

## **The Normed $S^*$ -Vector Lattice and Some Properties**

### **الحزم المتجهة المعيارية - $S^*$ وبعض خصائصها**

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#### **§Abstract**

In this paper, we will discuss some order and topological properties of normed  $S^*$ -vector lattices which are analogous to the corresponding properties of usual normed lattices, we established assertions contained on  $(o)$  and  $(no)$ -convergence remain valid for the  $(t)$  and  $(nt)$ -convergence and investigate the fundamental concepts used for understanding this work.

**Keywords:** Order convergence, vector lattice, Riesz space, normed  $S^*$ -vector lattice, Banach  $S^*$ -vector lattice.

#### **الخلاصة**

في هذا البحث سنناقش بعض الخصائص الترتيبية والتوبولوجية للحزم المتجهة المعيارية -  $S^*$  والتي تكون مماثلة لما يقابلها من خصائص في الحزم المعيارية الاعتيادية. أثبتنا إن الفرضيات التي تحتوي على تقارب  $(o)$  و  $(no)$  تبقى صحيحة بالنسبة للتقارب  $(t)$  و  $(nt)$ , علاوة على ذلك فقد استقصينا المفاهيم الأساسية و الضرورية لفهم الموضوع.

#### **§ 1. Introduction.**

The theory of vector lattices appeared in early thirties of last century and is connected with the names of L. V. Kantorovich, F. Riesz, and H. Freudenthal. The study of vector spaces equipped with an order relation compatible with a given norm structure was evidently motivated by the general circumstances that brought to life functional analysis in those years. Here the general inclination to abstraction and uniform approach to studying functions, operations on functions, and equations related to them should be noted. A remarkable circumstance was that the comparison of the elements could be added to the properties of functional objects under consideration. At the same time, the general concept of a Banach space ignored a specific aspect of the functional spaces-the existence of a natural order structure in them, which makes these spaces vector-lattice

Along with the theory of ordered spaces, the theory of Banach algebras was being developed almost at the same time. Although at the beginning these two theories advanced in parallel, soon their paths parted. Banach algebras were found to be effective in function theory, in the spectral theory of operators, and in other related fields. The theory of vector lattices was developing more slowly and its achievements related to the characterization of various types of ordered spaces and to the description of operators acting in them was rather unpretentious and specialized (see [1]).

In ordered vector spaces there are several natural ways to define convergence using only the ordering. We will refer to them as 'order-convergence'. Order-convergence of nets is widely used. In for example, the study of normed vector lattices, it is used for order continuous norms [2, 3]. It is also used in theory on operators between vector lattices to define order continuous operators which are operators that are continuous with respect to order-convergence [3]. The commonly used definition of order-convergence for nets [3] originates from the definition for sequences. Abramovich and Sirotkin [4] proposed, in the setting of vector lattices, a new and improved definition for order-convergence of nets. This definition is also used in [2], where some relations are given in the case of vector lattices.

Our interest in this work is to discuss some order and topological properties of normed  $S^*$ -vector lattices and established assertions contained on  $(o)$  and  $(no)$ -convergence remain valid for the  $(t)$  and  $(nt)$ -convergence. The main result in this work is the following:

**Theorem 3.1.** Let  $(X, \|\cdot\|)$  be a normed  $S^*$ -vector lattice .The following conditions are equivalent.

- a)  $X$  is a Banach  $S^*$ -vector lattice .
- b) If  $\{x_n\}$  is an  $(t)$ -Cauchy increasing sequence from  $X_+$ , then  $\{x_n\}$  is  $(nt)$ -converges in  $X$ .
- c) If  $\{x_n\}$  is an  $(t)$ -Cauchy increasing sequence from  $X_+$ , then there exists  $x = (\sup x_n) \in X$ .

**Theorem 3.2.** Let  $(X, \|\cdot\|)$  be a  $\sigma$ -complete normed  $S^*$ -vector lattice with the order continuous. Then  $X$  is of countable type and so  $X$  is a complete vector lattice.

## **§ 2. Convergence. Preliminaries**

This section consists of a collection of known notions, and facts related to the theory of vector lattices and some order and topological properties.

**Definition 2.1[5].** An order on a non-empty set  $X$  is a relation  $\leq$  such that:

- 1)  $x \leq x$  for all  $x \in X$  .
- 2)  $x \leq y$  and  $y \leq x$  imply that  $x = y$ .
- 3)  $x \leq y$  and  $y \leq z$  implies that  $x \leq z$ .

We use  $y \geq x$  as a synonym for  $x \leq y$ . If  $A$  is a non-empty subset of  $X$  then  $x$  is an upper bound for  $A$  if  $a \leq x$  for all  $a \in A$ . In this case we also say that  $A$  is bounded above. An upper bound  $x$  for  $A$  is its least upper bound or supremum if for any other upper bound  $y$  for  $A$  we have  $x \leq y$  .The terms lower bound, bounded below, greatest lower bound or infimum are defined analogously. Sets that are both bounded above and below are termed order bounded. An order interval in  $X$  is a set of the form  $[x, y] = \{m \in X: x \leq m \leq y\}$ .

**Definition 2.2[5].** A lattice is a non-empty set  $X$  with an order  $\leq$  such that every pair of elements  $x, y \in X$  has both a supremum  $x \vee y$  and an infimum  $x \wedge y$ . The supremum of a general subset  $A$  of  $X$ , when it exists, is denoted by any of  $\sup(A), \vee A, \sup\{a: a \in A\}, \vee\{a: a \in A\}$ .

**Definition 2.3[5].** An ordered vector space is a real vector space  $X$  which is also an ordered space with the linear and order structures connected by the implications

- 1) If  $x, y, z \in X$  and  $x \leq y$  then  $x + z \leq y + z$ .
- 2) If  $x, y \in X, x \leq y$  and  $0 \leq \alpha \in \mathbb{R}$  then  $\alpha x \leq \alpha y$ .

The set  $X_+ = \{x \in X: x \geq 0\}$  is termed the positive cone in  $X$  and its elements are termed positive (rather than non-negative).

An ordered vector space which is also a lattice is a vector lattice or Riesz space.

**Remark 2.4 [5].** For a vector lattice  $X$ , the positive part of  $x \in X$  is  $x^+ = x \vee 0$ , whilst the negative part is  $x^- = (-x) \vee 0$ . The modulus of  $x$  is  $|x| = x \vee (-x)$ . We say that  $x, y \in X$  are disjoint, written  $x \perp y$ , if  $|x| \wedge |y| = 0$ . It is elementary, but often useful, that  $x_+$  and  $x_-$  are disjoint and that  $x = x^+ - x^-$  whilst  $|x| = x^+ + x^-$ .

**Definition 2.5 [6].** An element from  $X_+$  is called a Freudenthal unit and denoted by  $\hat{1}$ , if it follows from  $x \wedge \hat{1} = 0, x \in X$  that  $x = 0$ .

**Definition 2.6[6].** A vector lattice  $X$  is called complete ( $\sigma$ -complete) if, for any (countable) subset  $A$  of  $X$ , there exist  $\sup A$  and  $\inf A$ .

**Definition 2.7[6].** We say that  $X$  be a bimodule over  $S^*=[0,1]$ , i.e.  $X$  is abelian group with respect to addition operation  $(+)$  and right and left multiplication by element from  $S^*$  are defined on  $X$  having the properties: For all  $x, y \in X, \lambda, \mu \in S^*$

- 1)  $\lambda(x + y) = \lambda x + \lambda y, (x + y)\lambda = x\lambda + y\lambda$  .
- 2)  $(\lambda + \mu)x = \lambda x + \mu x, x(\lambda + \mu) = x\lambda + x\mu$ .
- 3)  $\lambda(\mu x) = (\lambda\mu)x, (x\lambda)\mu = x(\lambda\mu)$ .
- 4)  $\hat{1} \cdot x = x. \hat{1}$  .

**Remark 2.8[6].** A bimodule  $X$  over  $S^*$  is called a normal  $S^*$ -module if:

- 1) For all  $x \in X, \lambda \in S^*$ , then  $\lambda x = x\lambda$ .
- 2) For any  $e \in \nabla(s^*), e \neq 0$ , there exists  $x \in X$  such that  $xe \neq 0$ .
- 3) For any decomposition of the identity  $\{e_i\} \subset \nabla(s^*)$  and for any  $\{x_i\} \subset X$  there exists  $x \in X$  such that  $xe_i = x_i e_i, i = 1, 2, \dots, n$ .
- 4) For any  $x \in X$  and any sequence  $\{e_n\}$  of mutually disjoint element from  $\nabla(s^*)$  it follows from the equalities  $e_n x = 0, n = 1, 2, 3 \dots$  that  $(\sup_{n \geq 1} e_n)x = 0$ .

It is clear that the condition 4 implies a validity of the analogous property for increasing sequences of idempotent from  $S^*$ .

**Definition 2.9[6].** A normal  $S^*$ - module is called a  $S^*$ - vector lattice if  $X$  is simultaneously, lattice, i.e. an ordered set in which for any two elements  $x, y \in X$  there exists their supremum  $x \vee y$ , infimum  $x \wedge y$  and ,in addition , the following algebraic operations and order agreement conditions are fulfilled:

- 1) For any  $z \in X$  it follows from  $x \leq y$  that  $x + z \leq y + z$ .
- 2) If  $x \geq 0, \lambda \in S^*, \lambda > 0$  then  $\lambda x \geq 0$ .

It is evident that any  $S^*$ - vector lattice  $X$  is a vector lattice in a usual sense (it is sufficient to consider  $X$  as a vector space over the field  $\{\alpha \cdot \hat{1} : \alpha \in \mathbb{R}\} \approx \mathbb{R}$ ).

**Remark 2.10[6].** If  $X$  is an  $S^*$ -vector lattice (where  $S^*$  is the ring of all real measureable function on the interval  $[0,1]$ ) and  $X$  is a  $\sigma$ -complete vector lattice, then we call  $X$  is a  $\sigma$ -complete  $S^*$ -vector lattice.

**Definition 2.11[7].** A sequence  $\{x_n\}$  of elements from  $X$  (o)-converges to an element  $x \in X$  if there exist two sequences  $\{a_n\}, \{b_n\}$  in  $X$  such that  $a_n \leq x_n \leq b_n$ , for each  $n \in N$ , and, in addition,  $a_n \uparrow x, b_n \uparrow x$ . We denote  $x = (o) - \lim_n x_n$  or  $x_n \xrightarrow{(o)} x$ .

**Definition 2.12[7].** If  $x_\alpha \xrightarrow{(o)} x$  and  $\{x_\alpha\}_{\alpha \in A}$  is increasing (decreasing), then we write  $x_\alpha \uparrow x$  (respectively,  $x_\alpha \downarrow x$ ).

**Remark 2.13[7].** Suppose that  $t$  is a measure topology in the metrizable topological vector lattice  $S^*$ , then  $x_n \xrightarrow{(t)} x$ , means that  $\{x_n\}$  converges to  $x$  in the topology  $t$  where  $x_n, x \in S^*$ . It is known that the convergence  $x_n \xrightarrow{(o)} x$  where  $x_n, x \in S^*$  implies the convergence  $x_n \xrightarrow{(t)} x$ . Besides, there exists a base of closed normal neighborhoods around zero in  $(S^*, t)$ .

**Definition 2.14[8].** A vector lattice  $X$  is called countable type if for any set of non-zero mutually disjoint elements from  $X$  is finite or countable.

**Lemma 2.15[9].** If  $\{x_n\}_{n=1}^\infty$  be a sequence of disjoint element, then  $x_n \xrightarrow{(o)} 0$ .

**Definition 2.16[10].** A mapping  $\|\cdot\|: X \rightarrow S^*$  from a normal  $S^*$ -module  $X$  into  $S^*$  is called a  $S^*$ -norm if:

- 1)  $\|x\| > 0$  for all  $x \in X$  and  $\|x\| = 0$  if and only if  $x = 0$ .
- 2)  $\|\lambda x\| = |\lambda| \|x\|$  for any  $x \in X, \lambda \in S^*$ .
- 3)  $\|x + y\| \leq \|x\| + \|y\|$  for any  $x, y \in X$ .

**Definition 2.17[10].** Let  $X$  be a normal  $S^*$ -module,  $\|\cdot\|$  is an  $S^*$ -norm on  $X$ , then the pair  $(X, \|\cdot\|)$  is called a normed  $S^*$ -module.

**Definition 2.18[10].** A sequence  $\{x_n\}$  from a normed  $S^*$ - module  $X$  is said to be (o) –Cauchy (t-Cauchy) if  $\|x_n - x_m\| \xrightarrow{(o)} 0$  i.e.  $(\sup_{n, m \geq k} \|x_n - x_m\|) \downarrow 0$  (respectively,  $\|x_n - x_m\| \xrightarrow{(t)} 0$ ). In the sequel the record  $x_n \xrightarrow{(no)} x$  (respectively,  $x_n \xrightarrow{(nt)} x$ ), means that  $x_n, x \in X$  and  $\|x_n - x\| \xrightarrow{(o)} 0$  (respectively,  $\|x_n - x\| \xrightarrow{(t)} 0$ ).

**Proposition 2.19[10].** Let  $(X, \|\cdot\|)$  be a normed  $S^*$ -module,  $x_n, x \in X$  the following statements are satisfied:

- 1) If  $x_n \xrightarrow{(no)} x$ , then  $x_n \xrightarrow{(nt)} x$ .
- 2) If  $x_n \xrightarrow{(nt)} x$ , then there exists a subsequence  $x_{n_k} \xrightarrow{(no)} x$ .
- 3) If  $\{x_n\}$  is an  $(o)$ -Cauchy sequence, then  $\{x_n\}$  is a  $(t)$ -Cauchy sequence.
- 4) If  $\{x_n\}$  is a  $(t)$ -Cauchy sequence, then there exists  $(o)$ -Cauchy subsequence of  $\{x_n\}$ .

**Remark 2.20[10].** Let  $(X, \|\cdot\|)$  be a normed  $S^*$ -module and,  $t$  be a measure topology in  $S^*$ ,  $U$  be the system of all the neighborhoods around zero in  $(S^*, t)$ . Denote by  $\Omega(U)$  the set of all  $x \in X$  such that  $\|x\| \in U$ , where  $U \in \mathcal{V}$ . If  $U$  is a normal neighborhood around zero in  $(S^*, t)$ ,  $V \in \mathcal{V}$  and  $V + V \in U$ , then  $\Omega(V) + \Omega(V) \subset \Omega(U)$ .

**Remark 2.21[10].** Let  $(X, \|\cdot\|)$  be a Banach  $S^*$ -module,  $\{x_n\}$  be a  $(t)$ -Cauchy sequence from  $X$ . Then there exists an  $(o)$ -Cauchy subsequence  $\{x_{n_k}\}$ , so there exists  $x \in X$  such that  $\|x_{n_k} - x\| \xrightarrow{(o)} 0$ .

Hence  $x_{n_k} \xrightarrow{(nt)} x$ , therefore  $\|x_n - x\| \leq \|x_n - x_{n_k}\| + \|x_{n_k} - x\| \xrightarrow{(t)} 0$ .

Thus  $(X, \|\cdot\|)$  is a  $t$ -Banach  $S^*$ -module. It is analogously established that a  $t$ -Banach  $S^*$ -module is a Banach  $S^*$ -module.

**Definition 2.22[6].** An  $S^*$ -vector lattice  $X$  with an  $S^*$ - norm  $\|\cdot\|$  is called a normed  $S^*$ - vector lattice, if it follows from  $|x| \leq |y|$ ,  $x, y \in X$  that  $\|x\| \leq \|y\|$ .

**Example 2.23**  $L_p(0,1)$  when  $(X, \Sigma, \mu)$  is the usual Lebesgue measure space on  $[0,1]$  is  $S^*$ -vector lattice. The norm is defined by

$$\|f\|_p = \left( \int_X |f(x)|^p d\mu \right)^{1/p}. \text{ Then } (L_p(0,1), \|\cdot\|_p) \text{ is a normed } S^* \text{- vector lattice.}$$

**Definition 2.24[10].** If a normed  $S^*$ -vector lattice  $(X, \|\cdot\|)$  is a Banach  $S^*$ -module, then  $(X, \|\cdot\|)$  is called a Banach  $S^*$ -vector lattice.

**Example 2.25** The space  $C(Q)$  of continuous functions on an arbitrary compact topological space  $Q$  with the natural (pointwise) order and with the ordinary (uniform) norm is a Banach lattice.

**Proposition 2.26[8].** Let  $(X, \|\cdot\|)$  be a normed  $S^*$ -vector lattices,  $\{x_n\}$  be an increasing (decreasing) sequence from  $X$  and  $x_n \xrightarrow{(nt)} x \in X$ . Then  $x = \sup x_n$  (respectively,  $x = \inf x_n$ ).

**Remark 2.27[8].** If  $(X, \|\cdot\|)$  be a normed  $S^*$ -vector lattice, we will say that the  $S^*$ -norm in  $X$  is order continuous if it follows from  $\{x_n\} \subset X_+$ ,  $x_n \downarrow 0$  that  $\|x_n\| \xrightarrow{(t)} 0$ .

**Theorem 2.28[10].** Let  $(X, \|\cdot\|)$  be a normed  $S^*$ -vector lattice. The following conditions are equivalent

- 1)  $X$  is a Banach  $S^*$ -vector lattice .
- 2) If  $\{x_n\}$  is an  $(o)$ -Cauchy increasing sequence from  $X_+$ , then  $\{x_n\}$  is  $(no)$ -converges in  $X$ .
- 3) If  $\{x_n\}$  is an  $(o)$ -Cauchy increasing sequence from  $X_+$ , then there exists  $x = (\sup x_n) \in X$ .

### § 3. The main results.

In this section, we shall prove some properties of normed  $S^*$ -vector lattices and we established assertions contained on  $(o)$  and  $(no)$  –convergence remain valid for the  $(t)$  and  $(nt)$  –convergence.

**Theorem 3.1.** Let  $(X, \|\cdot\|)$  be a normed  $S^*$ -vector lattice. The following conditions are equivalent.

- a)  $X$  is a Banach  $S^*$ -vector lattice .
- b) If  $\{x_n\}$  is an  $(t)$ -Cauchy increasing sequence from  $X_+$ , then  $\{x_n\}$  is  $(nt)$ -converges in  $X$ .
- c) If  $\{x_n\}$  is an  $(t)$ -Cauchy increasing sequence from  $X_+$ , then there exists  $x = (\sup x_n) \in X$ .

**Proof.**

**(a→b)** Let  $X$  is a Banach  $S^*$ -vector lattice and let  $\{x_n\}$  is an  $(t)$ -Cauchy increasing sequence from  $X_+$ . Then there exists an  $(o)$ -Cauchy subsequence  $\{x_{n_k}\}$ . By using the Remark 2.21, we have

$$x_n \xrightarrow{(nt)} x ,$$

i.e.  $\{x_n\}$  is  $(nt)$ -converges in  $X$ .

**(b→a)** Let  $\{x_n\}$  be a  $(t)$ -Cauchy increasing sequence from  $X_+$ . Choose a basis  $\{U_k\}$  of normal closed neighborhoods around zero in  $(s^*, t)$ , by using the Remark 2.20, we have

$$U_{n+1} + U_{n+1} \subset U_n , \quad n = 1, 2, \dots$$

Since  $\{x_n\}$  is an  $(t)$ -Cauchy sequence, we have

$$\left( \sup_{n, m \geq k} \|x_n - x_m\| \right) \downarrow 0 .$$

We may consider that  $\|x_{n+1} - x_n\| \in U_n$  for all  $n$ .

Put  $\beta_n = \|x_{n+1} - x_n\|$

We have for  $n > m$ ,

$$\left| \sum_{k=1}^n \beta_k - \sum_{k=1}^m \beta_k \right| \leq \sum_{k=m+1}^n \beta_k \in U_{m+1} + \dots + U_n \subset U_m ,$$

i.e.  $\left\{ \sum_{k=1}^n \beta_k \right\}$  is a  $(t)$ -Cauchy sequence, therefore the series

$$\sum_{k=1}^{\infty} \beta_k = \beta \in S^* \text{ converges in } (s^*, t).$$

In particular ,

$$\left( \sup_{n, m > k} \sum_{k=m}^n \beta_k \right) \downarrow 0 .$$

Let  $y_n = \sum_{k=1}^n (x_{k+1} - x_k)_+ , \quad z_n = \sum_{k=1}^n (x_{k+1} - x_k)_- .$

If  $n > m$ , then

$$\begin{aligned} \|y_n - y_m\| &= \left\| \sum_{k=1}^n (x_{k+1} - x_k)_+ - \sum_{k=1}^m (x_{k+1} - x_k)_+ \right\| = \left\| \sum_{k=m+1}^n (x_{k+1} - x_k)_+ \right\| \\ &\leq \sum_{k=m+1}^n \|(x_{k+1} - x_k)_+\| \leq \sum_{k=m+1}^n \beta_k \end{aligned}$$

Hence  $\{y_n\}$  is an  $(t)$ -Cauchy increasing subsequence from  $X_+$ . Therefore, there exists  $y \in X$  for which

$$\|y_n - y\| \xrightarrow{(t)} 0 . \quad (3.1)$$

Similarly, there exists  $z \in X$  such that

$$\|z_n - z\| \xrightarrow{(t)} 0 . \quad (3.2)$$

From (3.1) and (3.2), we obtain

$$\|(y_n - z_n) - (y - z)\| \xrightarrow{(t)} 0 .$$

It remains to note that

$$\begin{aligned} y_n - z_n &= \sum_{k=1}^n (x_{k+1} - x_k)_+ - \sum_{k=1}^n (x_{k+1} - x_k)_- \\ &= \sum_{k=1}^n (x_{k+1} - x_k)_+ - (x_{k+1} - x_k)_- \\ &= \sum_{k=1}^n (x_{k+1} - x_k) \end{aligned}$$

$$= x_{n+1} - x_1 ,$$

we get,

$$x_{n+1} = y_n - z_n + x_1 .$$

From above, we have  $y_n - z_n \xrightarrow{(nt)} y - z$

and  $x_{n+1} = y - z + x_1$ , imply that

$$\|x_{n+1} - x_n\| = \|y - z + x_1 - x_n\| \xrightarrow{(t)} 0 ,$$

Consequently  $x_n \xrightarrow{(nt)} y - z + x_1$ .

Thus,  $X$  is a Banach  $S^*$ -vector lattice .

**(b→c)** We have  $(X, \|\cdot\|)$  be a normed  $S^*$ -vector lattice. Let  $\{x_n\}$  is an  $(t)$ -Cauchy increasing sequence from  $X_+$ , and its  $(nt)$  -converges in  $X$ . We will use the proposition 2.26, to get

$$x = (\sup x_n) \in X .$$

**(c→b)** Let  $\{x_n\}$  is an  $(t)$ -Cauchy increasing sequence from  $X_+$ . By hypothesis of the theorem above , There exists

$$x = (\sup x_n) \in X .$$

We can consider that

$$\|x_{n+1} - x_n\| \in n^{-1}U_n ,$$

where  $\{U_k\}$  is the same basis of neighborhoods around Zero in  $(S^*, t)$  as shown above. Now we set

$$\alpha_n = n\|x_{n+1} - x_n\| ,$$

we have for  $n > m$ ,

$$\left| \sum_{k=1}^n \alpha_k - \sum_{k=1}^m \alpha_k \right| \leq \sum_{k=m+1}^n \alpha_k \in U_{m+1} + \dots + U_n \subset U_m$$

i.e.  $\left\{ \sum_{k=1}^n \alpha_k \right\}$  is a  $(t)$ -Cauchy sequence , therefore the series

$$\sum_{k=1}^{\infty} \alpha_k = \alpha \in S^* ,$$

In particular ,  $\left( \sup_{n,m>k} \sum_{k=m}^n \alpha_k \right) \downarrow 0$ .

Put  $y_n = x_1 + \sum_{k=1}^n k(x_{k+1} - x_k)$ . If  $n > m$  , then

$$\|y_n - y_m\| = \left\| x_1 + \sum_{k=1}^n k(x_{k+1} - x_k) - x_1 - \sum_{k=1}^m k(x_{k+1} - x_k) \right\|$$

$$\begin{aligned}
 &= \left\| \sum_{k=m+1}^n k (x_{k+1} - x_k) \right\| \\
 &\leq \sum_{k=m+1}^n k \left\| (x_{k+1} - x_k) \right\| \in U_{m+1} + \dots + U_n \subset U_m \\
 &\leq \sum_{k=m+1}^n \alpha_k .
 \end{aligned}$$

Hence  $\{y_n\}$  is an  $(t)$ -Cauchy sequence from  $X_+$ . Since this sequence is increasing, then by the condition (c), there exists  $y = \sup y_n$ . Furthermore,

$$\begin{aligned}
 n(x - x_n) &= \sup_{m>n} \sum_{k=n}^m n(x_{k+1} - x_k) \\
 &\leq \sup_{m>n} \sum_{k=n}^m k(x_{k+1} - x_k) \\
 &\leq \sup_{m>n} y_n = y .
 \end{aligned}$$

Hence  $0 \leq (x - x_n) \leq n^{-1}y$ .

Therefore  $\|x - x_n\| \leq n^{-1}\|y\|$ , we obtain

$$\|x - x_n\| \xrightarrow{(t)} 0 ,$$

From the definition 2.18,  $x_n \xrightarrow{(nt)} x$ . ■

**Theorem 3.2.** Let  $(X, \|\cdot\|)$  be a  $\sigma$ -complete normed  $S^*$ -vector lattice with the order continuous. Then  $X$  is of countable type and so  $X$  is a complete vector lattice.

**Proof.** Let  $E$  be an arbitrary infinite bounded set of non-zero mutually disjoint elements from  $X$ ,  $\{w_n\}_{n=1}^\infty$  be some basis of neighborhoods around zero in  $(S^*, t)$ .

Put  $E_n = \{x \in E : \|x\| \notin w_n\}$ .

If  $E_n$  is infinite, then there exists a sequence  $\{x_k\}$  of pairwise distinct elements from  $E_n$ . Since  $X$  is  $\sigma$ -complete,  $|x_k| \leq y$  for some  $y \in X$  and all  $k = 1, 2, \dots$ ,  $|x_k| \wedge |x_m| = 0$  for  $m \neq k$  and by using lemma 2.15, we have

$$x_k \xrightarrow{(o)} 0,$$

since order continuous is hold, we obtain  $\|x_k\| \xrightarrow{(t)} 0$ .

Thus  $\|x_k\| \in w_n$  beginning from some number. Therefore  $E_n$  is a finite set. Hence  $E = \bigcup_{n=1}^\infty E_n$  is countable. ■

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