SOME RESULTS OF MULTIVLUED CONTRACTIONS MAPPINGS

بعض النتائج للتطبيقات الانكماشية المتعددة القيم

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Abstract

In this paper, there are some results about fixed point theory for multivalued mappings. The reliable idea to get these results including some multivalued mappings which admit selections satisfying the assumptions of Carist's theorem, through they need not have continuous selections. Note that some results in above will be known and present here to complete the idea.

الخلاصة

يتضمن هذا البحث بعض نتائج النقطة الصامدة لدوال متعددة القيـــم حيث أن الفكرة المعتمدة لاستحصال هذه النتائج تعتمد على تقديم بعض الدوال المتعددة القيم التي تسمح بتطبيقات تحقق مبرهنة كراستي (Caristi's Theorem) بدون الأخذ بنظر الاعتبار كون هذه التطبيقات المختارة مستمرة أو لا . نود أن نشير إلى أن بعض النتائـــج أعلاه معروفه ولكن أضيفت للبحث لغرض استكمال عرض الفكرة .

Introduction and key wordes

The study of fixed points for multivalued contraction mappings using Hausdorff metric initiated by Nadler[6].After this ,fixed point theory has been developed further and applied to many disciplines to solve functional equations.The Banach contraction principle has been extended in different directions either by using generalized contractions for multi valued mappings and hybrid pairs of single and multi valued mappings, or by using more general spaces.

The purpose of this paper is two folds .First ,in section one, it shows that the restriction of Caristi's theorem to continuous function can be derived directly from the Zermelo's theorem. In section two, there is a simple extension for Caristi's theorem due to jachymjki [9] .We give a new proof for this result depending on the concept of ω-distance .

Second, using an idea of jachymjki[8]we establish to selection theorems selections satisfying the assumptions of Caristi's theorem or its extension. Though they need not have continuous selections

1- If X is a Banach space, **a net** in X is a function from A to X in the form (X,α),which expresses that the element α in A is amapped to the element in X.
   For example every non-empty totally ordered set is directed . Therfore every function on such asset is anet.
2- Let (X,d) metric space,T is **contraction mapping** with the property that ther is asome non negative real number K<1 such that for all x and y in X
   $d(T(x),T(y)) \leq k\, d(x,y)$.
3- **A function f** such that $|f(x)\_f(y)| \leq |x\_y|$ for all x and y ,where C is constant independent of x&y ,is called **lipschitz function.**for example any function with a bounded firste derivative must be lipschitz.

1-CARISTI'S THEOREM AND SELECTIONS OF MULTIVALUED CONTRATIONS

Introduction

The simplicity and usefulness of Banach contraction principle has inspired many authors to analyze it further. These studies have led to a number of generalizations and modifications of the principle, one of the deepest result is Caristi's theorem, [2] .

Theorem (Caristi's Therorem)(1.1) [7]

Let (X,d) be a complete metric space and $\phi: X \to \mathbb{R}^+$ be 1.s.c function. Suppose $T: X \to X$ is an arbitrary mapping which satisfies

$d(x,Tx) \le \phi(x) - \phi(Tx)$,for all $x \in X$ …………………...…….. *

Then T has a fixed point

An original proof of the above result is based an axiom of choice.
Subsequently ,a number of authors found simpler proofs. In particular, Kirk [4] .Using an idea of BrØndsted ordering , gave an elegant proof involving this ordering Kirk and Saliga [5]. Using Brezis-Browder ordering , gave a short proof.

Here we need to recall **BrØndsted ordering**[4]the partial ordering $\le$ defined on a metric space (X , d) by:

$x \le y$ if $d(x,y) \le \phi(x) - \phi(y)$, for all $x,y, \in X$……………………………....**

where ϕ is non-negative lower – semi continuous. The following result state that the Zermelo's theorem implies the restriction of Caristi's theorem to continuous function ϕ. This result is independed of the axiom of choice.

Theorem (1.2) [4]

Let (X,d)be a complete metric space and $T:X \to X$ such that (*) holds with continuous function $\phi: X \to \mathbb{R}^+$,then T and X endowed with BrØndsted ordering (**)satisfy the assumption of Zermelo's theorem In particular ,the Zemelo's theorem implies the Banach contraction principle.

Proof :

Let $\le$ be the ordering defined by (**).It is straight forward that $\le$ is partial ordering in particular the transitivity of $\le$ follows from the triangle inequality.
Clearly the inequality (*) means that T is a progressive mapping on (X,d) such that $d(x,Tx) \le \Phi(x) - \phi(Tx)$ then $x \le Tx$
We may treat C as a net. Putting $\sum: =C$ and $x_\sigma = \boldsymbol{\sigma}$ for $\boldsymbol{\sigma} \in C$. we may infer that $\{x_\sigma\}_{\sigma \epsilon \sum}$ is Cauchy. Obviously ,this net is increasing, i.e. ,if $\boldsymbol{\sigma}1 \le \boldsymbol{\sigma}2$, then:

$$0 \le d(x_{\sigma 1}, x_{\sigma 2}) \le \phi(x_{\sigma 1}) - \phi(x_{\sigma 2}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(***)$$

Hence $\phi(\boldsymbol{\sigma}2) \le \phi(\boldsymbol{\sigma}1)$, i.e, the net $\{\phi(x_\sigma)\}_{\sigma \epsilon \sum}$ is non-increasing and thus it is convergent if ϕ is bounded below.
Hence and by (***),$\{x_\sigma\}_{\sigma \epsilon \sum}$ satisfies Caushy's condition so it converges to some $x_0 \,\boldsymbol{\epsilon}\, X$. and $x_\sigma \le x_0$, i.e.,
$d(x_\sigma, x_0) \le \phi(x_\sigma) - \phi(x_o)$, for all $\boldsymbol{\sigma \epsilon} C$, which mean that x_0 is an upper bound of C. Now, we show that $x_0 = \sup C$.

Let x be an arbitrary upper bound of C . Then

$$d(x_0, x) \le d(x_0, x_\sigma) + d(x_\sigma, x) \le d(x_o, x_\sigma) + \phi(x_\sigma) - \phi(x)$$

Hence taking limit with respect to $\boldsymbol{\sigma}$, we obtain by continuity of ϕ that:
$\lim_{\sigma\to\infty}\phi(x_\sigma) = \phi(\lim_{\sigma\to\infty}x_\sigma)$
$d(x_0, x) \leq \phi(x_0) - \phi(x)$, i.e., $x_0 \leq x$,which yield that $x_0 = \sup c$.
To prove the last statement of theorem (**)
Let T:X→X be a Banach contraction with a constant K∈ (0,1) , given x∈X

$$\mathbf{d(x,Tx)}=\frac{d(x,Tx)}{1-K} - \frac{Kd(x,Tx)}{1-K} \leq \frac{d(x,Tx)}{1-K} - \frac{H(Tx,T^2x)}{1-k} \leq \phi(x) - \phi(Tx),$$

where $\boldsymbol{\phi(x)}=\frac{d(x,Tx)}{1-k}$
clearly ,ϕ is continuous, so each nonempty well-orderd subset in (X,≤) has a supremum as shown above.

Moreover T is a progressive on (X,≤), mean that T satisfies of assumptions of the restrictions Caristi's `theorem .■

Theorem (1.3)
Let T is Nadler's multivalued contraction, on a complete metric space (X,d), then T admits a selection f:X→X, which is a Caristi's mapping on (X,d) generated by a lipschitzain function ϕ.

Proof :
Choose a real k' such that k is non negative real number and k< k'< 1, then a given x∈X, then: A set $\{y \in Tx: k'd(x,y) \leq d(x,Tx)\}$ is a nonempty. By the axiom of choice there is a mapping f :X→X, such that fx ∈Tx for all x∈X and

$k'd(x,fx) \leq D(x,Tx)$.

$D(fx,T(fx)) \leq H(Tx,T(FX)) \leq kd(x,fx)$
Hence :

$$d(x,fx) = \frac{1}{k'-k}\left(k'd(x,fx) - kd(x,fx)\right)$$
$$\leq \frac{1}{k'-k}(D(x,Tx) - D(fx,T(fx)))$$

So we get that f. satisfies (1.1) with $\phi(x)=\frac{D(x,Tx)}{k'-k}$ .Moreover, ϕ is lipschitzain, since:

$$|\phi(x) - \phi(y)| = \frac{D(x,Tx)}{k'-k} - \frac{D(y,Ty)}{k'-k}$$

$$\leq \frac{d(x,y)+D(y,Ty)+H(Tx,Ty)-D(y,Ty)}{k'-k}$$

$$\leq \frac{d(x,y)+H(Tx,Ty)}{k'-k}$$

$$\leq \frac{1+k}{k'-k}d(x,y)$$

By Cariti's theorem has a fixed point x^* and $x^* \in Tx^*$. ■

The following example(mentioned without details) in[6]shows that under the assumptions of theorem (1.3) a Nadler's multivalued mapping T may have neither a lipschitzain selection nor even continuous selection, it is given here with all details since it seems that this example is not well known.

Example (1.1)

Let X be the unit circle in the complex plane and for z $\in$ X, let Tz be the set of two square roots of z. For $\alpha \in [0,2\pi)$ denote
$P_{\propto} = cos_{\propto} + i\ sin_{\propto}$. Then

$$\mathrm{T}\,\mathrm{P}_{\propto} = \{\mathrm{P}_{a/2}, \mathrm{P}_{a/2+\pi}\}.$$

If $\alpha, \beta \in [0,2\pi)$ and $\propto \leq \beta$, then it can be easily verified that

$$\mathrm{d}\,(\mathrm{P}_{\propto}, \mathrm{P}_{\beta}) = 2\sin(\beta - \alpha)/2, \text{ where}$$

$$\mathrm{H}(\mathrm{T}p_{\propto}, Tp_{\beta}) = \begin{cases} 2\sin(\beta-\alpha)/4, & if\ \beta-\alpha \leq \pi \\ 2\cos(\beta-\alpha)/4, & if\ \beta-\alpha > \pi \end{cases}$$

Hence we easily get that T is Nadler's set valued contraction with a contractive constant equal to $\sqrt{2}/2$.We give an elementary proof of the fact that T has nocontinuous selections.

Suppose,on the contrary that f is such a selection of T . Consider the case in which $\mathrm{fp}_0 = \mathrm{p}_0$. By the continuity of f at p_0

There exist $\propto_0 > 0$ such that $\mathrm{fp}_{\alpha} = \mathrm{p}_{\alpha/2}$for all $\propto \in [0, \alpha_0]$. Define

$$\alpha^* = \sup\{\alpha_0 \in (0,2\pi) \colon \mathrm{fp}_{\alpha} = \mathrm{p}_{\alpha/2} \text{ for all } \alpha \in [0, \alpha_0]\}$$

Suppose $\alpha^* < 2\pi$, then,by the definition of α^*

$\mathrm{fp}_{\propto} = \mathrm{p}_{\alpha/2}$for $\propto \in [0, \alpha^*)$ and there is a sequence $\{\propto_n\}_{n=1}^{\infty}$, such that:

$\alpha^* \leq \propto_n < 2\pi, \propto_{n\rightarrow}\ \alpha^*$ and $\mathrm{fp}_{\propto} = \mathrm{p}_{\propto n/2+\pi}$

By the continuity of f at $\mathrm{p}_{\propto^*}$.

$\mathrm{p}_{\propto^*/2 = \lim_{\alpha\rightarrow\alpha^{*-}} \mathrm{fp}_{\alpha} = \lim_{n\rightarrow\infty} \mathrm{fp}_{\propto n} = \mathrm{p}_{\propto^*/2+\pi}}$ contradiction

Then $\propto_* = 2\pi$, that is $\mathrm{fp}_{\propto} = \mathrm{p}_{\propto/2}$for all $\propto \in [0,2\pi)$

Then $\mathrm{fp}_{2\pi-1/\mathrm{n}} = \mathrm{p}_{\pi-1/2\mathrm{n}\rightarrow}\mathrm{p}_{\pi}$on the other hand $\mathrm{p}_{2\pi-1/\mathrm{n}} \rightarrow \mathrm{p}_0$, so by the continuity of f at p_0

$\mathrm{fp}_{2\pi-1/\mathrm{n}} \rightarrow \mathrm{fp}_0 = \mathrm{p}_0$ contradiction

A similar argument can be used to show that we will get a contradiction also in the case in which fp_0=p_{π}.

2-EXTENDING CARISTI'S THEOREM AND SELECYION OF MULTIVALUED CONTRACTIONS

Introduction

Here we establish a selection theorem stating that some multivalued contractions admit selections satisfying the assumptions of extending Cariti's theorem .The extension of Cariti's result involving a condition of the type :

$\eta(\mathrm{D}(\mathrm{x},\mathrm{Tx})) \leq \Phi(x) - \Phi(Tx)$, for all x$\in$ X ……………………….……………..(2.1)

Where η is a function from $\mathbb{R}^+$ into $\mathbb{R}^+$.

We begin with recalling the following definition:

Definition (2.1)

A function $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be a **subadditive** if:

$\eta(\mathrm{s}+\mathrm{t}) \leq \eta(\mathrm{s}) + \eta(\mathrm{t})$, for all s,t $\in \mathbb{R}^+$ ……………………….……(2.2)

and η is said to be **subadditive** if the revers inequality holds.

The following result is a simple extension of Cariste's theorem we can derive it showing that the composition function ηod is a ω-distance on X this proof is different from the original proof [2].

Theorem (2.1)

Let T be a self mapping of a complete metric space (X,d),Φbe a nonnegative 1.s.c.function on X and η: $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a non decreasing and subadditive function continuous at 0 and such that $\eta^{-1}(\{0\})=\{0\}$.If (2.1) holds, then T has a fixed point.

Proof:

We show that theorem (2.1) satisfy the assumptions of Cariste's theorem

Let $\rho = \eta od$, we must prove that $\rho = \eta od$ is ω-distance

$$\begin{aligned}\rho(x,y) &= \eta od\,(x,y)\\ &= \eta(d(x,y)) \leq \eta(d(x,z) + d(z,y))\\ &= \eta(d(x,z) + \eta d(z,y))\\ &\leq \eta od(x,z) + \eta od(z,y)\\ &\leq \rho(x,z) + \rho(z,y)\end{aligned}$$

Then the condition (i)of definition of ω-distance is satisfied .

It clearly $\eta od(x,0) : X \longrightarrow [0,\infty)$ is a1.s.c since d is continuous, and for any $\varepsilon > 0$, there exist $\delta = \frac{\varepsilon}{2}$ such that $\rho(z,x) = \frac{\varepsilon}{2}$ and $\rho(z,y) = \frac{\varepsilon}{2}$

$$\begin{aligned}d\,(x,y) &\leq d(x,z) + d(z,y) \leq \eta(d(x,z) + \eta(d(y,z)\\ &\leq \rho(x,z) + \rho(y,z)\\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon\end{aligned}$$

Then the condition 3 of definition of ω-distance is satisfied .

By (2.1). $\eta(d(x,Tx)) \leq \Phi(x) - \Phi(Tx)$. Let f = x

$$\eta(d(x,Tx)) \leq f(x) - f(Tx)$$

then $f(Tx) + \eta(d(x,Tx)) \leq f(x)$

then $f(Tx) + \rho(x,Tx) \leq f(x)$

which means that T has a fixed point . ∎

Theorem (2.2)

Let(X,d) be a complete metric space and $T:X \longrightarrow CL(X)$, is valued μ-contraction, such that μ is superadditive and there exist a function $t \longrightarrow t - \mu(t)$, for all $t \in \mathbb{R}^+$ is non decreasing .Then there exist a selection f of T and a function $\eta : \mathbb{R}^+ \longrightarrow R^+$,which satisfy the assumptions of theorem(2.1). Moreover,there is an equivalent metric ρ such that (X,ρ)is a complete and f is a Caristi's mapping on (X,ρ).

Proof:

Define $\sigma(t) = t - \mu(t)$, for $t \in \mathbb{R}^+$. Then σ is subadditive and non decreasing .since μ is superadditive and non negative ,it is non decreasing,

Hence

$$0 \leq \mu(0) \leq \mu(t) < t, \text{ for } t > o$$

Which implies the continuity of μ (hence σ) at 0. Define

$$v(t) = \frac{t+\mu(t)}{2} \text{ for } t \in \mathbb{R}^+$$

Then v is continuous and $v(t) < t$ for $t > 0$.Therefore, given an $x \in X$,the set $\{ y \in Tx : v(d(x,y)) \leq D(x,Tx) \}$ is nonempty.

By the axiom of choice., there is a mapping $f : X \longrightarrow X$ such that

$$v(d(x,fx)) \leq D(x,Tx) \text{ for } x \in X .$$

Hence

$$D(fx,T(fx)) \leq H(Tx,T(fx)) \leq \mu(d(x,fx)).$$

Let $\eta(t) = \frac{1-\mu(t)}{2}$ for $t \in \mathbb{R}^{+}$ and $\Phi(x) = D(x,Tx)$ for $x \in X$. Then

$$\eta(d(x,fx)) = \frac{d(x,fx) - \mu(d(x,fx))}{2}$$

$$= \frac{d(x,fx) + \mu(d(x,fx)) - \mu(d(x,fx)) - \mu(d(x,fx))}{2}$$

$$= v(d(x,fx) - \mu(d(x,fx)) \leq D(x,Tx) - D(fx,T(fx))$$

$$\leq \Phi(x) - \Phi(fx)$$

So (2.1) holds. Thus the assumption of theorem (2.1) are satisfied.■

EXAMPLE (2.1)

Let X=R with the usual metric. Define f:X→X ,T:X→CB(X).Let Ais a bounded subset in (X,T) if there exist x϶X and M≥0 such that for all a϶A, we have a϶(x,M),that is,T(x,a)≤T(a,a)+M.

For A,B϶CB(X) and x϶X

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T(X,A)=inf{ T(x,y):y϶A} ,T(A,B)=sup{T(a,B): a϶A}

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