New theorems in approximation theory

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Abstract:

The aim of this paper is to prove some results for equivalence of moduli of smoothnes in approximation theory , we used a"non uniform" modulus of smoothness $\varpi_{\phi}^{k}(f,t)=\sup_{0<h\leq t}\left\|\overset{-}{\Delta}{}_{\beta}^{k}(f,x)\right\|_{p}$ and the weighted Ditzian –Totik moduli of smoothness $\varpi_{k,v}^{\psi}(f,t)=\sup_{0<h\leq t}\left\|\overset{-}{\psi}(.)^{v}\Delta_{h\psi(.)}^{k}(f,x)\right\|_{p}$ in $L_p[-1,1], 0<p\leq\infty$ by spline functions ,several results are obtained .For example , it shown that ,for any $1\leq\phi\leq r+1,\ 1\leq p\leq\infty$ the inequality $\left|J\right|^{v}\varpi_{\phi-v}^{k}(S^{(v)},J)_{p}\approx\varpi_{\phi}^{k}(S,J)_{p}, 1\leq v\leq\min[\phi,m+1]$ is satisfied ,finally, similar result for chebyshev partition and weighted Ditzian –Totik moduli of smoothness are also obtained.

Key words: Univariate splines , moduli of smoothness , "non uniform" modulus of smoothness , weighted Ditzian –Totik moduli of smoothness.

Introduction:

Equivalence moduli of smoothness was proved by Hu and Yu [1] and by Hu [2] for uniform and quasi-uniform partition , respectively. In fact , if we set $m=k-1$, then corollary (1.5) (with an additional restriction $k\leq r$) becomes theorem 1 in [2] ,also in the case $k=r+1$ and $m=r-1$, corollary (1.5) follow from theorem 2 of [2] .

A function S is called a spline if : [3]

1- the domain is an interval [a,b].

2- $S,S',S'',...,S^{(r-1)}$ are all continuous function on [a,b]

3- there are point (the knot of S)such that $a<Z_0<Z_1<...<Z_{n-1}<Z_n=b$ and such that S is a polynomial of degree $\leq r$ on each subintrrval $[Z_i,Z_{i+1}]$

Usually piecewise polynomials from $S_r(Z_n)$ are called "splines" if they possess continuous $(r-1)st$ derivatives .We emphasize that if $S\in S_r(Z_n)$ and $m\in N$ then $S^{(m)}\in\delta_{\max\{r-m,o\}}(Z_n)$ and $S^{(m)}=0$ a.e if $m\geq r+1$.

Now a moduli of smoothness are intended for mathematicians working in approximation theory ,numerical analysis and real analysis .Measuring the smoothness of function by differentiability is too crude for many purposes in approximation theory . more suitable measurement are

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provided by moduli of smoothness. [4]

$L_p(J)$, $0 < p \le \infty$, denotes the space of all measurable function f on J Such that $\|f\|_{L_{p(J)}} < \infty$.

The kth symmetric of f is given by

$$\bar{\Delta}^k_\beta(f,x,[a,b]) = \begin{cases} \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f(x+i\beta) & \text{if } [x,x+k\beta]\subset[a,b] \\ 0 & \text{other wise} \end{cases}$$

then the kth "non uniform" modulus of smoothness of $f \in L_p[-1,1], 0 < p \le \infty$ is define by

$$\varpi^k_\phi(f,t) = \sup_{0<h\le t}\left\|\bar{\Delta}^k_\beta(f,x)\right\|_p$$

Where

$\beta = \beta(x,h) = \sqrt{1-x^2}h + h^2, \phi(x) = \sqrt{1-x^2}$ and

The weighted Ditzian –Totik moduli of smoothness of a function $f \in L_p[-1,1], 0 < p \le \infty$ is define by

$$\varpi^\psi_{k,v}(f,t) = \sup_{0<h\le t}\left\|\psi(.)^v \bar{\Delta}^k_{h\psi(.)}(f,x)\right\|_p$$,[5]

Where $\psi(x) = \sqrt{1-x^2}$

A partition $Z_n = \{Z_0,....Z_n\} =$ $-1 < Z_0 < Z_1 < ... < Z_{n-1} < Z_n = 1$ of the interval $[-1,1]$ denote the scale of parition Z_n by $\theta = \theta(Z_n) = \max\frac{|J_j \pm 1|}{|J_i|}$ where $J_j = [Z_j, Z_{j+1}]$ with $Z_j = -1, j < 0$ and $Z_j = 1, j > n$.

Auxiliary results

The following properties of the moduli of smoothness are well known[6],[7]

(i) for $f \in L_p(J), 0 < p \le \infty$, we have

$$\omega_{k+1}(f,t,J)_p \le 2^{\max\{1,1/p\}}\omega_k(f,t,J)_p, k \in N \quad (2.1)$$

(ii)

$$\omega_k(f,\lambda t,J)_p \le C(k,p)(\lambda+1)^{k-1+\max\{1,1/p\}}\omega_k(f,t,J)_p, \lambda > 0 \quad (2.2)$$

(iii) suppose that $f \in L_p[-1,1], 0 < p < \infty$ and $k,\mu \in N$ then

$$\sum_{j=0}^{n-\mu-1}\omega_k(f,\bigcup_{i=j}^{j+\mu}J_i)^p_p \le \begin{cases} c(k,\Delta,\mu,p)\omega_k(f,n^{-1})^p_p & \text{if } z_n = u^\Delta_n \\ c(k,\mu,p)\omega^\varphi_k(f,n^{-1})^p_p & \text{if } z_n = t_n \end{cases} \quad (2.3)$$

Lemma .1 (Whitney's inequality). [8]

For any $f \in L_p[a,b], 0 < p \le \infty$ such that $q_{k-1} \in \Pi_{k-1}$ such that

$$\|f - q_{k-1}\|_{L_p[a,b]} \le C\omega_k(f,[a,b]). \quad (2.4)$$

Lemma .2

For any polynomial $q_k \in \Pi_k$,$0 < p \le \infty$ and intervals I and J such that $I \subseteq J$ we have

$$|J|^{\frac{1}{p}}\|q_r^{(v)}\|_{L_\infty(J)} \approx \|q_r^{(v)}\|_{L_p(J)} \le C(r,|J|/|I|,p)|J|^{-v}\|q_r\|_{L_p(I)}, 0 \le v \le r \quad (2.5)$$

The main result

The following theorem is not satisfied for all $f \in L_p, 0 < p < \infty$ but satisfied for spline functions

Theorem .1 (local estimates)

Let $S \in S_r(Z_n), r \in N$ and $J = [Z_{M_1}, Z_{M_2}]$ with $M_2 - M_1 \le c_0$ for some constant c_0

Then for $1 \le \phi \le r+1,\ 0 \le p \le \infty$ we have

$$\varpi^{k}_{\phi-v}(S^{(v)},J)_p \le c(r,\theta,c_0,p)|J|^{-v}\varpi^{k}_{\phi}(S,J)_p$$

for all $v = 1,2,...,\phi.$

proof:

Every where in this proof q_{k-1} denote a polynomial of degree $\le k-1 \ni \|s-q_{k-1}\|_{L_{p(J)}} \le C\varpi^{k}_{\phi}(s,J)_p$ (2.4)

for any $1 \le v \le \phi.$ denoting $s_j = s/J_i$ and using [(2.5),[7]] we have $\left|\Delta^{k-v}_{h}(s^{(v)},x,J)\right| \le \left|\overleftarrow{\Delta}^{k-v}_{p}(s^{(v)},x,J)\right|$ since $\varpi^{k}_{\phi}(f,t) \approx \omega^{k}_{\phi}(f,t)_\infty$

$$= \left|\overleftarrow{\Delta}^{k-v}_{p}(s^{(v)} - q^{(v)}_{k-1},x,J)\right|$$

$$\le 2^{k-v}\left\|s^{(v)} - q^{(v)}_{k-1}\right\|_{L_\infty}(J)$$

$$\le C(k)\max_{M_1 \le j \le M_2-1}\left\|s^{(v)} - q^{(v)}_{k-1}\right\|_{L_\infty}(J_j)$$

$$\le C(r,p)\max\left|J_j\right|^{-v-\frac{1}{p}}\left\|s_j - q_{k-1}\right\|_{L_p}(Jj)$$

$$\le C(r,\theta,c_0,p)\left|J_j\right|^{-v-\frac{1}{p}}\left\|s - q_{k-1}\right\|_{L_p(J_j)}$$

$$\le C(r,\theta,c_0,p)\left|J_j\right|^{-v-\frac{1}{p}}\varpi^{k}_{\phi}(s,J)_p.$$

Corollary .2 (equivalaence of moduli of smoothness)

Let $S \in S_r(Z_n)\bigcap C^m[-1,1], r \in N$ and $J = \lfloor Z_{M_1}, Z_{M_2} \rfloor$ with $M_2 - M_1 \le c_0$ for some constant c_0

Then for $1 \le \phi \le r+1,\ 1 \le p \le \infty$ we have

$$|J|^{v}\varpi^{k}_{\phi-v}(S^{(v)},J)_p \approx \varpi^{k}_{\phi}(S,J)_p,$$

$1 \le v \le \min[\phi, m+1]$

Equivalaence constants above depend only on r,θ,c_0.

Suppose that $\delta_{\max} = \delta_{\max}(Z_n) = \max_{0 \le j \le n-1}|J_j|$ and $\delta_{\min}(Z_n) =$`$\min_{0 \le j \le n-1}|J_j|$ we say that Z_n is $\Delta-$quasi uniform if $\Delta = \delta_{\max}/\delta_{\min}$ is bounded by a constant independen of n, and denote such partition by u^{Δ}_{n} , if $Z_n = u^{\Delta}_{n}$ then $2/(n\Delta) \le \delta_{\min} \le 2/n < \delta_{\max} < 2\Delta/n$ and $\theta(Z_n) \le \Delta$

Therefore $\delta_{\min} \approx \delta_{\max} \approx n^{=1}$ with Equivalaence constants depending only on Δ.

Theorem.3 (quasi uniform partition)

Let $u^{\Delta}_{n},\ n \in N$ be a quasi uniform partition of $[-1,1]$ and let $S \in S_r(u^{\Delta}_{n}), r \in N$ then for any $1 \le \phi \le r+1,\ 0 < p \le \infty$ we have

$$\varpi^{k}_{\phi-v}(S^{(v)},n^{-1})_p \le c(r,\Delta,p)n^{v}\varpi^{k}_{\phi}(S,n^{-1})_p$$

for all $v = 1,2,...,\phi.$

proof

Since u^{Δ}_{n} is a Δ- quasi uniform $\delta_{\max} \approx \delta_{\min} \approx |J_i| \approx n^{-1}$ for all $0 \le j \le n-1$ with equivalence constants depending only Δ .

Using (2.2) we have

$$\varpi^{k}_{\phi-v}(s^{(v)},n^{-1})_p = \varpi^{k}_{\phi-v}(s^{(v)},\frac{\phi}{n\delta_{\min}},\frac{\delta_{\min}}{\phi})$$

$$\le C(\phi,\Delta,p)\varpi^{k}_{\phi-v}(s^{(v)},\delta_{\min}/\phi)_p$$

Now , for any $0 < \beta < \delta_{\min}/\phi$ and $x \in J_j, 0 \le j \le n-1$ such that $\Delta_{\beta}^{\phi-v}(s^{(v)}, x) \ne 0$

All point $x - (\phi - v)\beta/2 + i\beta$, $0 \le i \le \phi - v$ are either in $J_{j-1} \bigcup J_j$ or $J_j \bigcup J_{j+1}$

$$\varpi_{\phi-v}^{k}(s^{(v)}, \delta_{\min}/\phi)_p^p = \sup_{0 \le \beta \le \delta_{\min}/\phi} \left\| \overline{\Delta}_{\beta}^{k(\phi-v)}(s^{(v)}, .) \right\|_p^p$$

$$= \sup_{0 \le \beta \le \delta_{\min}/\phi} \sum_{j=0}^{n-1} \left\| \overline{\Delta}_{\beta}^{k(\phi-v)}(s^{(v)}, .) \right\|_p^p$$

$$\le 2 \sup_{0 \le \beta \le \delta_{\min}/\phi} \sum_{j=0}^{n-1} \left\| \overline{\Delta}_{\beta}^{k(\phi-v)}(s^{(v)}, ., J_{j-1} \bigcup J_j) \right\|_{L_p(J_{j-1} \bigcup J_j)}^p$$

$$\le 2 \sum_{j=1}^{n-1} \varpi_{\phi-v}^{k}(s^{(v)}, J_{j-1} \bigcup J_j)_p^p$$

$$\le C(r, \Delta, p) \sum_{j=1}^{n-1} \left| J_{j-1} \bigcup J_j \right|^{-vp} \varpi_{\phi}^{k}(s, J_{j-1} \bigcup J_j)_p^p$$

$$\le C(r, \Delta, p) n^{pv} \sum_{j=1}^{n-1} \varpi_{\phi}^{k}(s, J_{j-1} \bigcup J_j)_p^p$$

$$\le C(r, \Delta, p) n^{pv} \varpi_{\phi}^{k}(s, n^{-1})_p^p .$$

Where the last inequality follow from (2.3)

Corollary .4

Let u_n^{Δ}, $n \in N$ be a quasi uniform partition of $[-1,1]$ and let $S \in S_r(u_n^{\Delta}) \bigcap C^m[-1,1], r \in N, 0 \le m \le r-1$ then for any $1 \le \phi \le r+1$, $1 \le p \le \infty$ we have

$$n^{-v} \varpi_{\phi-v}^{k}(S^{(v)}, n^{-1})_p \approx \varpi_{\phi}^{k}(S, n^{-1})_p ,$$

$1 \le v \le \min[\phi, m+1]$

Equivalaence constants above depend only on $r, \Delta.$.

Let Z_n is a chebyshev partition , if $Z_n = t_n = (t_i)_{i=0}^{n}$ where $t_i = \cos(\frac{(n-i)\Pi}{n}), 0 \le i \le n.$

Theorem.5 (chebyshev knote)

Let t_n be a chebyshev partition of $[-1,1]$ if $s \in S_r(t_n), r \in N$ then , for any $1 \le \psi \le r+1$, $0 < p < 1$ we have $\varpi_{\psi-v}^{k}(S^{(v)}, n^{-1})_p \le c(r, p) n^{v} \varpi_{\psi}^{k}(S, n^{-1})_p$ for all $1 \le v \le \psi$.

Proof:

Chebyshev knote: let $J_{j,\phi-v} = [t_{j-3(\psi-v)}, t_{j+4+3(\psi-v)}]$, denote $J_{j,\psi-v}$, and , for each $0 \le j \le n-1$

Let $q_j \in \Pi_{\phi-v-1}$ be such that $\left\| s^{(v)} - q_j \right\|_{L_p(J_j)} \le C \varpi_{\phi-v}(s^{(v)}, J_j)_p$

[6], and we assume that $q_j = 0$ if $v = \psi$. Then using theorem 3.1 and the inequality $\psi(x) \le n|J_j|, x \in J_j$, we have

$$\omega_{k-v,v}^{\psi}(s^{(v)}, n^{-1})_p = \sup_{0<h<n^{-1}} \left\| \psi^{v} \Delta_{h\psi}^{k-v}(s^{(v)}, .) \right\|_p$$

$$= \sup_{0<h<n^{-1}} \sum_{j=0}^{n-1} \left\| \psi^{v} \Delta_{h\psi}^{k-v}(s^{(v)}, .) \right\|_{L_p(J_j)}$$

$$\le \sum n^{vp} |J_j|^{vp} \sup_{0<h<n^{-1}} \left\| \psi^{v} \Delta_{h\psi}^{k-v}(s^{(v)} - q_j, .) \right\|_{L_p(J_j)}$$

$$\le C(k, p) n^{vp} \sum_{j=0}^{n-1} |J_j|^{vp} \left\| s^{(v)} - q_j \right\|_{L_p(J_j)}$$

$$\le C(k, p) n^{vp} \sum_{j=0}^{n-1} |J_j|^{vp} \omega_{k-v,v}(s^{(v)}, J_j)_p$$

$$\leq C(r,p)n^{vp}\sum_{j=0}^{n-1}\ \omega_k(s,J_j)_p$$

$$\leq C(r,p)n^{vp}\omega_k^{\psi}(s,n^{-1})_p$$

where the last inequality follows from (2.3)

Corollary.6

Let $s \in S_r(t_n)\bigcap C^m[-1,1]$ $r \in N, 0 \leq m \leq r-1$ then , for any $1 \leq \psi \leq r+1,\ 0 < p < 1$ we have $n^{-v}\varpi_{\psi-v}^k(S^{(v)},n^{-1})_p \approx \varpi_{\psi}^k(S,n^{-1})_p$ for all $1 \leq v \leq \min\{\psi, m+1\}$.

Equivalaence constants above depend only on r.

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مبرهنات جديدة في نظرية التقريب

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الخلاصة

الهدف من هذا البحث هو برهان نتائج للتكافؤات بين مقايس النعومة في نظرية التقريب ،استخدمنا مقياس النعومة غير الطبيعي $\varpi_{\phi}^k(f,t)=\sup_{0<h\leq t}\left\|\bar{\Delta}_{\beta}^{k}(f,x)\right\|_p$ ومقياس نعومة ديتيزين الثقيل $\varpi_{k,v}^{\psi}(f,t)=\sup_{0<h\leq t}\left\|\psi(.)^v\bar{\Delta}_{h\psi(.)}^{k}(f,x)\right\|_p$ في الفضاءات $L_p[-1,1], 0<p\leq\infty$ لدوال السيلاني ، نتائج عديدة حصلنا عليها مثلا، بينا المتراجحة $|J|^v\varpi_{\phi-v}^k(S^{(v)},J)_p \approx \varpi_{\phi}^k(S,J)_p$ عندما $1\leq\phi\leq r+1,\ 1\leq p\leq\infty$ ، $1\leq v\leq\min[\phi,m+1]$ متحققة واخيرا بينا تجزئة شيبشيف متحققة ايضا باستخدام مقياس نعومة ديتزين الثقيل.