Quasi-posinormal operators

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Abstract:

In this paper, we introduce a class of operators on a Hilbert space namely quasi-posinormal operators that contain properly the classes of normal operator, hyponormal operators, M–hyponormal operators, dominant operators and posinormal operators . We study some basic properties of these operators .Also we are looking at the relationship between invertibility operator and quasi-posinormal operator .

Key words: posinormal operators , Hyponormal operators ,M- hyponormal operators, dominant operators.

Introduction:

Let B(H) denote the set of all bounded linear operators on a Hilbert space H., an operator T is said to be posinormal operator if there exists a positive operator P $\in$B(H), such that $TT^* = T^*PT$ .Also, T is posinormal operator if and only if $Range(T) \subseteq Range(T^*)$, [1,2].An operator T is called hyponormal operator if $T^*T - TT^* \geq 0$,or equivalently $\|T^*x\| \leq \|Tx\|$ for all x in H [3] ,and T is called dominant operator if for each $\lambda \in$ ¢ there exists a number $M_\lambda > 0$ such that $\|(T-\lambda)^* x\| \leq M_\lambda \|(T-\lambda)x\|$ for all x$\in$ H. Furthermore , if the set of constants M_λ are bounded by a positive number M then T is called M-hyponormal operator [4,5,6,p480]. Let $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$ and $r(T) = \sup\{|\lambda|, \lambda \in \sigma(T)\}$ denote the spectrum, the point spectrum , the approximate point spectrum of T and the spectral radius of T, [6,p196,502] . An operator is said to be normaloid if $\|T\| = r(T)$,[7,8 ,p267].In this paper , we give some types of operators namely quasi-posinormal operators .

1- Some basic properties of quasi- posinormal operator .

We start this section by giving the definition of quasi-posinormal operator ,and we give some basic properties of these operators

Definition 1.1

Let T$\in$B(H). We call T is a quasi-posinormal operator if $Range(T^2) \subseteq Range(T^*)$.

Example 1.2

Let H= ℓ_2(¢) ={x:x=(x_1,x_2,x_3,…,x_n,..): $\sum_{i=1}^{\infty} |x_i|^2 < \infty$} , the Unilateral shift operator on H is defined by U(x_1, x_2, x_3,) =(0,x_1, x_2, x_3,…).It is known that U^*(x_1, x_2, x_3,…) =(x_2,x_3, x_4,……) and

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U^2 ($x_1, x_2, x_3,\ldots$) =U($0,x_1, x_2, x_3,\ldots$) =($0,0,x_1, x_2, x_3,\ldots$).
Now let y∈Range(U^2) then y=($0,0,x_1, x_2, x_3,\ldots$) for some x in H .If we assume x=($0,0,0,x_1, x_2, x_3,\ldots$) then U^*(x)= ($0,0,x_1, x_2, x_3,\ldots$) = y, and y ∈Range(U^*) . Hence U is quasi-posinormal operator .

Now we give an operator that is not quasi-posinormal operator.

Example 1.3

Let H= $\ell_2(¢)$ ={x:x=($x_1,x_2,x_3..,x_n$..): $\sum_{i=1}^{\infty}|x_i|^2 < \infty$} , the Bilateral shift operator on H is defined by B($x_1, x_2, x_3,\ldots$) =($x_2,x_3, x_4,\ldots$). It is known that B^*($x_1, x_2, x_3,\ldots$) =($0,x_1, x_2, x_3,\ldots$). Now let y= (1,0,0,0,…,0,…) then y∈Range(B^2) and $B^*(x) \neq y$ for all x in H .Hence y∉ $Range(B^*)$ and therefore B is not quasi-posinormal operator.

The above example also shows that if T is quasi-posinormal operator then T^* is not quasi -posinormal operator .

In [9]. Douglas proved the following theorem

Theorem 1.4 [9]

For A,B ∈ B(H) the following statements are equivalent :

1-Range (A)⊆ Range (B)

2-$AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$

3-there exists a T∈ B(H) such that A=BT.

Moreover if one of 1, 2, and 3 holds then there is a unique operator T such that

a-$\|T\|^2$ = inf { μ: $\mu \geq 0$ and $AA^* \leq \mu BB^*$}

b-KerA = KerT; and

c-Rnage(T) ⊆ $\overline{Range(B^*)}$.

If we put A= T^2 and B= T^* we get a special case from Douglas theorem Which gives a characterization of totally quasi-posinormal operator.

Theorem 1.5

Let T∈B(H), the following statement are equivalent ;

1- $Range(T^2) \subseteq Range(T^*)$, i.e. T is quasi-posinormal operator.

2- $T^2 T^{*2} \leq \lambda^2 T^* T$ for some $\lambda \geq 0$; and

3- there exists an operator C_T ∈B(H) , such that $T^2 = T^* C_T$

Moreover if 1,2,and 3 hold then there is a unique operator C_T ∈B(H) such that

a- $\|C_T\|^2 = \inf\{\mu, T^2 T^{*2} \leq \mu T^* T\}$.

b-KerT^2 =ker C_T ; and

c-$Range(C_T) \subseteq \overline{Range(T)}$.

Let [T]={AT : A∈B(H) } the left ideals in B(H) generated by T. We have the following corollary.

Corollary 1.6

T is quasi-posinormal operator if and only if $T^{*2} \in [T]$.

Proof :

Let T be a quasi-posinormal operator then $T^2 = T^*$C for some bounded operator C∈B(H) and $T^{*2} = C^* T$ implies $T^{*2} \in$[T].Conversely, if $T^{*2} \in$[T] then T^{*2} =KT for some K∈B(H), and hence $T^2 = T^* K^*$ so T is quasi-posinormal operator .

Proposition 1.7

Let T∈B(H) ,then T is quasi-posinormal operator if and only if for each x in H , there exists a constant M≥ 0 such that $\|T^{*2}x\| \leq M\|Tx\|$.

Proof :

Let T be a quasi-posinormal then

$$\|T^{*2}x\|^2 = \langle T^* T^* x, T^* T^* x\rangle = \langle T^2 T^{*2} x, x\rangle$$

$\leq M < T^{*}Tx, x >= M < Tx, Tx >= M \|Tx\|^{2}$

.for some M≥ 0.

Conversely ,let $\|T^{*2}x\| \leq M \|Tx\|$

$< T^{2}T^{*2}x, x >=< T^{*2}x, T^{*2}x >= \|T^{*2}x\|^{2}$

$\leq M^{2}\|Tx\|^{2} = M^{2} < Tx, Tx >= M^{2} < T^{*}Tx, x >$

,this implies for each x in H , then $T^{2}T^{*2} \leq M^{2}T^{*}T$, hence T is quasi-posinormal operator .

Proposition 1.8

Let T∈B(H),if T is posinormal operator then T is quasi-posinormal.

Proof:

Since

$Range(T^{2}) \subseteq Range(T) \subseteq Range(T^{*})$

then T is quasi-posinormal .

Corollary 1.9

Every Dominant operator in particular every M-hyponormal operator, hyponormal operator , normal operator are quasi-posinormal operators.

The converse of the above Proposition is not true, see the following example .

Example 1.10

Let

H=ℓ₂(¢)={x:x=(x₁,x₂,x₃,…x$_n$,..): $\sum_{i=l}^{\infty}|x_i|^{2} < \infty$} ,we define T by

T(x₁, x₂, x₃,…) =(x₂,0,0,0,..).

It is easy to check that

T^{*}(x₁, x₂, x₃,…) =(0,x₁,0,0,0,0…) but T^{2}(x₁, x₂, x₃,…) =T(x₂,0,0,0,……) =(0,0,0,0,0,0…)and $Range(T^{2}) \subseteq Range(T^{*})$,

hence T is quasi-posinormal operator.

Easily we see that $Range(T) \not\subset Range(T^{*})$.

Therefore T is not posinormal operator.

2- Invertibility, translates and quasi-posinormal operator

In this section we are looking at the relationship between invertibilty operators and quasi -posinormal operator .A quasi- posinormal operator need not be an invertible operator (see example 1.2) ,we start this section by the following theorem

Theorem 2.1

Let T∈B(H),be an invertible operator then

1- T is quasi-posinormal operator .
2- T^{-1} is quasi-posinormal operator.

Proof :

1- $\|T^{*2}x\| \leq \|T^{*2}\|\|x\| \leq \|T^{*2}\|\|T^{-1}\|\|Tx\|$

for all x in H ,we take M=$\|T^{*2}\|\|T^{-1}\|$,

hence T is quasi-posinormal operator

2-

$\|(T^{-1})^{*2}x\| \leq \|(T^{-1})^{*2}\|\|x\| \leq \|(T^{-1})^{*2}\|\|T\|\|T^{-1}x\|$

for all x in H, we take M=$\|(T^{-1})^{*2}\|\|T\|$,

hence T^{-1} is quasi-posinormal operator.

Corollary 2.2

Let T∈B(H), and $\lambda \notin \sigma(T)$ then T-λI is quasi-posinormal operator.

Before we state the next theorem we need the following lemma which appeared in [10].

Lemma 2.3

Let $\{a_n\}$ be a sequence of positive numbers , which satisfy the relation $a_1^2 \leq a_2$ and $a_n^2 \leq a_{n-1}a_{n+1}$ for n=2,3,… then $a_1^n \leq a_n$ for n=1,2,3,4,5,… .

Theorem 2.4

Let T be an invertible operator and $\|T^{-1}\| \leq 1$ then

1- $\|T^{2}x\|^{n+1} \leq M^{n(n+1)/2}\|T^{n+2}x\|$ for $\|x\| = 1$ and n=1,2,…., there exists a constant M>0 such that

2- if T^{n+1}x =0 then T^2x =0 for all x in H .

Proof :

1-Let k=n+1. We want to show that $\|T^2x\|^k \le M^{k(k-1)/2}\|T^{k+1}x\|$

Let $a_1 = \|T^2x\|$,and $a_k = M^{k(k-1)/2}\|T^{k+1}x\|$ k=2,3,…

Since

$\|T^2x\|^2 = <T^2x, T^2x> = <x, T^{*2}T^2x>$

$\le \|T^{*2}T^2x\|\|x\| \le M\|T^3x\|$ then $a_1^2 \le a_2$.

Now

$a_k^2 = M^{k(k-1)}\|T^{k+1}x\|^2 = M^{k(k-1)} <T^{k+1}x, T^{k+1}x>$

$= M^{k(k-1)} <T^{*2}T^{k+1}x, T^{k-1}x>$

$\le M^{k(k-1)}\|T^{*2}T^{k+1}x\|\|T^{k-1}x\|$

$\le M^{k(k-1)}M\|T^{k+2}x\|\|T^{k-1}x\|$

$\le M^{k^2-k+1}\|T^{-1}\|\|T^{k+2}x\|\|T^kx\|$

$\le a_{k+1}a_{k-1}$ then by Lemma 2.3

$a_1^k \le a_k$ and $\|T^2x\|^k \le M^{k(k-1)/2}\|T^{k+1}x\|$

2-

$$\|T^2x\|^n = \|x\|^n\left\|T^2\frac{x}{\|x\|}\right\|^n$$

$$\le \|x\|^n M^{n(n-1)/2}\left\|T^{n+1}\frac{x}{\|x\|}\right\| \le 0$$, hence

T^2x=0 for all x in H .

Theorem 2.5

Let T be a quasi-posinormal operator and then

1-λT is a quasi-posinormal operator for λ ∈¢

2- the translate T+λI need not be a quasi-posinormal operator

Proof :

1-

$\|(\lambda T)^{*2}x\| = |\lambda|^2\|T^{*2}x\| \le M|\lambda|^2\|Tx\| \le M|\lambda|\|\lambda Tx\|$

for all x in H .

2- consider the case T=B-5I (where B is the operator defined in example1.3). Since $5 \notin \sigma(B)$, then T is an invertible operator by theorem 2.1 T is quasi-posinormal operator. But T+5I=B is not quasi posinormal operator

Definition 2.6

Let T∈B(H), the quasi-spectrum of T , denoted Q(T) is the set {λ: T -λI is not quasi-posinormal operator }

Proposition 2.7

let T∈B(H),be a quasi- posinormal operator then

1- Q(T) ⊆ $\sigma(T)$.

2- If $\lambda \in \sigma_p(T)$ and $(T-\lambda)^{*2}x \ne 0$ for all x≠0 ∈ H then λ ∈Q(T).

3- If $\lambda \in \sigma_{ap}(T)$ and $(T-\lambda)^{*2}x \ne 0$ for all x≠0 ∈ H then λ ∈Q(T).

Proof :

(1) By corollary 2.2 makes that Q(T) is a subset of $\sigma(T)$.

(2) Suppose λ∉Q(T) then T-λI is quasi-posinormal operator and $\|(T-\lambda I)^{*2}x\| \le M\|(T-\lambda)x\|$ for all x in H. Now $\lambda \in \sigma_{ap}(T)$ then there exists x≠0such that $(T-\lambda)x = 0$ and $(T-\lambda)^{*2}x = 0$ contradiction , hence λ ∈Q(T)

(3) by the same way we can prove it .

Remark 2.8

The sum and the product of two quasi-posinormal operators need not be quasi-posinormal operator. We can see that by the following examples

1- Let H= ℓ2(¢), Let T_1 =U the unilateral shift operator and T_2 is the operator defined on H by T_2(x1,x2,x3,…)= $(0,0,0,-x_3,-x_4,-x_5,.......)$ it is clear that T_2 is hyponormal operator hence T_2 quasi-posinormal operator .Now (T_1+T_2) (x_1, x_2, x_3,…)= T_1(x_1, x_2, x_3,…)+T_2(x_1, x_2, x_3,…)=

(0, x_1, x_2,0,0,0,0,0….),and $(T_1+T_2)^*$(x_1, x_2, x_3,…)=(x_2, x_3,0,0,0,…..).If we take x=(0,0, x_3, $x_4, x_5 \ldots$) such that $x_3 \neq 0$,then $\|(T_1+T_2)x\|^2 = \|0\|^2$ which implies $\|(T_1+T_2)x\| = 0$, but $\|(T_1+T_2)^{*2} x\|^2 = \|(T_1+T_2)^*(0, x_3, 0, 0, 0,)\|^2 =$ $\|(x_3, 0, 0, 0, 0, ...)\|^2 = |x_3|^2$ then for all M >0 that $\|(T_1+T_2)^{*2} x\| \geq M \|(T_1+T_2)x\|$ and (T_1+T_2) is not quasi-posinormal operator.

2- Let H= ℓ_2(¢), T_1 =U the unilateral shift operator and T_2 be the operator defined on H by

T_2(x_1, x_2, x_3,…)= (x_1, x_2,0,0,0,…) then T_2 is self-adjoint operator hence is quasi-posinormal operator but T_1T_2(x_1, x_2, x_3,…)= T_1(x_1, x_2,0,0,0,…)= (0,x_1, x_2,0,0,0,0,…) and T_1T_2 is not quasi-posinormal operator by above example (1) .

Remark 2.9

Let T∈B(H) be a quasi-posinormal operator then T is not normaloid operator. i.e. the spectral radius of T is not necessarily equal to $\|T\|$,for example let $\{e_n\}_{n=1}^{\infty}$ be an orthogonal basis of a Hilbert space H and T be the a weighted shift defined by $Te_1 = e_2$, $Te_2 = 2e_3$ and $Te_i = e_{i+1}$ for i≥3 ,in [11]. Wadhwa.B.L proved that T is M-hyponormal operator, and not normaloid operator but by Corollary 1.9 T is quasi posinormal operator and not normaloid operator.

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المؤثرات الشبه السوية الموجبة

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الخلاصة :

في هذا البحث سندرس صنفاً من المؤثرات المعرفة على فضاء هلبرت سوف نطلق على عناصره اسم المؤثر شبه السوي الموجب ويضم كـلا من صنف المؤثرات السوية، المؤثرات فوق السوية و المؤثرات فوق السوية من النمط M ،المؤثرات المهيمنة والمؤثرات السوية الموجبة و سوف ندرس بعض الصفات الاساسية لهذا الصنف من المؤثرات وكذلك البحث عن العلاقة التي تربط هذا الصنف بالمؤثرات التي لها نظير .