On Semi-p-Compact Space[1]

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Abstract:

The purpose of this paper is to introduce a new type of compact spaces, namely semi-p-compact spaces which are stronger than compact spaces; we give properties and characterizations of semi-p-compact spaces.

Key words: semi-p-open set, pre-open set and compact space.

Introduction:

Let (X,τ) be a topological space and let A be a subset of X. We denote the closure of A (the interior of A) by cl A (int A) respectively.

A subset A of (X,τ) is called pre-open set, see [1], [2] and [3], if $A \subseteq \text{int}(\text{cl } A)$. The complement of a pre-open set is called a pre-closed set; see [1], [2] and [3]. The intersection of all pre-closed sets containing A is called the pre-closure of A and is denoted by pre-clA, [2].

A subset A of (X,τ) is called semi-p-open, [1] if there exists a pre-open subset U of X such that $U \subseteq A \subseteq$ pre-clU. The complement of semi-p-open set is called semi p-closed set, see [3].

The family of all semi-p-open subsets of X is denoted by S-P-O(X). The intersection of all semi-p-closed sets containing A is called the semi-p-closure of A and is denoted by semi-p-cl A, see [1,3].

We study and define many concepts in this paper in order to give properties and characterizations of semi-p-compact spaces, like cluster and semi-p-cluster points, compact spaces, nets, filters, T_2 and semi-p-T_2 spaces, regular [redacted] spaces, almost [redacted] and semi-p-irrsolute functions. For more details of these concepts see [4], [2], [5], [6], [7] and [8].

Semi-p-Compact Spaces:

In this section, we define and study the concept of semi-p-compactness.

1 Definition

A family $\tilde{A}$ of semi-p-open subsets of a topological space (X,τ) which covers X is called semi-p-open cover of X.

2 Definition

A topological space (X,τ) is said to be semi-p-compact space if and only if every semi-p-open cover of X has a finite semi-p-open subcover.

Notice that every semi-p-compact space is compact, since every open subset of X is semi-p-open, but the converse is not true in general as the following example shows:

3 Example

Let $X = \mathbb{N} \cup \{0\}$

$\tau = \{U \subseteq X \mid U \subseteq \mathbb{N} \text{ or } (0 \in U \wedge U^c \text{ is finite})\}$

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$\mathcal{F}$ = {F ⊆ X | 0 ∈ F or (F ⊆ ℕ ∧ F is finite)}
S-P-O(X) = ℙ(X)\ {{0}}
Then (X,τ) is a compact space but not semi-p-compact space.

Semi-p-compactness is a weak hereditary property, as shown in the following proposition.

4 Proposition

A semi-p-closed subset of a semi-p-compact space is a semi-p-compact subspace.

Proof:

Let A be a semi-p-closed subset of a semi-p-compact space (X,τ) and let {G_α: G_α is semi-p-open subset of X, α∈∧} be a semi-p-open cover of A. Since A^c is a semi-p-open set in X, so {G_α:α∈∧} ∪ {A^c} forms a semi-p-open cover of X which is a semi-p-compact space, then there exist finitely many members of index set ∧ say α_1, α_2, …, α_n such that $X = \bigcup_{i=1}^{n} G_{\alpha_i} \cup A^c$. But A ⊆ X and A ∩ A^c = ϕ, therefore $A \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$. Thus A is semi-p-compact.

In the following theorem we give a characterization of definition of semi-p-closure of a set.

5 Definition

Let A be a subset of a topological space (X,τ). The semi-p-closure of A (semi-p-cl A) is the intersection of all semi-p-closed subsets of X which contain A.
We shall call *x*, where *x* ∈ X, a semi-p-closure point of A if *x* ∈ semi-p-cl A.

6 Theorem

Let (X,τ) be a topological space and let A be a subset of X. A point *x* in X is a semi-p-closure point of A if and only if every semi-p-open nbd (neighborhood) of *x* intersects A.

Proof:

The "only if"part

Assume that *x* is a semi-p-closure point of A, then *x* ∈ K = ∩ {F | F is a semi-p-closed subset of X containing A}. Suppose that there exists a semi-p-open nbd U of *x* such that U∩A=ϕ, therefore A ⊆ U^c where U^c is a semi-p-closed subset of X with *x* ∉ U^c, that is, *x* ∉ K which is a contradiction. Hence every semi-p-open nbd of *x* must intersects A.

The "if" part

Assume that every semi-p-open nbd of *x* intersects A, and suppose that X is not a semi-p-closure point of A, therefore *x* ∉ K, that is there exists a semi-p-closed subset F of X with A ⊆ F such that *x* ∉ F, it follows that *x* ∈ F^c which is a semi-p-open set in X and A ∩ F^c = ϕ. That implies a contradiction with our assumption. Hence *x* must be a semi-p-closure point of A. ■

7 Definition

Let (X,τ) be a topological space and let (*f*, X,A,≥) be a net in X. A point x_0 in X is called a "semi-p-cluster point of *f* " if for each *a* ∈ A and for each semi-p-open nbd U of x_0 there exists *b* ∈ A such that *b* ≥ *a* and *f* (*b*) ∈ U.

8 Definition

Let (X,τ) be a topological space and let (*f*, X,A,≥) be a net in X, then *f* is said to be "semi-p-convergent" to a point x_0 in X if for each semi-p-open nbd. N of x_0 there exists an element a_0∈ A such that f_a ∈ N for each $a \geq a_0$.

9 Theorem

Let (X,τ) be a topological space and let (*f*, X,A,≥) be a net in X. For each *a*∈A, let M_a = { *f* (*x*) : *x* ≥ *a* in A} then a point p of X is a semi-p-cluster point of f if and only if $p \in$ semi-p-cl M_a for each $a \in A$.

Proof:

The "only if"part

Assume that p is a semi-p-cluster point of f and let N be a semi-p-open nbd. of p, then for each $a \in A$, there exists an element $x \geq a$ in A such that $f(x) \in N$.

Hence $M_a \cap N \neq \phi$ for each $a \in A$. Since N is an arbitrary nbd., so by theorem 2.6 $p \in$ semi-p- cl M_a for each $a \in A$.

The "if" part

Assume that $p \in$ semi-p-cl M_a for each $a \in A$ and suppose, if possible, p is not a semi-p-cluster point of f, then there exists a semi-p-open nbd. N of p and an element $a \in A$ such that $f(x) \notin N$ for every $x \geq a$ in A. This implies that $N \cap M_a = \phi$, it follows that $p \notin$ semi-p-cl M_a for this a which is a contradiction. Hence p must be a semi-p-cluster point of the net f. ■

10 Definition

Let (X,τ) be a topological space and let $\mathcal{F}$ be a filter on X. A point x in X is called a "semi-p-cluster point of $\mathcal{F}$ " if each semi-p-open nbd. of x intersects every member of $\mathcal{F}$.

Notice that, every semi-p-cluster point of a filter is a cluster point.

11 Theorem

Let (X,τ) be a topological space and let $\mathcal{F}$ be a filter on X. A point p in X is a semi-p-cluster point of $\mathcal{F}$ if and only if $p \in$ semi-p-cl F for each $F \in \mathcal{F}$.

Proof:

The "only if"part

Let p be a semi-p-cluster point of $\mathcal{F}$, then each semi-p-open nbd. of p intersects every member of $\mathcal{F}$, that is, for each semi-p-open nbd. U of p, $U \cap F \neq \phi$ for each $F \in \mathcal{F}$. It follows that, $p \in$ semi-p-cl F for each $F \in \mathcal{F}$, by theorem 6.

The "if" part

Assume that $p \in$ semi-p-cl F for each $F \in \mathcal{F}$, then by theorem 6 every semi-p-open nbd. of p intersects F for each $F \in \mathcal{F}$, that is every semi-p-open nbd. of p intersects every member of $\mathcal{F}$. Hence p is a semi-p-cluster point of $\mathcal{F}$. ■

In the next theorem we give two characterizations of semi-p-compact spaces.

12 Theorem

Let (X,τ) be a topological space then the following statements are equivalent:

1. X is a semi-p-compact space,
2. Every collection of semi-p-closed subsets of X with the FIP (finite intersection property) has a non-empty intersection,
3. Every filter on X has a semi-p-cluster point.

Proof:

$(1 \Rightarrow 2)$ Assume that X is a semi-p-compact space and let $\{F_\alpha : \alpha \in \wedge\}$ be a collection of semi-p-closed subsets of X with FIP. Suppose that $\bigcap_{\alpha \in \wedge} F_\alpha = \phi$, then by De-Morgan Laws $X = \bigcup_{\alpha \in \wedge} F_\alpha^c$ where F_α^c is a semi-p-open set for each $\alpha \in \wedge$. Therefore $\{F_\alpha^c : \alpha \in \wedge\}$ is a semi-p-open cover of X which is a semi-p-compact space, then there exist finitely many members $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $X = \bigcup_{i=1}^{n} F_{\alpha_i}^c$, it follows by De-Morgan Laws that $\bigcap_{i=1}^{n} F_{\alpha_i} = \phi$ which is a contradiction with our assumption that $\{F_\alpha : \alpha \in \wedge\}$ has a FIP. Hence $\bigcap_{\alpha \in \wedge} F_\alpha \neq \phi$.

$(2 \Rightarrow 3)$

Let $\mathcal{F}$ be a filter on X, then $\mathcal{F}$ has a FIP. In particular the collection {semi-p-cl F:F∈$\mathcal{F}$} of semi-p-closed subset of X has the FIP, so by 2 there exists at least one point x∈∩{semi-p-cl F: F∈$\mathcal{F}$}, that is, x∈semi-p-cl F for each F∈$\mathcal{F}$. Hence by theorem 11 x is a semi-p-cluster point of $\mathcal{F}$.

(3 ⇒ 1)

Assume that every filter on X has a semi-p-cluster point. To prove X is a semi-p-compact space. Let $\Im$ be a semi-p-open cover of X and suppose, if possible, $\Im$ has no finite subcover. The collection $\wp$ = {X – G : G∈$\Im$} has the FIP. For if there exists a finite subcollection {X – G_i | 1 ≤ i ≤ n} of $\wp$ such that ∩ {X – G_i | 1 ≤ i ≤ n} = ϕ. This implies that ∪ {G_i | 1 ≤ i ≤ n} = X which contradicts our supposition that $\Im$ has no finite subcover. Thus $\wp$ must have the FIP. It follows that there exists an ultra filter $\mathcal{F}$ on X containing $\wp$. By 3 $\mathcal{F}$ has a semi-p-cluster point x ∈ X, then by theorem 11 x ∈ semi-p-cl F for each F ∈ $\mathcal{F}$. In particular x ∈ semi-p-cl (X – G) for each G ∈$\Im$. But X – G is a semi-p-closed subset of X for each G ∈ $\Im$, then semi-p- cl (X – G) = X – G for each G ∈ $\Im$. This implies x ∈ ∩ { X – G: G ∈ $\Im$} = X – ∪ {G| G ∈ $\Im$}. Hence x ∉ ∪ {G| G ∈ $\Im$} which contradicts the fact that $\Im$ is a semi-p-open cover of X. Thus $\Im$ must have a finite subcover and consequently X is semi-p-compact space. ■

13 Proposition

Let (X,τ) be a topological space. If X is a semi-p-compact space then every net in X has a semi-p-cluster point.

Proof:

Let (f, X,A,≥) be a net in X. For each a ∈ A, let M_a={f (x):x ≥ a} since A is directed by ≥, so the collection { M_a: a ∈ A} has the FIP, in particular the collection {semi-p-cl M_a: a ∈ A} of semi-p-closed subsets of X is also has the FIP. It follows by theorem 12 that ∩ {semi-p-cl M_a: a∈A} ≠ϕ, let p∈∩{semi-p-cl M_a: a ∈ A}, then p∈semi-p-cl M_a for each a ∈ A, thus by theorem 9 p is a semi-p-cluster point of f. ■

It seems that the converse of proposition 13 is not true in general, but we could not get a counter example.

14 Definition [3]

Let f :(X,τ) ⟶ (Y,τ′) be any function, then f is said to be "semi-p-irresolute function" if the inverse image of any semi-p-open subset of Y is a semi-p-open subset of X.

15 Proposition

The semi-p-irresolute image of a semi-p-compact space is a semi-p-compact.

Proof:

Let f be a semi-p-irresolute function from a semi-p-compact space (X,τ) onto a topological space (Y,τ′). To prove Y is a semi-p-compact space let {G_α:α∈∧} be a semi-p-open cover of Y, then {$f^{-1}(G_\alpha)$:α∈∧} is a semi-p-open cover of X which is semi-p-compact space, then there exist finitely many members of ∧ say $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that X= $\bigcup_{i=1}^{n} f^{-1}(G_{\alpha_i})$, it follows that Y = $\bigcup_{i=1}^{n} G_{\alpha_i}$. Thus Y is a semi-p-compact space. ■

16 Corollaries

1. The semi-p-irresolute image of a semi-p-compact space is a compact space.

2. Semi-p-compactness is a topological property.

17 Definition

A topological space (X,τ) is said to be "semi-p-T_2-space" if for each two distant points x and y in X, there exists two semi-p-open subsets U and V of X, such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

18 Proposition

A semi-p-compact subset of a T_2-space is semi-p-closed.

Proof:

Let A be a semi-p-compact subset of the T_2-space (X,τ), so A is compact since every semi-p-compact is compact, but X is a T_2-space (given) so A is closed in X [5,p.156,prop.11] but every closed subset of A is semi-p-closed, so A is semi-p-closed. ■

Notice that, a semi-p-compact subset of semi-p-T_2-space need not be semi-p-closed as the following example shows:

19 Example

Let $X=\{1,2,3\}$, $\tau = \{X,\phi,\{2,3\}\}$, $F = \{x,\phi,\{1\}\}$.

$S\text{-}P\text{-}O(X) = \{X,\phi,\{2,3\},\{2\},\{3\},\{1,3\},\{1,2\}\}$

$S\text{-}P\text{-}C(X) = \{X,\phi,\{1\},\{1,3\},\{1,2\},\{2\},\{3\}\}$

Clear that X is semi-p-T_2 space. If $A = \{2,3\}$ then A is semi-p-compact subset of X, but not semi-p-closed.

20 Definition [3]

A topological space (X,τ) is said to be:

1. 'semi-p-regular space" if and only if for each point $x \in X$ and for each closed subset F of X such that $x \notin F$, there exist two disjoint semi-p-open subsets U and V of X such that $x \in U$ and $F \subseteq V$.

2. "Almost semi-p-regular space" if and only if for each point x in X and for each semi-p-closed subset F of X such that $x \notin F$, there exist two semi-p-open disjoint subsets U and V of X such that $x \in U$ and $F \subseteq V$.

3. "semi-p-normal space" if and only if for each two disjoint closed subsets F_1 and F_2 of X, there exist two disjoint semi-p-open subsets U and V of X such that $F_1 \subseteq U$ and $F_2 \subseteq V$.

Notice that, every regular space is a semi-p-regular and every normal space is a semi-p-normal.

21 Proposition

A compact T_2 – space is a semi-p-regular space.

Proof:

Clear.

22 Corollary

A semi-p-compact T_2-space is a semi-p-regular.

Proof:

Clear.

23 Proposition

A semi-p-compact T_2-space is an almost semi-p-regular space.

Proof:

Let (X,τ) be a semi-p-compact T_2-space and let F be a semi-p-closed subset of X and x be any point in X with $x \notin F$, then $x \neq y$ for each $y \in F$. Since X is a T_2-space, so there exist two disjoint open subsets U_y and V_y of X such that $x \in U_y$ and $y \in V_y$. Then the family $\{ V_y : y \in F\}$ forms an open cover of F, but it is compact set, since every semi-p-compact set is compact and F is semi-p-compact by proposition 4 therefore, we get finitly many elements $y_1, \dots, y_n$ of F such that $F \subseteq \bigcup_{i=1}^{n} V_{y_i}$. Now, let $V = \bigcup_{i=1}^{n} V_{y_i}$ and $U = \bigcap_{i=1}^{n} U_{y_i}$, then U and V are two disjoint open subset of X such that $x \in U$ and $F \subseteq V$. But every open set is semi-p-open, so X is an almost semi-p-regular space.

24 Proposition

A compact T_2 – space is a semi-p-normal space.

Proof:

Clear.

25 Corollary

A semi-p-compact T_2-space is a semi-p-normal (normal) space.

Proof:

Clear.

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فضاءات الرص شبه - p

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الخلاصة:

الغرض من هذا البحث تقديم نوع جديد من فضاءات الرص وهو فضاء الرص شبه-p وهو اقوى من فضاءات الرص، وكذلك اعطينا خواصاً ومميزات لفضاء الرص شبه – p.