Jordan θ-Centralizers of Prime and Semiprime Rings

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Abstract:

The purpose of this paper is to prove the following result: Let R be a 2-torsion free ring and T: R→R an additive mapping such that T is left (right) Jordan θ-centralizers on R. Then T is a left (right) θ-centralizer of R, if one of the following conditions hold (i) R is a semiprime ring has a commutator which is not a zero divisor . (ii) R is a non commutative prime ring . (iii) R is a commutative semiprime ring, where θ be surjective endomorphism of R . It is also proved that if T(xoy)=T(x)oθ(y)=θ(x)oT(y) for all x, y ∈ R and θ-centralizers of R coincide under same condition and θ(Z(R)) = Z(R) .

Key words: prime ring, semiprime ring, left (right) centralizer, centralizer, Jordan centralizer, left (right) θ -centralizer, θ-centralizer , Jordan θ-centralizer.

Introduction:

Throughout this paper, R will represent an associative ring with the center Z. R is called prime if aRb = (0) implies a = 0 or b = 0 and semiprime if aRa = (0) implies a = 0. A mapping D: R→R is called derivation if D(xy) = D(x)y+xD(y) holds for all x, y ∈ R. A left (right) centralizer of R is an additive mapping T: R→ R which satisfies T(xy) = T(x)y (T(xy) = xT(y)) for all x, y ∈ R. A centralizer of R is an additive mapping which is both left and right centralizer . If a ∈ R, then $L_a(x) = ax$ is a left centralizer and $R_a(x) = xa$ is a right centralizer.

A mapping D: R→R is called (θ,θ) derivation if D(xy) = D(x)θ(y)+ θ(x)D(y) holds for all x, y ∈ R[1]. A left (right) θ-centralizer of R is an additive mapping T: R→R which satisfies T(xy) = T(x)θ(y) (T(xy) = θ(x)T(y)) for all x, y ∈ R. A θ-centralizer of R is an additive mapping which is both left and right θ-centralizer. If a ∈ R, then $L_a(x) = aθ(x)$ is a left θ-centralizer and $R_a(x) = θ(x)a$ is a right θ-centralizer[2][3].

A mapping D: R→R is called Jordan (θ,θ) derivation if $D(x^2) = D(x)θ(x)+ θ(x)D(x)$ holds for all x ∈ R[7]. A Jordan left (right) θ-centralizer of R is an additive mapping T: R→R which satisfies $T(x^2) = T(x)θ(x)$ ($T(x^2) = θ(x)T(x)$) for all x ∈ R. A Jordan θ-centralizer of R is an additive mapping which is Jordan both left and right θ-centralizer[2,3] .

If R is a ring with involution *, then every additive mapping E: R→R which satisfies $E(x^2) = E(x)x^* + xE(x)$ for all x ∈ R is called Jordan *-derivation. These mappings are closely connected with a question of representability of quadratic forms by bilinear forms. Some algebraic properties of Jordan *-derivations are

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considered in [4], where further references can be found. For quadratic forms see [5].

Brešar and Zalar obtained a representation of Jordan *-derivations in terms of left and right centralizers on the algebra of compact operators on a Hilbert space. They arrived at a problem whether an additive mapping T which satisfies a weaker condition $T(x^2) = T(x)x$ is automatically a left centralizer. They proved that this is in fact so if R is a prime ring (generally without involution). In [6] Zalar generalize this result on semiprime rings. In [7] A. H. Majeed and H. A. Shaker extended the results of Zalar [6].

An easy computation shows that every centralizer T is satisfies T(xoy)=T(x)oy= xoT(y). B. Zalar in [6] prove that every additive mapping T: R→R which satisfies T(xoy)=T(x)oy= xoT(y) of a semiprime ring is a centralizer.

An easy computation shows that every θ-centralizer T satisfies T(xoy)=T(x)oθ(y)=θ(x)oT(y) .

In the present paper we generalize results of Zalar[6] to θ-centralizer .

1. The first result.

To prove our first result, we need two lemmas which we now state.

Lemma 1.1.[6]

Let R be a semiprime ring. If a,b ∈R such that axb=0 for all x ∈ R, then ab=ba=0.

Lemma 1.2.[6]

Let R be a semiprime ring and A,B: R×R→R biadditive mappings. If A(x,y) w B(x,y) = 0 for all x, y,w ∈ R, then A(x,y) w B(u,v) = 0 for all x, y, u, v,w ∈ R.

Theorem 1.3

Let R be a 2-torsion free ring, Then every Jordan left (right) θ-centralizer is a left (right) θ-centralizer, if one of the following statements hold:-

(i)   R is a semiprime ring has a commutator which is not a zero divisor.

(ii)   R is a non commutative prime ring .

(iii)   R is a commutative semiprime ring .

Where θ be surjective endomorphism of R

Proof:

$T(x^2) = T(x)\, \theta(x)$ for all x ∈ R …(1)

If we replace x by x + y, we get

$T(xy + yx) = T(x)\, \theta(y) + T(y)\, \theta(x)$...(2)

By replacing y with xy + yx and using (2), we arrive at

$T(x(xy+yx)+(xy+yx)x)=T(x)\theta(xy)+2T$

$(x)\theta(yx) +T(y)\theta(x2)$ ...(3)

But this can also be calculated in a different way.

$T(x^2y + yx^2) + 2T(xyx) = T(x)\theta(xy) +$

$T(y)\theta(x^2) + 2T(xyx)$ ...(4)

Comparing (3) and (4), we obtain

$T(xyx) = T(x)\theta(yx)$ for all x,y ∈ R …(5)

If we linearize (5), we get

$T(xyz + zyx) = T(x)\, \theta(yz) + T(z)\, \theta(yx)$ ...(6)

Now we shall compute j = T(xyzyx + yxzxy) for all x,y,z ∈ R in two different ways. Using (5), we have

$j = T(x)\, \theta(yzyx) + T(y)\, \theta(xzxy)$…(7)

Using (6), we have

$j = T(xy) \, \theta(zyx) + T(yx) \, \theta(zxy)...(8)$

Comparing (7) and (8) and introducing a biadditive mapping $B(x,y) = T(xy) - T(x)\theta(y)$, we arrive at
$B(x,y) \, \theta(zyx) + B(y, x) \, \theta(zxy) = 0$

for all $x,y,z \in R…(9)$

Equality (2) can be rewritten in this notation as $B(x,y) = -B(y,x)$ for all $x,y \in R$. Using this fact and equality (9), we obtain

$B(x,y) \, \theta(z) \, [\, \theta(x), \, \theta(y) \,] = 0$

for all $x,y,z \in R…(10)$

Using Lemma 1.2, we have

$B(x, y) \, \theta(z) \, [\theta(u), \, \theta(v)] = 0$

for all $x,y,z,u,v \in R$ …(11)

Using Lemma 1.1, we have

$B(x, y)[\theta(u), \, \theta(v)] = 0$ for   all

$x,y,,u,v \in R$ …(12)

If R has a commutator which is not a zero divisor
Using (12) and θ is onto, we have
  $B(x,y) = 0$ for all $x,y \in R$

If R is a non commutative prime ring
Using (11) and θ is onto, we have
  $B(x,y) = 0$ for all $x,y \in R$

If R is a commutative semiprime ring

Now we shall compute $j = T(xyzyx)$ in two different ways.
Using (5) we have

  $j = T(x) \, \theta(yzyx)…(13)$

  $j = T(xy) \, \theta(zyx)…(14)$

Comparing (13) and (14) , we arrive at

$B(x, y) \, \theta(z) \, \theta(yx) = 0$ for   all

$x,y,z \in R$ …   (15)

Let $\Psi(x,y)=\theta(x)\theta(y)$, it's clear that Ψ is a biadditive mapping, therefore

  $B(x, y) \, \theta(z) \, \Psi(y,x) = 0$ for all

      $x,y,z \in R$

Using Lemma 1.2, we have

  $B(x, y) \, \theta(z) \, \Psi(u,v) = 0$ for all

      $x,y,z,u,v \in R$

Implies that

    $B(x, y) \, \theta(z) \, \theta(uv) = 0$ for   all

$x,y,z,u,v \in R$ ….(16)

By replacing θ(v) with $B(x, y) \, \theta(z)$, θ is onto , and R is a semiprime ring, we have

  $B(x,y) = 0$ for all $x,y \in R$

If $T(x^2) = \theta(x)T(x)$, we obtain the assertion of the theorem with similar approach as above, the proof is complete.□

Corollary 1.4
    Let R be a 2-torsion free prime ring .Then every Jordan left (right) θ-centralizer is a left (right) θ-centralizer, where θ be surjective endomorphism of R .

2. The second result.
    We again divide the proof in few lemmas.

Lemma 2.1.
    Let R be a semiprime ring, D a θ-derivation of R and a ∈ R some fixed element. Where θ be surjective endomorphism of R

(i) $D(x)D(y) = 0$ for all x, y $\in$ R implies D = 0.

(ii) $a\theta(x) - \theta(x)a \in Z$ for all x $\in$ R implies $a \in Z(R)$.

Proof :

(i) $D(x)\,\theta(y)D(x) = D(x)D(yx) -$

$D(x)D(y)\,\theta(x) = 0$ for all x,y $\in$ R

But θ is onto, and R is a semiprime

ring, we have D=0

(ii)   Define $D(x) = a\theta(x) - \theta(x)a$. It is easy to see that D is a (θ,θ)-derivation. Since $D(x) \in Z(R)$ for all $x \in R$, we have $D(y)\theta(x) = \theta(x)D(y)$ and also $D(yz)\theta(x) = \theta(x)D(yz)$.

Hence

$D(y)\theta(zx) + \theta(y)D(z)\theta(x) =$

$\theta(x)D(y)\theta(z) + \theta(xy)D(z)$

$D(y)[\theta(z),\theta(x)] = D(z)[\theta(x),\theta(y)]$

Since θ is surjective take $a=\theta(z)$.

Obviously $D(z) = 0$, so we obtain

$0 = D(y)[a,\theta(x)] = D(y)D(x)$ for all x,y

$\in$ R

From (i) we get D = 0 and hence a $\in$

Z(R).$\square$

Lemma 2.2.

Let R be a semiprime ring and a $\in$ R some fixed element. If $T(x) = a\theta(x)+\theta(x)a$,and $T(xoy) = T(x)o\theta(y) = \theta(x)oT(y)$ for all x, y $\in$ R then $a \in Z$. Where θ be surjective endomorphism of R

Proof :

$T(xy + yx) = T(x)\theta(y) + \theta(y)T(x)$

          for all x,y $\in$ R

gives us

$a\theta(xy) + a\theta(yx) + \theta(xy)a + \theta(yx)a =$

$(a\theta(x) + \theta(x)a)\,\theta(y) + \theta(y)(a\theta(x) +$

         $\theta(x)a)$

Implies that

$a\theta(yx) + \theta(xy)a - \theta(x)a\theta(y) - \theta(y)a\theta(x)$

$= 0 = (a\theta(y) - \theta(y)a)\theta(x) - \theta(x)(a\theta(y)$

       $- \theta(y)a)$ for all x,y $\in$ R

The second part of Lemma 2.1 now

gives us a $\in$ Z(R).$\square$

Lemma 2.3.

Let R be a semiprime ring, and T: R$\rightarrow$R an additive mapping which satisfies $T(xoy) = T(x)o\theta(y) = \theta(x)oT(y)$ for all x, y $\in$ R. Then T maps from Z(R) into Z(R). Where θ be surjective endomorphism of R

Proof :

Take any c $\in$ Z and denote a = T(c).

$2T(cx) = T(cx + xc) = T(c)\theta(x) +$

$\theta(x)T(c) = a\theta(x) + \theta(x)a$

A straightforward verification shows that $S(x) = 2T(cx)$ is satisfies $S(xoy) = S(x)o\theta(y) = \theta(x)oS(y)$ for all x, y $\in$ R

By Lemma 2.2, we have T(c) $\in$ Z(R).$\square$

Theorem 2.4.

Let R be a 2-torsion free ring and T: R$\rightarrow$R an additive mapping which satisfies $T(xoy) = T(x)o\theta(y) = \theta(x)oT(y)$ for all x, y $\in$ R. Then T is a θ-centralizer of R, if one of the following statements hold :-

(i)   R is a semiprime ring has a commutator which is not a zero divisor .

(ii)   R is a non commutative prime ring .

(iii)   R is a commutative semiprime ring .

Where θ be surjective endomorphism of R , and $\theta(Z(R)) = Z(R)$

Proof :

$T(xy + yx) = T(x)\theta(y) + \theta(y)T(x) = \theta(x)T(y) + T(y)\theta(x)$ for all $x,y \in R$

If we replace y by $xy + yx$, we get

$T(x)\theta(xy + yx) + \theta(xy + yx)T(x) = T(xy + yx)\theta(x) + \theta(x)T(xy + yx) = (T(x)\theta(y) + \theta(y)T(x))\theta(x) + \theta(x)(T(x)\theta(y) + \theta(y)T(x))$ for all $x,y \in R$

Now it follows that $[T(x),\theta(x)]\theta(y) = \theta(y)[T(x),\theta(x)]$ holds for all $x,y \in R$, but θ is surjective, then we get $[T(x), \theta(x)] \in Z(R)$

The next goal is to show that $[T(x),\theta(x)] = 0$ holds. Take any $c \in Z(R)$.

$2T(cx) = T(cx + xc) = T(c)\theta(x) + \theta(x)T(c) = 2T(x)\,\theta(c)$, for all $x \in R$

Using Lemma 2.3, we get

$T(cx) = T(x)\theta(c) = T(c)\theta(x)$ for all $x \in R$,

$[T(x),\theta(x)]\theta(c) = T(x)\theta(x)\theta(c) - \theta(x)T(x)\theta(c) = T(c)\theta(x2) - \theta(x)T(c)\theta(x) = 0$

Since R is semiprime, $\theta(Z(R)) = Z(R)$, and $[T(x),\theta(x)]$ itself is central element , our goal is achieved.

$2T(x^2) = T(xx + xx) = (x)\theta(x) + \theta(x)T(x)$

$2T(x)\theta(x) = 2\theta(x)T(x)$ for all $x \in R$

Theorem 1.3 now concludes the proof.□

Corollary 2.5.

Let R be a 2-torsion free prime ring and T: R→R an additive mapping which satisfies $T(x o y) = T(x) o \theta(y) = \theta(x) o T(y)$ for all $x, y \in R$. Then T is a θ-centralizer of R, where θ be surjective endomorphism of R , and $\theta(Z(R)) = Z(R)$ or $\theta(Z(R)) \neq 0$.

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تمركزات جوردنθ في الحلقات الأولية وشبه الأولية

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الخلاصة:

الهدف من البحث هو برهان النتيجـة الآتيـة : لتكون R حلقة طليقة الالتواء من الدرجـة الثانيـة و $T:$ $R{\rightarrow}R$ دالـة جمعيـة بحيث إن T تكون تمركـز جـوردن θ يسـاري (يمينـي) علـى R ، فـإن T تكون تمركز θ يساري (يميني) على R إذا تحقق أحد الشـروط الآتيـة:- (i) R تكون حلقة شبه أولية تحتوي على مبادل غير قاسم للعناصر غير الصفري . (ii) R تكون حلقة أولية غير أبدالية . (iii) R تكون حلقة شبه أولية أبدالية . وإن θ دالة تشاكلية شاملة على R. وأيضا نبرهن إذا كان $T(xoy){=}T(x)o\theta(y){=}\theta(x)oT(y)$ لكل x,y في R فإن T تكون تمركز θ تحت الشروط نفسها أعلاه و $\theta(Z(R)){=}Z(R)$.