Construction of Complete (k,n)-arcs in the Projective Plane PG(2,11) Over Galois Field GF(11), $3 \leq n \leq 11$

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Abstract

The purpose of this work is to construct complete (k,n)-arcs in the projective 2-space PG(2,q) over Galois field GF(11) by adding some points of index zero to complete (k,n–1)-arcs $3 \leq n \leq 11$.

A (k,n)-arcs is a set of k points no n + 1 of which are collinear.

A (k,n)-arcs is complete if it is not contained in a (k + 1,n)-arcs.

Introduction

Mayssa 2004 (4), constructed of complete (k,n)-arcs in PG(2,17) and Sawsan 2001 (6), showed the classification and construction of (k,n)-arcs from (k,m)-arcs in PG(2,q) m < n. And Ban, (8) showed the classification and construction of (k,4)-arc, k = 17, 18,…, 34, in PG(2,11).

This paper is divided into two sections, section one consists of proving basic, theorems and giving some definitions of projective plane, (k,n)-arcs, maximal and complete arcs…ets. Section two consists of the projective plane of order eleven. The construction of complete (k,2)-arcs call it c_1, c_2, c_3, …, c_9 and the construction of complete (k,n)-arcs from complete (k,n – 1)-arcs in PG(2,11), where n = 3, 4, …, 9, 10 gave the points P_i and lines L_i.in PG(2,11) are determined in the table (1,1).

Section One

1.1 Definition "Projective Plane" (1)

A projective plane PG(2,q) over Galois field GF(q) is a two-dimensional projective space, which consists of points and lines with incidence relation between them. In PG(2,q) there are $q^2 + q + 1$ points, and $q^2 + q + 1$ lines, every line contains 1 + q points and every point is on 1 + q lines, all these points in PG(2,q) have the form of a triple (a_1,a_2,a_3) where a_1, a_2, $a_3 \in GF(q)$; such that $(a_1,a_2,a_3) \neq (0,0,0)$. Two points (a_1,a_2,a_3) and (b_1,b_2,b_3) represent the same point if there exists $\lambda \in GF(q)\backslash\{0\}$, such that $(b_1,b_2,b_3) = \lambda (a_1,a_2,a_3)$.

There exists one point of the form (1,0,0). There exists q points of the form (x,1,0). There exists q^2 points of the form (x,y,1), similarly for the lines.

A point $p(x_1,x_2,x_3)$ is incident with the line $L[a_1,a_2,a_3]$ if and only if $a_1x_1 + a_2x_2 + a_3x_3 = 0$, i.e.

A point represented by (x_1,x_2,x_3) is incident with the line represented by $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$. if

$$(x_1,x_2,x_3) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \Rightarrow a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

The projective plane PG(2,q) satisfying the following axioms:

1. Any two distinct lines intersected in a unique point.
2. Any two distinct points are contained in a unique line.
3. There exists at least four points such that no three of them are collinear.

1.2 Definition (1)

Two lines $[a_1,a_2,a_3]$ and $[b_1,b_2,b_3]$ represent the same line if there exists $\lambda \in GF(q)\backslash\{0\}$, such that $[b_1,b_2,b_3] = \lambda [a_1,a_2,a_3]$.

1.3 Definition "Quadric" (1)

A quadric Q in PG(n – 1,q) is a primal of order two, so Q is a quadric, then Q = V(F), where F is a quadric form, that is:

$$F = \sum_{\substack{i \le j \\ i,j=1}}^{n} a_{ij} x_i x_j = a_{11} x_1^2 + a_{12} x_1 x_2 + \ldots + a_{nn} x_n^2$$

1.4 Definition "Conics"(1)

Let Q(2,q) be the set of quadrics in PG(2,q), that is the varieties V(F), where:

$$F = a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + a_{12} x_1 x_2 + a_{13} x_1 x_3 + a_{23} x_2 x_3$$

If V(F) is non-singular, then quadric is conic.

1.5 Definition "(k,n)-arcs"

A (k,n)-arc, K in PG(2,q) is a set of K points such that some line in PG(2,q) meets K in n points but such that no line meets K in more that n points, where $n \geq 2$.

A line L in PG(2,q) is an i-secant of a (k,n)-arc K if $|L \cap K| = i$.

Let T_i denoted the total number of i-secants to K in PG(2,q).

0-secant is called an external line, a 1-secant is called a unisecant, a 2-secant is called a bisecant.

1.6 Definition "Complete (k,n)-arcs" (1)

A (k,n)-arc in PG(2,q) is complete if there is no (k+1,n)-arc containing it.

1.7 Definition (1)

A point N not in (k,n)-arc K is said to be has index i if there exists exactly i (2-secants) through N.

$C_i = |N_i|$ = the number of points of index i.

1.8 Definition "Maximal (k,n)-arcs" (2)

A (k,n)-arc K in PG(2,q) is a maximal arc if $k = (n - 1) q + n$.

1.9 Theorem (2)

Let M be a point of (k,2)-arc A in PG(2,q), then the number of unisecant through M is $u = q + 2 - k$.

Proof:

There exists exactly q + 1 lines through a point M in a(k,2)-arc A of PG(2,q), which are the bisecants and the unisecants of the arc. There exists exactly (k – 1) bisecants of the arc A through M and the other (k – 1) points of the arc, since the arc contains exactly k points. The number of unisecants through M is u, then

$u = q + 1 - (k - 1) = q + 1 - k + 1 = q + 2 - k$.

1.10 Theorem (2)

Let T_i be the number of the i-secants of a (k,n)-arc A in PG(2,q), then:

(a) $T_2 = k (k - 1) / 2$

(b) $T_1 = k\, u$, u is the number of unisecants of each point of A.

(c) $T_0 = q(q - 1) / 2 + u(u - 1) / 2$.

Proof (a):

T_2 = the number of bisecants of the (k,n)-arc A, the (k,n)-arc A contains k points, each two of them determine a bisecant line, so:

$T_2 = \binom{k}{2} = k! / (k-2)! \cdot 2! = k(k-1)/2$

Proof (b):

T_1 = the number of unisecants to the (k,n)-arc A. By Theorem (1.6) there exists exactly $u = q + 2 - k$ lines through any point M in (k,n)-arc A, since the number of points on (k,n)-arc is k.

Then there exists $ku = k(q + 2 - k)$ unisecants of the (k,n)-arc A.

Proof (c):

T_0 be the number of the external lines to the (k,n)-arc A, then;

$T_0 + T_1 + T_2 = q^2 + q + 1$ represents all the lines in PG(2,q) then,

$T_0 = q^2 + q + 1 - T_1 - T_2$ from part (a) and (b)

$T_0 = q^2 + q + 1 - k\,u - k(k-1)/2$

Since, $u = q + 2 - k \Rightarrow k = q + 2 - u$, then

$T_0 = q^2 + q + 1 - u\,(q + 2 - u) - (q + 2 - u)(q + 1 - u)/2$

$T_0 = \frac{1}{2}[2q^2 + 2q + 2 - 2u(q + 2 - u) - (q + 2 - u)(q + 1 - u)]$

$T_0 = \frac{1}{2}[2q^2 + 2q + 2 - 2uq - 4u + 2u^2 - q^2 - q + uq - 2q - 2 + 2u + uq + u - u^2]$

$T_0 = \frac{1}{2}[2q^2 + 2q - 4u + 2u^2 - q^2 - 3q + 3u - u^2]$

$T_0 = \frac{1}{2}[q^2 - q + u^2 - u]$

$T_0 = q\,(q-1)/2 + u(u-1)/2$

1.11 Theorem (3)

A (k,n)-arc A in PG(2,q) is complete if and only if $C_0 = 0$.

Proof: $\Rightarrow$

Let A be a complete (k,n)-arc in PG(2,q) and suppose that $C_0 \neq 0$, then $\exists$ at least one point say N has an index zero and $N \notin A$. Then $A \cup \{N\}$ is an arc in PG(2,q). Hence $A \subseteq A \cup \{N\}$. Which implies that the (k,n)-arc A is incomplete (contradicts the hypothesis).

$\Leftarrow$ suppose that $C_0 = 0$ for the (k,n)-arc A then there are no points of index zero, for A, so the (k,n)-arc A is a complete.

1.12 Theorem (3)

If a (k,n)-arc A is maximal arc in PG(2,q), then,

(a) if $n = q + 1$, then A = PG(2,q)

(b) if $n = q$, then A = PG(2,q) \ L , where L is line

(c) if $2 \leq n \leq q$, then $n|q$ and the dual of the complements of (k,n)-arc A forms a $(q(q + 1 - n)/n, q/n)$-arc, also maximal.

Proof (a):

A (k,n)-arc A is a maximal in PG(2,q), then $k = (n-1)q + n$, and if $n = q + 1$, then

$k = ((q + 1) - 1)\,q + (q + 1) = q^2 + q + 1$ points

$A = (q^2 + q + 1, q + 1) = PG(2,q)$.

Proof (b):

When $n = q$, since A is a maximal arc, then $A = (n + 1)\,q + n$, $A = (q-1)\,q + q = q^2$

$|PG(2,q)| = q^2 + q + 1$

$|PG(2,q) \setminus L| = |PG(2,q)| - |L| = q^2 + q + 1 - (q + 1) = q^2 = A$. Then

$A = PG(2,q) \setminus L$.

Proof (c):

When $2 \le n \le q$, there exists a point M not in A, so the number of 0-secants through M is q / n, it follows that n / q. the dual of complement of (k,n)-arc A is $(T_0, q / n)$-arc is maximal. Then $(q(q+1-n) / n, q / n)$-arc is maximal.

1.13 Lemma (4)

For a (k,n)-arc in PG(2,q), the following equation hold:

1. $\sum_{i=0}^{n} T_i = q^2 + q + 1$

2. $\sum_{i=1}^{n} iT_i = k(q+1)$

3. $\sum_{i=2}^{n} i(i-1)T_i / 2 = k(k-1)/2$

4. $\sum_{i=2}^{n} (i-1)p_i = k-1$

Note: T_i denote the total number of i-secants to the arc in PG(2,q).

1.14 Theorem (5)

A (k,n)-arc A In PG(2,q) is maximal if and only if every line in PG(2,q) is a 0-secant or n-secant.

Proof: ⇒ Suppose that (k,n)-arc A is maximal arc in PG(2,q), then the result was proved in the theorem.

⇐ Suppose every line in PG(2,q) is a 0-secant or n-secant.

If $T_1 = T_2 = T_3 = \ldots = T_{n-1} = 0$, then

$\sum_{i=1}^{n} iT_i = k(q+1)$ (by Lemma (1.13), (2))

$T_1 + 2T_2 + \ldots + (n-1)T_{n-1} + nT_n = k(q+1)$

$nT_n = k(q+1)$ …[1]

$\sum_{i=2}^{n} i(i-1)T_i / 2 = k(k-1)/2$ (Lemma (1.13), (3))

$T_2 + 3T_3 + \ldots + n(n-1)T_n / 2 = k(k-1)/2$

$n(n-1)T_n / 2 = k(k-1)/2$

$n(n-1)T_n = k(k-1)$ …[2]

From equation [1], we get:

$nT_n / k = q + 1$ …[3]

From equation [2], we get:

$nT_n / k = (k-1)/(n-1)$ …[4]

From equations [3] and [4], we get

$(k-1)/(n-1) = q+1 \Rightarrow (k-1) = (q+1)(n-1) \Rightarrow (k-1) = (n-1)q + (n-1)$

$\Rightarrow k = (n-1)q + n$

(k,n)-arc A is maximal arc (by definition 1.5)

Section Two

The projective plane PG(2,11) contains 133 points, 133 lines, every line contains 12 points and every points is on 12 points. The points and lines of PG(2,11) are shown in table (1,1).

2.1 The Construction of (k,2)-arc in PG(2,11) (2)

Let A = (1,2,13,25) be the set of unit and reference points in PG(2,11) as in the table (1,1) such that:

1 = (1,0,0), 2 = (0,1,0), 13 = (0,0,1), 25 = (1,1,1), A is (4,2)-arc, since no three points of A are collinear, the points of A are the vertices of a quadrangle whose sides are the lines.

L_1 = [1,2] = {1,2,3,4,5,6,7,8,9,10,11,12}

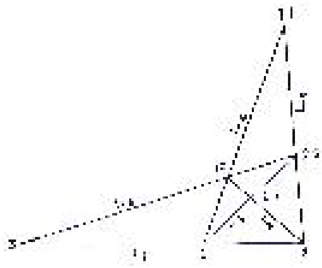
L_2 = [1,13] = {1,13,14,15,16,17,18,19,20,21,22,23}

L_3 = [1,25] = {1,24,25,26,27,28,29,30,31,32,33,34}

L_4 = [2,13]= {2,13,24,35,46,57,68,79,90,101,112,123}

L_5 = [2,25]= {2,14,25,36,47,58,69,80,91,102,113,124}

L_6 = [13,25] = {3,13,25,37,49,61,73,85,97,109,121,133}



The diagonal points of A are the points {3,14,24} where,

$L_1 \cap L_6 = 3$; $L_2 \cap L_5 = 14$; $L_3 \cap L_4 = 24$.

Which are the intersection of pairs of the opposite sides, then there are 61 points on the sides of the quadrangle, four of them are points of the arc A and three of them are the diagonal points of A, so there are 72 points not on the sides of quadrangle which are the points of index zero for A, these points are: 38, 39, 40, 41, 42, 43, 44, 45, 48, 50, 51, 52, 53, 54, 55, 56, 59, 60, 62, 63, 64, 65, 66, 67, 70, 71, 72, 74, 75, 76, 77, 78, 81, 82, 83, 84, 86, 87, 88, 89, 92, 93, 94, 95, 96, 98, 99, 100, 103, 104, 105, 106, 107, 108, 110, 111, 114, 115, 116, 117, 118, 119, 120, 122, 125, 126, 127, 128, 129, 130, 131, 132. Hence A is incomplete (4,2)-arc.

2.2 The Conics in PG(2,11) Through the Reference and Unit Points (1)

The general equation of the conic is:

$$F = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_1x_2 + a_5x_1x_3 + a_6x_2x_3 = 0 \quad \ldots[1]$$

By substituting the points of the arc A in [1], then:

1 = (1,0,0) implies that $a_1 = 0$, 2 = (0,1,0), then $a_2 = 0$, 13 = (0,0,1), then $a_3 = 0$, 25 = (1,1,1), then $a_4 + a_5 + a_6 = 0$.

Hence, from equation [1]

$$a_4 x_1 x_2 + a_5 x_1 x_3 + a_6 x_2 x_3 = 0 \quad \ldots[2]$$

If $a_4 = 0$, then $a_5 x_1 x_3 + a_6 x_2 x_3 = 0$, and hence $x_3(a_5 x_1 + a_6 x_2) = 0$, then $x_3 = 0$ or $a_5 x_1 + a_6 x_2 = 0$, which is a pair of lines, then the conic is degenerated, therefore for $a_4 \neq 0$, similarly $a_5 \neq 0$ and $a_6 \neq 0$.

Dividing equation [2] by a_4, one can get:

$$x_1x_2 + \frac{a_5}{a_4}x_1x_3 + \frac{a_6}{a_4}x_2x_3 = 0, \text{ then } x_1 x_2 + \alpha x_1 x_3 + \beta x_2 x_3 = 0 \quad \ldots[3]$$

where $\alpha = \frac{a_5}{a_4}, \beta = \frac{a_6}{a_4}$, so that $1 + \alpha + \beta = 0$ (mod.11)

$\beta = -(1 + \alpha)$, then [3] can be written as: $x_1 x_2 + \alpha x_1 x_3 - (1 + \alpha) x_2 x_3 = 0$

where $\alpha \neq 0$ and $\alpha \neq 10$ for if $\alpha = 0$ or $\alpha = 10$, then degenerated conics, can be obtained thus α = 1, 2, 3, 4, 5, 6, 7, 8, 9.

2.3 The Equation and the Points of the Conics of PG(2,11) Through the Reference and Unit Points (1)

1. If $\alpha = 1$, then the equation of the conic C_1 is $x_1 x_2 + x_1 x_3 + 9 x_2 x_3 = 0$, the points of C_1 are: {1, 2, 13, 25, 40, 53, 63, 77, 87, 100, 104, 116}, which is a complete (12,2)-arc, since there are no points of index zero for C_1.
2. If $\alpha = 2$, then the equation of the conic C_2 is $x_1 x_2 + 2x_1 x_3 + 8 x_2 x_3 = 0$, the points of C_2 are: {1, 2, 13, 25, 42, 50, 59, 78, 84, 96, 110, 131}, which is a complete (12,2)-arc, since there are no points of index zero for C_2.

3. If $\alpha = 3$, then the equation of the conic C_3 is $x_1 x_2 + 3x_1 x_3 + 7 x_2 x_3 = 0$, the points of C_3 are: {1, 2, 13, 25, 41, 48, 64, 76, 89, 95, 115, 132}, which is a complete (12,2)-arc, since there are no points of index zero for C_3.

4. If $\alpha = 4$, then the equation of the conic C_4 is $x_1 x_2 + 4x_1 x_3 + 6 x_2 x_3 = 0$, the points of C_4 are: {1, 2, 13, 25, 44, 56, 65, 72, 82, 108, 118, 125}, which is a complete (12,2)-arc, since there are no points of index zero for C_4.
5. If $\alpha = 5$, then the equation of the conic C_5 is $x_1 x_2 + 5x_1 x_3 + 5 x_2 x_3 = 0$, the points of C_5 are: {1, 2, 13, 25, 43, 51, 67, 71, 99, 103, 119, 127}, which is a complete (12,2)-arc, since there are no point of index zero for C_5.
6. If $\alpha = 6$, then the equation of the conic C_6 is $x_1 x_2 + 6x_1 x_3 + 4x_2 x_3 = 0$, the points of C_6 are: {1, 2, 13, 25, 45, 62, 88, 98, 105, 114, 126}, which is a complete (12,2)-arc, since there are no points of index zero for C_6.
7. If $\alpha = 7$, then the equation of the conic C_7 is $x_1 x_2 + 7x_1 x_3 + 3x_2 x_3 = 0$, the points of C_7 are: {1, 2, 13, 25, 38, 55, 75, 81, 94, 106, 122, 129}, which is a complete (12,2)-arc, since there are no points of index zero for C_7.
8. If $\alpha = 8$, then the equation of the conic C_8 is $x_1 x_2 + 8x_1 x_3 + 2x_2 x_3 = 0$, the points of C_8 are: {1, 2, 13, 25, 39, 60, 74, 86, 92, 111, 120, 128}, which is a complete (12,2)-arc, since there are no points of index zero for C_8.
9. If $\alpha = 9$, then the equation of the conic C_9 is $x_1 x_2 + 9x_1 x_3 + 1x_2 x_3 = 0$, the points of C_9 are: {1, 2, 13, 25, 54, 66, 70, 83, 93, 107, 117, 130}, which is a complete (12,2)-arc, since there are no points of index zero for C_9.

Thus there are nine complete (12,2)-arcs (conics) in PG(2,11) through the reference and the unit points. Hence each arc is a maximum arc, since contains (12) points.

2.4 The Construction of Complete (k,n)-arcs in PG(2,11) (2)

1. The construction of complete arcs of degree 3

In 2.3, we found nine complete (k,2)-arcs which are $C_1, C_2, C_3, \ldots, C_9$, so the complete arcs of degree 3 can be constructed from some complete arcs of degree 2, say C_1, C_1 = {1, 2, 13, 25, 40, 53, 63, 77, 87, 100, 104, 116}. C_1 is not complete (k,3)-arc, since there exist some points of index zero for C_1 which are {3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 54, 55, 56, 57, 58, 59, 60, 61, 62, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81,82, 83,84, 85, 86, 88, 89, 90, 91, 92, 93, 94, 95, 96, 98, 99, 101, 102, 103, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133}, one can add to C_1 seven points of index zero which are:{12, 14, 45, 49, 57, 70, 128}, then it can be obtained a complete (19,3)-arc, H_1 = {1, 2, 12, 13, 14, 25, 40, 45, 49, 53, 57, 63, 70, 77, 87, 100, 104, 116, 128} since each point not in H_1 is on at least one 3-secant and H_1 intersect each line in at most 3 points, thus $C_0 = 0$, since there are no points of index zero for H_1. Similarly one can find complete arcs of degree 3 from $C_2, C_3, \ldots, C_9$, by adding some points of index zero to each one of them, call them: $H_2, H_3, \ldots, H_9$.

2. The construction of complete arcs of degree 4

One will try to construct complete arcs of degree 4 from the complete arcs of degree 3, taken the complete (19,3)-arc: H_1 = {1, 2, 12, 13, 14, 25, 40, 45, 49, 53, 57, 63, 70, 77, 87, 100, 104, 116, 128}, since there exist some points of index zero for H_1 which are {3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 41, 42, 43, 44, 46, 47, 48, 50, 51, 52, 54, 55, 56, 57, 58, 59, 60, 61, 62, 64, 65, 66, 67, 68, 69, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81,82, 83,84, 85, 86, 88, 89, 90, 91, 92, 93, 94,

95, 96, 97, 98, 99, 101, 102, 103, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 133}.
the arc H_1 is incomplete (19,4)-arc, one can add to H_1 eight of these points which are:{10, 23, 32, 38, 47, 84, 90, 105}, then it can be obtained a complete (27,4)-arc S_1, S_1 = {1, 2, 10, 12, 13, 14, 23, 25, 32, 38, 40, 45, 47, 49, 53, 57, 63, 70, 77, 84, 87, 90, 100, 104, 105, 116, 128}, S_1 is a complete (27,4)-arc, since every point not on S_1 is on at least one 4-secant, there are no points of index zero for S_1 intersect each line in at most 4 points. Similarly one can find

complete arcs of degree 4 from by adding some points of index zero to $H_2, H_3, \ldots, H_9$ to obtain complete arcs of degree 4, call them $S_1, S_2, \ldots, S_9$.

3. The construction of complete arcs of degree 5

In the same method in 1 and 2, one can construct complete arcs of degree 5 by adding some points of index zero to complete arcs of degree 4, for example by taking S_1, and the points of index zero for S_1 : {3, 4, 5, 6, 7, 8, 9, 11, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 29, 30, 31, 33, 34, 35, 36, 37, 39, 41, 42, 43, 44, 46, 48, 50, 51, 52, 54, 55, 56, 58, 59, 60, 61, 62, 64, 65, 66, 67, 68, 69, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81, 82, 83, 85, 86, 88, 89, 91, 92, 93, 94, 95, 96, 97, 98, 99, 101, 102, 103, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 133}, by adding to S_1 nine of these points which are: {8, 22, 27, 43, 56, 62, 74, 85,112 }, so one can get a complete arc of degree 5 call M_1 , M_1 ={1, 2, 8, 10, 12, 13, 14, 22, 23, 25, 27, 32, 38, 40, 43, 45, 47, 49, 53, 56, 57, 62, 63, 70, 74, 77, 84, 85, 87, 90, 100, 104, 105, 112, 116, 128}, M_1 is complete (36,5)- arc, since there are no point of index zero; i.e. $C_0 = 0$, so every points not in M_1 is on at least one 5-secant, and M_1 intersects each line in at most 5 points, Similarly one can find complete arcs of degree 5 by adding some point of index zero to : $S_2, S_3, \ldots, S_9$, to obtain complete arcs of degree 5, call them, $M_1, M_3, \ldots, M_9$.

4. The construction of complete arcs of degree 6

Complete arcs of degree 6 can be obtained from the complete arcs of degree 5 by adding some points of index zero, for example, one takes the (36,6)-arc, The points of index zero for M_1 are:{3, 4, 5, 6, 7, 9, 11, 15, 16, 17, 18, 19, 20, 21, 24, 26, 28, 29, 30, 31, 33, 34, 35, 36, 37, 39, 41, 42, 44, 46, 48, 50, 51, 52, 54, 55, 58, 59, 60, 61, 64, 65, 66, 67, 68, 69, 71, 72, 73, 75, 76, 78, 79, 80, 81, 82, 83, 86, 88, 89, 91, 92, 93, 94, 95, 96, 97, 98, 99, 101, 102, 103, 106, 107, 108, 109, 110, 111, 113, 114, 115, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 133}, and M_1 ={1, 2, 8, 10, 12, 13, 14, 22, 23, 25, 27, 32, 38, 40, 43, 45, 47, 49, 53, 56, 57, 62, 63, 70, 74, 77, 84, 85, 87, 90, 100, 104, 105, 112, 116, 128 }, by adding to M_1 eleven of these points which are {6, 30, 54, 67, 69, 75, 79, 92, 93, 107, 120}, so we have N_1 ={1, 2, 6, 8, 10, 12, 13, 14, 22, 23, 25, 27, 30, 32, 38, 40, 43, 45, 47, 49, 53, 54, 56, 57, 62, 63, 67, 69, 70, 74, 75, 77, 79, 84, 85, 87, 90, 92, 93, 100, 104, 105, 107, 112, 116, 120, 128}, then N_1 is complete (47,6)-arc, since There are no points of index zero for N_1. Similarly one can construct complete arcs of degree 6 by adding some points of index zero to M_2 , M_3, …, M_9, then complete of degree 6 can be obtained,and call them $N_2, N_3, \ldots, N_9$.

5. The construction of complete arcs of degree 7

Complete arcs of degree 7 can be constructed from the complete arcs of degree 6, one can take the (47,6)-arc, N_1 is complete arc of degree 7, since there exist some points of index zero which are:{3, 4, 5, 7, 9, 11, 15, 16, 17, 18, 19, 20, 21, 24, 26, 28, 29, 31, 33, 34, 35, 36, 37, 39, 41, 42, 44, 46, 48, 50, 51, 52, 55, 58, 59, 60, 61, 64, 65, 66, 68, 71, 72, 73, 76, 78, 80, 81, 82, 83, 86, 88, 89, 91, 94, 95, 96, 97, 98, 99, 101, 102, 103, 106, 108, 109, 110, 111, 113, 114, 115, 117, 118, 119, 121, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 133}. By adding to N_1 eleven of these points which are:{5, 21, 51, 58, 61, 64, 82, 83, 111, 117, 121}, then K_1 ={1, 2, 5, 6, 8, 10, 12, 13, 14, 21, 22, 23, 25, 27, 30, 32, 38, 40, 43, 45, 47, 49, 51, 53, 54, 56, 57, 58, 61, 62, 63, 64, 67, 69, 70, 74, 75, 77, 79, 82, 83, 84, 85, 87, 90, 92, 93, 100, 104, 105, 107, 111, 112, 116, 117, 120, 121, 128} is a complete (58,7)-arc, since there are no points of index zero, thus every point not in K_1 is on at least one 7-secant and K_1 intersects each line in at most 7 points. Similarly , constructed arcs of degree 7can be contructed from $N_2, N_3, \ldots, N_9$, call them $K_2, K_3, \ldots, K_9$.

6. The construction of complete arcs of degree 8

Complete arcs of degree 8 can be constructed from the complete arcs of degree 7, one can take the (58,7)-arc, k_1 is complete (58,7)-arc, since there exist some points of index zero which are:{3, 4, 5, 7, 9, 11, 15, 16, 17, 18, 19, 20, 24, 26, 28, 29, 31, 33, 34, 35, 36, 37, 39, 41, 42, 44, 46, 48, 50, 52, 55, 59, 60, 61, 64, 65, 66, 68, 71, 72, 73, 76, 78, 80, 81, 86, 88, 89, 91, 92, 94, 95, 96, 97, 98, 99, 101, 102, 103, 106, 108, 109, 110, 113, 114, 115, 118, 119, 122, 123, 124, 125, 126, 127, 129, 130, 131, 132, 133}. By adding to k_1 thirteen of these points which are:{3, 16, 24, 26, 28, 35, 37, 41, 48, 59, 78, 98, 125}, to obtain a complete (71,8)-arc L_1 and L_1 ={1, 2, 3, 5, 6, 8, 10, 12, 13, 14, 16, 21, 22, 23, 24, 25, 26, 27, 28, 30, 32, 35, 37, 38, 40, 41, 43, 45, 46, 47, 49, 51, 53, 54, 56, 57, 58, 59, 61, 62, 63, 64, 67, 69, 70, 74, 75, 77, 79, 82, 83, 84, 85, 87, 90, 92, 93, 98, 100, 104, 105, 107, 111, 112, 116, 117, 120, 121, 125, 128} is a complete (71,8)-arc, since there are no points of index zero, thus every point on L_1 is on at least one 8-secant and L_1 intersects any line in at most 8 points. Similarly arcs of degree 8 can be constructed from $K_2, K_3, \ldots, K_9$, call them $L_2, L_3, \ldots, L_9$.

7. The construction of complete arcs of degree 9

Complete arcs of degree 9 can be constructed from the complete arcs of degree 8, the complete (71,8)-arc L_1 is taken, L_1 is in complete (71,9)-arc, the points of index zero of L_1 are:{4, 7, 9, 11, 15, 17, 18, 19, 20, 29, 34, 36, 39, 42, 44, 46, 50, 52, 55, 60, 65, 66, 68, 71, 72, 73, 76, 80, 81, 86, 88, 89, 91, 94, 95, 96, 97, 99, 101, 102, 103, 106, 108, 109, 110, 113, 114, 115, 118, 119, 122, 123, 124, 126, 127, 129, 130, 131, 132, 133}. By adding to L_1 twelve of these points which are:{4, 15, 29, 36, 44, 52, 65, 71, 80, 88, 119, 133}, then a complete (83,9)-arc call it O_1 is obtained (83,9)-arc and O_1 ={1, 2, 3, 4, 5, 6, 8, 10, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32, 35, 36, 37, 38, 40, 41, 43, 44, 45, 47, 48, 49, 51, 52, 53, 54, 56, 57, 58, 59, 61, 62, 63, 64, 65, 67, 69, 70, 74, 75, 77, 78, 79, 80, 82, 83, 84, 85, 87, 88, 90, 92, 93, 98, 100, 104, 105, 107, 111, 112, 116, 117, 119, 120, 121, 125, 128, 133} is a complete (83,9)-arc, since there are no points of index zero, thus every point on O_1 is on at least one 9-secant and O_1 intersects any line in at most 9 points. In the same way complete arcs of degree 9 can be obtained from arcs of degree 8, $L_2, L_3, \ldots, L_9$, call them $O_2, O_3, \ldots, O_9$.

8. The construction of complete arcs of degree 10

Complete arcs of degree 10 can be constructed from the complete arcs of degree 9 as the following:

The complete arc of degree 9, O_1 is complete (83,10)-arc, since there exist some points of index zero for O_1 which are:{7, 9, 11, 17, 18, 19, 20, 31, 33, 34, 39, 42, 44, 46, 50, 55, 60, 66, 68, 72, 73, 76, 81, 86, 89, 91, 94, 95, 96, 97, 99, 101, 102, 103, 106, 108, 109, 110, 113, 114, 115, 118, 122, 123, 124, 126, 127, 129, 130, 131, 132}. Twelve of these points are added to O_1 which are:{9, 17, 31, 42, 46, 73, 86, 95, 96, 99, 103, 113}, then a complete (95,10)-arc call it B_1, is obtained B_1 ={1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 16, 17, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 51, 52, 53, 54, 56, 57, 58, 59, 61, 62, 63, 64, 65, 67, 69, 70, 71, 73, 74, 75, 77, 78, 79, 80, 82, 83, 84, 85, 86, 87, 88, 90, 92, 93, 95, 96, 98, 99, 100, 103, 104, 105, 107, 111, 112, 113, 116, 117, 119, 120, 121, 125, 128, 133} is a complete (95,10)-arc, since there are no points of index zero, i.e. $C_0 = 0$.

Similarly complete arcs of degree 10 can be constructed, call it $B_2, B_3, \ldots, B_9$ from $O_2, O_3, \ldots, O_9$.

9. Them construction of complete arcs of degree 11

Complete arcs of degree 11 can be constructed from complete arcs of degree 10.
The complete arcs of degree 10 B_1 is taken. B_1 is in complete (95,11)-arc, since there exist some points of index zero for B_1 which are:{7, 11, 18, 19, 20, 33, 34, 39, 50, 55, 60, 66, 68, 72, 76, 81, 89, 91, 94, 97, 101, 102, 106, 108, 109, 110, 114, 115, 118, 122, 123, 124, 126, 127, 129, 130, 131, 132}, by adding to B_1 (26) points of these points which are :{11, 19, 20,

33, 50, 66, 68, 89, 91, 94, 96, 101, 106, 108, 109, 110, 114, 115, 122, 124, 126, 127, 129, 130, 131, 132}, so we get a complete (121,11)-arc, call it Z_1 = {1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 56, 57, 58, 59, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 77, 78, 79, 80, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 98, 99, 100, 101, 103, 104, 105, 106, 107, 108, 109, 110, 112, 113, 114, 115, 116, 117, 119, 120, 122, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133}, The Z_1 is complete (121,11)-arc, since There are no point of index zero ,i.e. Co = 0. Similarly complete arcs of degree 11, $Z_2, Z_3, \ldots, Z_9$ can be constructed from complete arcs of degree 10.

Refrence

1. Hirschfeld, J.W.P., (1979), Oxford University press, Oxford.
2. Mayssa, G. M., (2004), M.Sc. Thesis, Baghdad University, College of Education Ibn-Al-Haitham, Iraq.
3. Aziz, S.M., (2001), M.Sc. Thesis, Mosul University.
4. Sawsan, J. K., (2001), M.Sc. Thesis, University of Baghdad, College of Education Ibn-Al-Haitham, Iraq.
5. بان عبد الكريم، (2001)، رسالة ماجستير، جامعة الموصل، العراق.
6. Rashad, (1999), M.Sc. Thesis, University of Baghdad, College of Education Ibn-Al-Haitham

Table :(1,1) of the points and lines of PG(2,11)

i	P_i	L_i											
1	1 0 0	2	13	24	35	46	57	68	79	90	101	112	123
2	0 1 0	1	13	14	15	16	17	18	19	20	21	22	23
3	1 1 0	12	13	34	44	54	64	74	84	94	104	114	124
4	2 1 0	7	13	29	45	50	66	71	87	92	108	113	129
5	3 1 0	9	13	31	38	56	63	70	88	95	102	120	127
6	4 1 0	10	13	32	40	48	67	75	83	91	110	118	126
7	5 1 0	4	13	26	39	52	65	78	80	93	106	119	132
8	6 1 0	11	13	33	42	51	60	69	89	98	107	116	125
9	7 1 0	5	13	27	41	55	58	72	86	100	103	117	131
10	8 1 0	6	13	28	43	47	62	77	81	96	111	115	130
11	9 1 0	8	13	30	36	53	59	76	82	99	105	122	128
12	10 1 0	3	13	25	37	49	61	73	85	97	109	121	133
13	0 0 1	1	2	3	4	5	6	7	8	9	10	11	12
14	1 0 1	2	23	34	45	56	67	78	89	100	111	122	133
15	2 0 1	2	18	29	40	51	62	73	84	95	106	117	128
16	3 0 1	2	20	31	42	53	64	75	86	97	108	119	130
17	4 0 1	2	21	32	43	54	65	76	87	98	109	120	131
18	5 0 1	2	15	26	37	48	59	70	81	92	103	114	125
19	6 0 1	2	22	33	44	55	66	77	88	99	110	121	132
20	7 0 1	2	16	27	38	49	60	71	82	93	104	115	126
21	8 0 1	2	17	28	39	50	61	72	83	94	105	116	127
22	9 0 1	2	19	30	41	52	63	74	85	96	107	118	129
23	10 0 1	2	14	25	36	47	58	69	80	91	102	113	124
24	0 1 1	1	123	124	125	126	127	128	129	130	131	132	133
25	1 1 1	12	23	33	43	53	63	73	83	93	103	113	123
26	2 1 1	7	18	34	39	55	60	76	81	97	102	118	123
27	3 1 1	9	20	27	45	52	59	77	84	91	109	116	123

28	4 1 1	10	21	29	37	56	64	72	80	99	107	115	123
29	5 1 1	4	15	28	41	54	67	69	82	95	108	121	123
30	6 1 1	11	22	31	40	49	58	78	87	96	105	114	123
31	7 1 1	5	16	30	44	47	61	75	89	92	106	120	123
32	8 1 1	6	17	32	36	51	66	70	85	100	104	119	123
33	9 1 1	8	19	25	42	48	65	71	88	94	111	117	123
34	10 1 1	3	14	26	38	50	62	74	86	98	110	122	123
35	0 2 1	1	68	69	70	71	72	73	74	75	76	77	78
36	1 2 1	11	23	32	41	50	59	68	88	97	106	115	124
37	2 2 1	12	18	28	38	48	58	68	89	99	109	119	129
38	3 2 1	5	20	34	37	51	65	68	82	96	110	113	127
39	4 2 1	7	21	26	42	47	63	68	84	100	105	121	126
40	5 2 1	6	15	30	45	49	64	68	83	98	102	117	132
41	6 2 1	9	22	29	36	54	61	68	86	93	111	118	125
42	7 2 1	8	16	33	39	56	62	68	85	91	108	114	131
43	8 2 1	10	17	25	44	52	60	68	87	95	103	122	130
44	9 2 1	3	19	31	43	55	67	68	80	92	104	116	128
45	10 2 1	4	14	27	40	53	66	68	81	94	107	120	133
46	0 3 1	1	90	91	92	93	94	95	96	97	98	99	100
47	1 3 1	10	23	31	39	47	66	74	82	90	109	117	125
48	2 3 1	6	18	33	37	52	67	71	86	90	105	120	124
49	3 3 1	12	20	30	40	50	60	70	80	90	111	121	131
50	4 3 1	4	21	34	36	49	62	75	88	90	103	116	129
51	5 3 1	8	15	32	38	55	61	78	84	90	107	113	130
52	6 3 1	7	22	27	43	48	64	69	85	90	106	122	127
53	7 3 1	11	16	25	45	54	63	72	81	90	110	119	128
54	8 3 1	3	17	29	41	53	65	77	89	90	102	114	126
55	9 3 1	9	19	26	44	51	58	76	83	90	108	115	133
56	10 3 1	5	14	28	42	56	59	73	87	90	104	118	132
57	0 4 1	1	101	102	103	104	105	106	107	108	109	110	111
58	1 4 1	9	23	30	37	55	62	69	87	94	101	119	126
59	2 4 1	11	18	27	36	56	65	74	83	92	101	121	130
60	3 4 1	8	20	26	43	49	66	72	89	95	101	118	124
61	4 4 1	12	21	31	41	51	61	71	81	91	101	122	132
62	5 4 1	10	15	34	42	50	58	77	85	93	101	120	128
63	6 4 1	5	22	25	39	53	67	70	84	98	101	115	129
64	7 4 1	3	16	28	40	52	64	76	88	100	101	113	125
65	8 4 1	7	17	33	38	54	59	75	80	96	101	117	133
66	9 4 1	4	19	32	45	47	60	73	86	99	101	114	127
67	10 4 1	6	14	29	44	48	63	78	82	97	101	116	131
68	0 5 1	1	35	36	37	38	39	40	41	42	43	44	45
69	1 5 1	8	23	29	35	52	58	75	81	98	104	121	127
70	2 5 1	5	18	32	35	49	63	77	80	94	108	122	125
71	3 5 1	4	20	33	35	48	61	74	87	100	102	115	128
72	4 5 1	9	21	28	35	53	60	78	85	92	110	117	124
73	5 5 1	12	15	25	35	56	66	76	86	96	106	116	126
74	6 5 1	3	22	34	35	47	59	71	83	95	107	119	131
75	7 5 1	6	16	31	35	50	65	69	84	99	103	118	133
76	8 5 1	11	17	26	35	55	64	73	82	91	111	120	129
77	9 5 1	10	19	27	35	54	62	70	89	97	105	113	132
78	10 5 1	7	14	30	35	51	67	72	88	93	109	114	130
79	0 6 1	1	112	113	114	115	116	117	118	119	120	121	122
80	1 6 1	7	23	28	44	49	65	70	86	91	107	112	128
81	2 6 1	10	18	26	45	53	61	69	88	96	104	112	131
82	3 6 1	11	20	29	38	47	67	76	85	94	103	112	132
83	4 6 1	6	21	25	40	55	59	74	89	93	108	112	127
84	5 6 1	3	15	27	39	51	63	75	87	99	111	112	124
85	6 6 1	12	22	32	42	52	62	72	82	92	102	112	133
86	7 6 1	9	16	34	41	48	66	73	80	98	105	132	130
87	8 6 1	4	17	30	43	56	58	71	84	97	110	112	125
88	9 6 1	5	19	33	36	50	64	78	81	95	109	112	126
89	10 6 1	8	14	31	37	54	60	77	83	100	106	112	129
90	0 7 1	1	46	47	48	49	50	51	52	53	54	55	56
91	1 7 1	6	23	27	42	46	61	76	80	95	110	114	129
92	2 7 1	4	18	31	44	46	59	72	85	98	111	113	126
93	3 7 1	7	20	25	41	46	62	78	83	99	104	120	125
94	4 7 1	3	21	33	45	46	58	70	82	94	106	118	130
95	5 7 1	5	15	29	43	46	60	74	88	91	105	119	133
96	6 7 1	10	22	30	38	46	65	73	81	100	108	116	124
97	7 7 1	12	16	26	36	46	67	77	87	97	107	117	127
98	8 7 1	8	17	34	40	46	63	69	86	92	109	115	132
99	9 7 1	11	19	28	37	46	66	75	84	93	102	122	131
100	10 7 1	9	14	32	39	46	64	71	89	96	103	121	128
101	0 8 1	1	57	58	59	60	61	62	63	64	65	66	67
102	1 8 1	5	23	26	40	54	57	71	85	99	102	116	130
103	2 8 1	9	18	25	43	50	57	75	82	100	107	114	132
104	3 8 1	3	20	32	44	56	57	69	81	93	105	117	129

105	4 8 1	11	21	30	39	48	57	77	86	95	104	113	133
106	5 8 1	7	15	31	36	52	57	73	89	94	110	115	131
107	6 8 1	8	22	28	45	51	57	74	80	97	103	120	126
108	7 8 1	4	16	29	42	55	57	70	83	96	109	122	124
109	8 8 1	12	17	27	37	47	57	78	88	98	108	118	128
110	9 8 1	6	19	34	38	53	57	72	87	91	106	121	125
111	10 8 1	10	14	33	41	49	57	76	84	92	11	119	127
112	0 9 1	1	79	80	81	82	83	84	85	86	87	88	89
113	1 9 1	4	23	25	38	51	64	77	79	92	105	118	131
114	2 9 1	3	18	30	42	54	66	78	79	91	103	115	127
115	3 9 1	10	20	28	36	55	63	71	79	98	106	114	133
116	4 9 1	8	21	27	44	50	67	73	79	96	102	119	125
117	5 9 1	9	15	33	40	47	65	72	79	97	104	122	129
118	6 9 1	6	22	26	41	56	60	75	79	94	109	113	128
119	7 9 1	7	16	32	37	53	58	74	79	95	111	116	132
12	8 9 1	5	17	31	45	48	62	76	79	93	107	121	124
121	9 9 1	12	19	29	39	49	59	69	79	100	110	120	130
122	10 9 1	11	14	34	43	52	61	70	79	99	108	117	126
123	0 10 1	1	24	25	26	27	28	29	30	31	32	33	34
124	1 10 1	3	23	24	36	48	60	72	84	96	108	120	132
125	2 10 1	8	18	24	41	47	64	70	87	93	110	116	133
126	3 10 1	6	20	24	39	54	58	73	88	92	107	122	126
127	4 10 1	5	21	24	38	52	66	69	83	97	111	114	128
128	5 10 1	11	15	24	44	53	62	71	80	100	109	118	127
129	6 10 1	4	22	24	37	50	63	76	89	91	104	117	130
130	7 10 1	10	16	24	43	51	59	78	86	94	102	121	129
131	8 10 1	9	17	24	42	49	67	74	81	99	106	113	131
132	9 10 1	7	19	24	40	56	61	77	82	98	103	119	124
133	10 10 1	12	14	24	45	55	65	75	85	95	105	115	125

إنشاء الأقواس الكاملة (k,n) في المستوي الاسقاطي $PG(2,11)$ حول حقل كالوا $GF(11)$ حيث ان $3 \leq n \leq 11$

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الخلاصة

ان الغاية الاساسية من هذا البحث هو ايجاد قوس كامل (k,n) في الفضاء الاسقاطي الثنائي $PG(2,q)$ حول حقل كالو $GF(11)$ وذلك بواسطة اضافة بعض النقاط دليلها صفر الى القوس الكامل $(k,n-1)$ حيث $3 \leq n \leq 11$.

القوس (k,n) هو مجموعة k من النقاط ليس هنالك $n+1$ على استقامة واحدة.

القوس الكامل (k,n) هو قوس لا يمكن ان يكون محتوى في القوس $(k+1,n)$.