The Approximated Solution for The Nonlinear Second Order Delay Multi-Value Problems

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Abstract:

This paper is attempt to study the nonlinear second order delay multi-value problems. We want to say that the properties of such kind of problems are the same as the properties of those with out delay just more technically involved. Our results discuss several known properties, introduce some notations and definitions. We also give an approximate solution to the coined problems using the Galerkin's method.

Key words: Nonlinear Delay Multi-value Problems

Introduction:

Nonlinear second order delay multi-value problems are now recognized as an excellent source of models to simulate processes and phenomena observed in control theory, population dynamics, physics, chemistry, industrial robotics and many other fields, [1]. Due to this reason, they have been an object of active research. We shall note that, in spite of the great number of investigations of this kind of problems, their solutions has not yet elaborated like that of nonlinear second order multi-value problems, [2]. In recent years, there has been increasing interest on the solution of the nonlinear second order delay multi-value problems, [3].

In this work we use the Galerkin's method to solve the investigated problems.

Basic Notations and Definitions

In this section we introduce some notations and definitions that needed in this work. We start the section by the following definition.

Definition 1.1

The delay differential equation is an equation that the unknown function or it's derivative evaluated at arguments that differ by any fixed number of values, [1].

Definition 1.2

The delay differential equations are said to be nonlinear in case they are nonlinear with respect to the unknown function that enter with different arguments and their derivatives that appeared in them, [2].

The following definition introduce the new concepts of our results.

Definition 1.3

A nonlinear delay multi-value problem is a problem in which the unknown multi-function and some of its derivatives, evaluated at arguments which are different by any fixed number of values.

Consider the following nonlinear second order delay multi-value problem:

$$- (p_i(x)y'_i(x))' + q_i(x)y_i(x-\tau) - f(x,\{\lambda_{ij}\}^n_{j=1}, y_i(x-\tau)) = 0, \quad x \in [a,b] \dots [1.1]$$

with the boundary conditions:

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$a_{i1}\, y_i(a) = a_{i2}\, y_i{}'(a)$

$$, x \in [a - \tau, a] \dots [1.2]$$

$b_{i1}\, y_i(b) = b_{i2}\, y_i{}'(b)$

$$y_i(x - \tau) = \varphi_i(x - \tau), \text{if } x - \tau < a$$

where p_i, $p_i{}'$ and q_i are given real-valued continuous functions defined on the interval $[a,b]$, p_i is positive, not both coefficients in one condition are zero, $\tau > 0$ is the time delay, f_i is a known nonlinear function with respect to y_i, φ_i is the initial function defined on $x \in [x_0 - \tau, x_0]$. The problem here is to determine the eigen-value λ_j in which a nontrivial solution y_i for the problem given by equations [1.1]-[1.2] occurs. In this case λ_j is said to be a delay eigen-value and y_i is the associated delay eigen-function.

More generally, if we rewrite the problem given by equations[1.1]-[1.2] as L y=0 where

$$L = \begin{bmatrix} L_1 & 0 & \dots & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & 0 & \dots & L_n \end{bmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

and

$$L_i = -\frac{d}{dx^2} p_i(x) - \frac{d}{dx} p_i'(x) + q_i(x) A_i(x), \; i = 1, 2, \dots, n$$
$$\dots [1.3]$$

where $A_i(x)$ is an operator defined by

$$A_i(x) y_i(x) = y_i(x - \tau).$$

Thus, L_i is a linear operator, which implies that L is linear operator also, [3].

Like the nonlinear second order multi-value problems, the problem given by equations [1.1]-[1.2] satisfies the following notation, [4].

Notation 1.1
The linear operator given by equation [1.3] is self-adjoint, [5].

Proposition 1.1
The   linear   operator

$$L = \begin{bmatrix} L_1 & 0 & \dots & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & 0 & \dots & L_n \end{bmatrix}$$ is self-adjoint.

Proof
Directly from notation (1.1).

Notation 1.2
There are infinite number of eigen-values forming a monotone increasing sequence with $\lambda_{ij} \to \infty$ as $j \to \infty$. Moreover, the eigen-functions corresponding to the eigen-values has exactly j roots on the interval (a,b) for each $i = 1,2,\dots,n$,[6].

While the following remarks are also hold for the problem given by equations [1.1]-[1.2],[7].

Remarks1.1
1. All the delay eigen-values are real.
2. The delay eigen-functions are orthogonal.
3. The delay eigen-functions are complete and normal in $L^2[a,b]$.
4. Each delay eigen-value corresponds only one delay eigen-function in $L^2[a,b]$.

To check the above remarks see [8].

The Galerkin's Method
In this section, we use the Galerkin's method to solve such a type of problems. To do this fix some i, say $i = k$, where $1 \le k \le n$. This method is based on approximating the unknown function y_k as a linear combination of n linearly independent functions $\{\phi_i\}^n_{i=1}$, that is write

$$y_k = \sum_{i=1}^{n} \phi_i(x) \dots [2.1]$$

which implies that

$$y_k(x - \tau) = \sum_{i=1}^{n} \phi_i(x - \tau), \text{if } x - \tau < a$$

But, this approximated solution must satisfy the boundary conditions given by equations [1.2] to get a new

approximated solution. By substituting this approximated solution into equation [1.1], one can get:

$$R_k(x, \{\lambda_{ij}\}_{j=1}^n, \vec{c}) =$$

$$-\left(p_k(x)\sum_{i=1}^m \phi_i'(x)\right)'$$

$$+ q_k(x)\sum_{i=1}^m \phi_i(x-\tau)$$

$$- f_k\left(x, \{\lambda_{ij}\}_{j=1}^n, \sum_{i=1}^m \phi_i(x-\tau)\right) \dots [2.2]$$

Where R_k is the error in the approximation of equation [2.2] and $\vec{c}$ is the vector of p elements of c_i, $i=1,2,\dots,n$, where $m-2 \le p \le m$.

Next, choose $p+n$ linearly independent functions $\{\psi_t\}_{t=1}^{p+n}$ to be orthogonal to R_k on the interval (a,b). That is,

$$\int_a^b \psi_i(x) R_k(x, \{\lambda_{ij}\}_{j=1}^n, \vec{c}) dx = 0 \quad , i=1,2,\dots,n$$

By evaluating the above equation at each $i=1,2,\dots,n$ one can get a system of $p+n$ nonlinear equations with $p+n$ unknowns which can be solved by any suitable method to get the values of $\{\lambda_{ij}\}_{j=1}^n$ and $\vec{c}$, [9].

To illustrate the above method we shall give the following example.

Example 2.1

Consider the following nonlinear delay multi-value problem:

$$-\quad y_1''(x) + (x-1)^2 y_1(x-1) - (2\lambda_1 y_1^2(x-1) - \lambda_2) = 0 \dots [2.3,a]$$
$$, x \in [1,2]$$
$$-y_2''(x) + y_2(x-1) - (2\lambda_1 y_2(x-1) - \lambda_2) = 0 \dots [2.3,b]$$

with the boundary conditions:

$$y_1(1) = 0 \quad y_1(2) = \frac{1}{2} y_1'(2) \dots [2.4,a]$$
$$, x \in [0,1]$$
$$y_2(1) = 0 \quad 2 y_2(2) = y_2'(2) \dots [2.4,b]$$

$$y_i(x-1) = x-1, \text{ if } x-1 < 1.$$

First, approximate the unknown function y_1 as a polynomial of degree three, which implies that $y_1(x-1) = c_1 + c_2(x-1) + c_3(x-1)^2 + c_4(x-1)^3$, but this approximated solution must satisfy the boundary conditions given by equations [2.4,a]. Thus, $y_1(x-1) = c_3(x-1)^2$, by substituting this approximated solution into equation [2.3,a] one can get:
$$R_1(x,\lambda_1,\lambda_2,c_3) = -2 c_3 + (x-1)^2 c_3(x-1)^2 - (2\lambda_1 c_3(x-1)^2 - \lambda_2) \dots [2.5]$$

Choose the functions $1, x, x^2$ and x^3 to be orthogonal to the error function R_1 to get the following system of equations:

$$\int_1^2 R_1 dx = 0$$

$$\int_1^2 R_1 x \, dx = 0$$

$$\int_1^2 R_1 x^2 dx = 0$$

$$\int_1^2 R_1 x^3 dx = 0$$

Thus, the nontrivial solution of the above system is $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 2$ and $c_3 = 1$.

Second, substitute the values of λ_1 and λ_2 into equation [2.3,b] to get that the solution of this equation is $y_2(x-1) = d_1 + d_2(x-1) + d_3(x-1)^2$.

But, this approximated solution must satisfy the boundary conditions given by equation [2.4,b]. Thus $d_3=1$.

Hence, $((\frac{1}{2}, 2), ((x-1)^2, (x-1)^2))$ is the double delay eigen-pair of the problem given by equations [2.3]-[2.4].

Results:

1. The nonlinear second order delay multi-value problems satisfies the

same properties as those with out delay.

2. The Galerkin's method is a good choice to solve such a type of problems.

Hint

The above problems has been solved with the help of Math-Cad software package, and the programs are so easy that omitted.

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الحـل التقريبي للمسـائل اللاخطية التباطؤيه ازدواجية القيم الذاتية ذات الرتبة الثانية

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الخلاصة:

كرس هذا البحث لدراسة المسائل اللاخطية التباطؤيه ازدواجية القيم الذاتية ذات الرتبة الثانية. أردنا القول أن هذا النوع من المسائل يحتفظ بخواص المسائل نفسها في حالة عدم وجود التباطؤ و لكن بخطوات تكتيكية أكثر. كانت النتائج مناقشة العديد من الخواص مع تقديم بعض الملاحظات والتعاريف. كما قمنا بتقديم الحل التقريبي للمسائل المشار إليها باستخدام طريقة گاليركن.