

## Stability of Nonlinear Systems of Fractional Order Differential Equations

*Alaaldeen N. Ahmed\**

*Rasheed A. Abdul-Satar\**

Received 3, April, 2009  
Accepted 12, October, 2009

### Abstract:

In this paper, a sufficient condition for stability of a system of nonlinear multi-fractional order differential equations on a finite time interval with an illustrative example, has been presented to demonstrate our result. Also, an idea to extend our result on such system on an infinite time interval is suggested.

**Key words:** Fractional Calculus, Nonlinear Differential Equations, Stability.

### Introduction:

Fractional derivatives have a long mathematical history, for many years they were not used in many different sciences, but in recent years, growing attention has been focused on the importance of fractional derivatives and integrals in science. For more details, see [1].

Problems of stability appeared for the first time in mechanics during the investigation of an equilibrium state of a system. A simple reflection may show that some equilibrium state of a system are stable with respect to small perturbations. The existing methods developed so far for stability check are mainly for integer order systems. However, for the fractional order systems, it is difficult to evaluate the stability by simple examining its characteristic equation either by finding its dominate roots or by using other algebraic methods. Direct check of the stability of fractional order system using polynomial criteria is not possible, because the characteristic equation of the system is, in general, not a polynomial but a pseudo polynomial function of fractional

powers of the complex variables. For more details, see [2].

The study of stability of such systems focuses a great interest. We can cite in this domain, the works in [3] and [2] for the stability of linear fractional systems, while the works in [4], and [5] are for the stability of fractional systems with time delay.

### Preliminaries

In this paper, we are concerned with the stability of nonlinear fractional order system of differential equations of the form

$$(1) \quad y_i^{\alpha_i}(t) = f_i(t, Y(t)) \quad i=1, \dots, m.$$

where  $Y(t) = (y_1(t), \dots, y_m(t))^T$  is its solution,  $0 < \alpha_i < 1$ , and  $f_i \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^{n+})$  are continuous positive functions defined on a finite interval.

There are several definitions of a fractional derivative of order  $\alpha_i > 0$  (see [1]). The two most commonly used definitions are the *Riemann-Liouville* and *Caputo*. Each definition uses *Riemann-Liouville* fractional integration and derivatives of whole order. In this paper, we are using

---

\*AL- Nahrain University, College of Science.

Riemann-Liouville fractional integration of order  $\alpha_i$  which is defined as:

$$y_i(t) = I^{\alpha_i} f_i(t, Y(t)) = \frac{y_0(t-t_0)^{\alpha_i-1}}{\Gamma(\alpha_i)} + \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} f(s, Y(s)) ds,$$

$$(2) \dots = \frac{t^{\alpha_i-1}}{\Gamma(\alpha_i)} \otimes f(t, Y(t)),$$

$\alpha_i > 0, t > 0.$

where  $\otimes$  is a convolution operator, in which

$$(3) \dots f \otimes g(t) = \int_0^t f(t-s)g(s) ds$$

Set  $I=[0, t_f]$ , say,  $t_f$  is a finite suitable positive number, and define the norm

$$(4) \dots \|Y(t)\| = \sum_{i=1}^m \|y_i(t)\|, \text{ where}$$

$$(5) \dots \|y_i(t)\| = \int_0^{t_f} |y_i(t)| dt$$

In order to study the  $L_1$  - stability solution of the system (1), we must prove that each  $y_i(t)$ , ( $i=1, \dots, m$ ) is bounded, i.e.  $\|y_i(t)\| < \infty$ , for all  $i=1, \dots, m$ .

Therefore, the solution  $y_i(t)$  may be rewritten using the product operator  $\otimes$  as follows

$$(6) \dots y_i(t) = (k_{\alpha_i} \otimes f_i)(t)$$

Where  $k_{\alpha_i}$  with  $0 < \alpha_i \leq 1$  is so called convolution kernel defined by

$$(7) \dots k_{\alpha_i} = \frac{t^{\alpha_i-1}}{\Gamma(\alpha_i)}, \text{ for all } i=1, \dots, m.$$

Now, we can state and prove the following theorem:

**Theorem:**

Let

$$k_{\alpha_i}, f_i \in C(\mathbb{R}^+, \mathbb{R}^+) \cap L_1([0, t_f]), \text{ with}$$

$t_f > 0$  and finite, then the solution of

(1) is stable, providing that

$$(8) \dots \left| \frac{t^{\alpha_i}}{\alpha_i \Gamma(\alpha_i)} \right| < \epsilon \text{ for all } i = 1, \dots, m. \text{ where}$$

$\epsilon > 0$ , and finite.

**Proof:**

With no loss of generality we can take

$y_0 = 0$  and  $t_0 = 0$ . This reduces (2) into

$$y_i(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i-1} f(s, Y(s)) ds,$$

First, we prove that  $\|Y(t)\| < \infty$ . We have

$$\|Y(t)\| = \sum_{i=1}^m \|y_i(t)\| = \sum_{i=1}^m \left( \int_0^{t_f} (k_{\alpha_i} \otimes f_i)(t) dt \right)$$

So, we need only to prove that  $\|y_i(t)\| < \infty$ .

$$\|y_i(t)\| = \int_0^{t_f} f_{\alpha_i} \otimes f_i(t, Y(t)) dt$$

$$= \int_0^{t_f} \int_0^t k_{\alpha_i}(t-s) f_i(s, Y(s)) ds dt \text{ (by Fubini Theorem for positive functions)}$$

$$= \int_0^{t_f} \int_0^{t_f} k_{\alpha_i}(t-s) f_i(s, Y(s)) ds dt$$

$$= \int_0^{t_f} f_i(s, Y(s)) \left( \int_0^{t_f-s} k_{\alpha_i}(t-s) d(t-s) \right) ds$$

(by using the change of variables)

$$= \int_0^{t_f} f_i(s, Y(s)) \left( \int_0^{t_f-s} k_{\alpha_i}(t) dt \right) ds$$

Since all  $k_{\alpha_i}$  is positive, we have

$$\int_0^{t_f-s} k_{\alpha_i}(t) dt \leq \int_0^{t_f} k_{\alpha_i}(t) dt$$

and

$$\|y_i(t)\| \leq \left( \int_0^{t_f} k_{\alpha_i}(t) dt \right) \left( \int_0^{t_f} f_i(s, Y(s)) ds \right)$$

Now, since  $t_f$  is finite, then the right side of the inequality is converge, and we get

$$(9) \dots \|y_i(t)\| \leq \left( \frac{(t_f)^{\alpha_i}}{\alpha_i \Gamma(\alpha_i)} \right) \left( \int_0^{t_f} f_i(s, Y(s)) ds \right) < \infty$$

Provided that, the condition (8) is satisfied.

Second, if we consider two solutions  $Y(t)$  and  $X(t)$  with different initial points  $Y_0$  and  $X_0$  with  $|Y_0 - X_0| < \delta(t_f)$ .

Perform the same above steps for  $\|Y(t) - X(t)\|$ , we get

$$\|Y(t) - X(t)\| = \sum_{i=1}^m \|y_i(t) - x_i(t)\| = \sum_{i=1}^m \int_0^{t_f} |y_i(t) - x_i(t)| dt$$

$$\begin{aligned}
 &= \sum_{i=1}^m \left| \left( \int_0^{t_f} \frac{(y_0-x_0)}{\Gamma_{\alpha_i}} t^{\alpha_i-1} + \left[ \left( \int_0^{t_f} k_{\alpha_i} \otimes (f_i(s, Y(t)) - f_i(s, X(t))) dt \right) \right] \right) \right| \\
 (10) \quad &\leq \sum_{i=1}^m \left| \left( \int_0^{t_f} \frac{(y_0-x_0)}{\Gamma_{\alpha_i}} t^{\alpha_i-1} + \left[ \left( \int_0^{t_f} |k_{\alpha_i} \otimes (f_i(s, Y(t)) - f_i(s, X(t))) dt \right) \right] \right) \right|
 \end{aligned}$$

As we have seen from (9), the second term of (10) is converge and bounded as  $(\alpha_i - 1) < 0$ . Also the first term of (10) is converge and bounded, providing that condition (8) is satisfied. Therefore we get

$$\int_0^{t_f} \| Y(t) - X(t) \| < \infty \blacksquare$$

Now, to demonstrate the stability of such system, we consider the following example of nonlinear system of fractional differential equations:

$$\begin{aligned}
 \frac{d^{\alpha_1} x}{dt^{\alpha_1}} &= -x \\
 \frac{d^{\alpha_2} y}{dt^{\alpha_2}} &= x - y^2 \\
 \frac{d^{\alpha_3} z}{dt^{\alpha_3}} &= y^2
 \end{aligned}$$

subject to the initial condition  $x(0) = 1, y(0) = z(0) = 0$ ,

Where  $0 < \alpha_i \leq 1, i = 1, 2, 3$ . and defined over the finite time interval. Since all the corresponding  $\int_0^T f_i(t) dt < \infty$ , then  $\| x(t) \| < \infty, \| y(t) \| < \infty, \text{ and } \| z(t) \| < \infty$ , therefore the system is stable. For special case when  $\alpha=1$ , this system represents a nonlinear reaction and was found in [6]. To verify the stability of this system, we are solving this system, using the appropriate successive approximation to generate the recursive relationship. The first four terms are given by

$$\begin{aligned}
 x &= 1 - \frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} - \frac{t^{3\alpha_1}}{\Gamma(3\alpha_1+1)} \\
 y &= \frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{t^{2\alpha_2}}{\Gamma(2\alpha_2+1)} + \left( 1 - \frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2+1)} \right) \frac{t^{3\alpha_2}}{\Gamma(3\alpha_2+1)} \\
 z &= \frac{\Gamma(2\alpha_3+1)}{\Gamma(\alpha_3+1)} \frac{t^{3\alpha_3}}{\Gamma(3\alpha_3+1)}
 \end{aligned}$$

Setting  $\alpha_i = 1, (i = 1, 2, 3)$ , we obtain the solution obtained by [10], which corresponding to a system of ordinary

differential equations. One can see that, the above solution is stable when “ $T$ ” is finite.

The generalization to the infinite case where  $t \in R^+$  is not possible because the kernel  $k_{\alpha_i}(t)$  does not belong to  $L_1(R^+)$ , and then  $\int_0^{\infty} k_{\alpha_i} \otimes f_i(t, Y(t)) dt$  does not converge, and  $\int_0^{\infty} k_{\alpha_i} \otimes f_i(t, Y(t)) dt \leq \int_0^{\infty} f_i(s, Y(s)) \left( \int_0^{\infty} k_{\alpha_i}(t) dt \right) ds$  does not true. Also, even if the system is defined since  $t_0 > 0$ , the generalization to the infinite case where  $t \in [t_0, \infty)$  is not possible, since the solution of the system (1) given by (2) cannot be defined by using a convolution product which is commutative. However, our kernel function given by (7) does not depend on the initial time. Therefore, a new approach for the stability analysis of a class of nonlinear fractional order systems is opened.

**References:**

1. Kilbas A., H. Sriastava and Trujillo J. 2006. Theory and Applications of Fractional Differential Equations. Amsterdam, Netherlands : Elsevier.
2. Kilbas A., H. Sriastava and Trujillo 1996. Stability Properties for Generalized Fractional Differential System, Proceeding of Fractional Differential Liverpool 2-4, September, 1:101-119.
3. Matignon D. 1996. Stability Result on Fractional Differential Equations with Applications to Control Processing, In IMACS\_SMC proceeding, July France, pp. 963-968.
4. Khusainov T. D. 2001. Stability Analysis of a Linear Fractional Delay System. Diff. Equs., 37(8): 1184-1188.

5. Lazarevic M. P. 2006. Finite time stability analysis of  $PD^\alpha$  fractional control of robotic time –delay systems. Mechanics Res. Comm. .33(1): 269-279.
6. Hull T.E. et. al. 1972. Comparing numerical methods for ordinary differential equations. SIAM J. Numer. Anal. 9.
7. Diethelm K., N.J. Ford 2002. Numerical solution of the Bagley\_Torvik equation. BIT 42(3): 490-507.
8. Diethelm K., Y. Luchko 2006. Numerical solution of linear multi\_term differential equations of fractional order. J. Comput. Anal Appl.10(2):242-248.
9. Shawagfeh N.T. 2002. Analytical approximation solutions for nonlinear fractional differential equations. Appl. Math. Comput. 13(1): 517-529.
10. Kaya D. 2006. A reliable method for the numerical solution of the Kinetics problems. Appl. Math. Comput.15(1):62-69.
11. EL-Sayed A. and et. al. 2004. Numerical solution for multi-term fractional (arbitrary) orders differential equations. Computational and Applied Mathematics, 23( 1): 33-54.
12. Momani S.M. 1999. On the existence of solutions of a system of ordinary differential equations of fractional order. Far East J. Math. Sci. 1(2): 265-270.

## استقرارية الانظمة الغير خطية للمعادلات التفاضلية ذات الرتب الكسرية

رشيد عبدالمجيد عبدالستار\*

علاء الدين نوري احمد\*

\*جامعة النهريين/ كلية العلوم

### الخلاصة:

في هذا البحث تم تقديم مبرهنة بينت فيها الشرط الكافي لاستقرارية نظام غير خطي من المعادلات التفاضلية ذات الرتبة الكسرية على مدة زمنية منتهية وقد تم تقديم مثال توضيحي وبعض الافكار لتوسيع النتائج المستخلصة في هذا البحث حول مدة زمنية غير منتهية.