

3- The numerical results which we got it from the test problem are consistent with the theorem of DMP.

4-All schemes are convergent , in the semi-discrete with $O(h^r)$ and in the full discrete converge with error of $O(h^r, \tau)$

5- For our study we show that if any scheme satisfies the DMP the then this will give us a guaranty that the approximate solution converges to the exact solution.

Reference:

[1] AL-Humedi, H. O. M., (2006) "*An improvement of finite element methods for solving the convection-diffusion-reaction equation*", Ph.D. thesis, Department of Mathematics, College of Science, University of Basrah.

[2] Codina, R., (1998) "*Comparison of some finite element diffusion-convection-reaction equation*", Comput. Appl. Mech. Eng. Vol. **156**, pp. 185-210.

[3] Codina, R. and Blasco, J., "*Analysis of a stabilized finite element approximation of the transient convection-diffusion-reaction equation using orthogonal subscales*", (to Appear in Computing and Visualization in Science Manuscript No. Will be inserted by the editor))

[4] Faragó, I., Horváth, R., Korotov, S., (2005) "*Discrete maximum principle for linear parabolic problems solved on hybrid meshes*", Applied Numerical Mathematics **53**, pp.249-264.

[5] Heitmann, N.F., (2002) "*Subgridscale eddy viscosity for convection dominated diffusion transport*", Department of Mathematics, Pittsburgh University, November.

[6] Johnson, C., (1987) "*Numerical solution of partial differential equation by the finite element method*", Cambridge University Press.

[7] Kashkool, H. A., (2002) "*Upwind finite element scheme for nonlinear convection-diffusion equation*" Act Scientiarum Naturalium Universitatis Nankaiensis vol. **35**, No. 2.

[8] Murray H. P. and Hans F. W., (1967) "*Maximum principles in differential equations*" Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

[9] Tyler, T.F.,(2000) "*Maximum curves and isolated points of entire functions*", Proceedings of the American mathematical society, Vol. 128, No. 9, pp. 2561-2568.

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