Applications of the Operator ${}_r\Phi_s$ in q-Integrals

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ARTICLE INFO	ABSTRACT

Keywords
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We establish general q-operator ${}_r\Phi_s$ and subsequently discover some of its operator identities, which we use to generalize various q-integrals such as the Andrews-Askey integral, Gasper integral, and Askey-Wilson integral. In q-integrals, we specify exact values for achieving certain new results or reproofing others.

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1. Introduction

In this paper, we employ the notations and terminology from [10], assuming that $0 < q < 1$. Let a be a complex variable. The q-shifted factorial is defined as follows:

$$(a;q)_0 = 1, \quad (a;q)_m = \prod_{k=0}^{m-1} (1 - aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

The multiple q-shifted factorial is given by:

$$(a_1, a_2, \dots, a_r; q)_m = (a_1;q)_m (a_2;q)_m \cdots (a_r;q)_m,$$

where $m \in \mathbb{Z}\ or\ \infty$, $r \in \mathbb{Z}^+\ and\ a_1, \dots, a_r \in \mathbb{C}$.

For non-negative integers r and s, the basic hypergeometric series ${}_r\phi_s$ is known as [10]

$${}_r\Phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, x\right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} x^k,$$

where $r, s \in \mathbb{N}$; $a_1, \dots, a_r, b_1, \dots, b_s \in \mathbb{C}$.

In this paper, we will use the following identities [10]:

$$(a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k} \ (-1)^k q^{\binom{k}{2}-nk} \ (\frac{q}{a})^k. \tag{1.1}$$

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}} \ (-1)^k q^{\binom{k}{2}-nk}. \tag{1.2}$$

$$(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}. \tag{1.3}$$

The q-binomial coefficient is presented as follows [10]:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k \, (q;q)_{n-k}}, & if \ \ 0 \le k \le n; \\ 0, & otherwise, \end{cases}$$

where n, $k \in \mathbb{N}$.

The Cauchy identity is

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1.$$

Euler provided the following special case of Cauchy identity [10]

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} x^n = (x;q)_\infty.$$

-Chu-Vandermonde's identity is given as [10]: q

$${}_2\phi_1\left(\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, q\right) = \frac{(c/b;q)_n}{(c;q)_n} b^n. \tag{1.4}$$

The q-Gauss sum is given as [10]:

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, c/ab\right) = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty}. \tag{1.5}$$

The transformation of ${}_3\phi_2$ is given by [10, Appendix III, equation (III.9)]:

$${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, de/abc\right) = \frac{(e/a, de/bc;q)_\infty}{(e, de/abc;q)_\infty} {}_3\phi_2\left(\begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; q, e/a\right). \tag{1.6}$$

The transformation of ${}_3\phi_2$ is given by [10, Appendix III, equation (III.12)]:

$${}_3\phi_2\left(\begin{matrix} q^{-n}, b, c \\ d, e \end{matrix}; q, q\right) = \frac{(e/c;q)_n}{(e;q)_n} c^n {}_3\phi_2\left(\begin{matrix} q^{-n}, c, d/b \\ d, cq^{1-n}/e \end{matrix}; qbq/e\right). \tag{1.7}$$

Definition 1.1 [5]. *The q-differential operator, or q-derivative, is defined by*

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}. \tag{1.8}$$

Theorem 1.2 [5]. *For* $n \geq 0$, *we have*

$$D_q^n\{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{f(a)\} D_q^{n-k}\{g(aq^k)\}. \tag{1.9}$$

Theorem 1.3 [19]. *Let* D_q *be defined as in* (1.8). *then*

$$D_q^k\left\{\frac{(av;q)_\infty}{(at;q)_\infty}\right\} = t^k (v/t;q)_k \frac{(avq^k;q)_\infty}{(at;q)_\infty}, \quad |at| < 1. \tag{1.10}$$

The Andrews-Askey integral is given by [1, 2]:

$$\int_c^d \frac{(qt/c, qt/d;q)_\infty}{(at, bt;q)_\infty} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd;q)_\infty}{(ac, ad, bc, bd;q)_\infty}, \tag{1.11}$$

where $\max\{|a|, |b|, |c|, |d|\} < 1$, $cd \neq 0$.

The Gasper integral [9] is given by:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfe^{i\theta};q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta};q)_\infty} d\theta$$

$$= \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf; q)_\infty}{(q, ac, ad, bc, bd, cf, df; q)_\infty}, \tag{1.12}$$

where $\max\{|a|, |b|, |c|, |d|\} < 1$, $cd\rho \neq 0$.

The Askey-Wilson integral is stated as follows [3, 11, 12]:

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; a, b, c, d)} d\theta = \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, \tag{1.13}$$

where $\max\{|a|, |b|, |c|, |d|\} < 1$.

In 1997, Chen and Liu [5] constructed the following q-exponential operator*:*

Definition 1.4 [5]. *The q-exponential operator $T(bD_q)$ is defined by*

$$T(bD_q) = \sum_{k=0}^{\infty} \frac{(bD_q)^k}{(q; q)_k}. \tag{1.14}$$

In light of parameter augmentation, they employed the q-exponential operator $T(bD_q)$ to obtain an extension to the Andrews-Askey integral (1.11) as follows:

Theorem 1.5 [5]. *Let $T(bD_q)$ be defined as in* (1.14), *then*

$$\int_c^d \frac{(qt/c, qt/d, abcdet; q)_\infty}{(at, bt, et; q)_\infty} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd, bcde, acde; q)_\infty}{(ac, ad, bc, bd, ce, de; q)_\infty}. \tag{1.15}$$

In 2005, Zhang and Wang [18] employed the q-exponential operator $T(bD_q)$ to offer the following extension to the Gasper integral (1.12):

Theorem 1.6 [18]. *Let $T(bD_q)$ be defined as in* (1.14), *then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfe^{i\theta}, bcdfge^{i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, fe^{i\theta}, ce^{-i\theta}, de^{-i\theta}, ge^{i\theta}; q)_\infty}$$

$$\times\ {}_3\phi_2\left(\begin{matrix} bcdf, fe^{i\theta}, be^{i\theta} \\ abcdfe^{i\theta}, bcdfge^{i\theta} \end{matrix}; q, acdg\right) d\theta$$

$$= \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf, bcdg, cdfg; q)_\infty}{(q, ac, ad, bc, bd, cf, df, cg, dg; q)_\infty}. \tag{1.16}$$

In 2008, Chen and Gu [4] introduced the Cauchy operator $T(a,b;D_q)$ as follows:

Definition 1.7 [4]. *The Cauchy operator $T(a,b;D_q)$ is defined by*

$$T(a,b;D_q)=\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}(bD_q)^k. \tag{1.17}$$

Chen and Gu [4] used the operator $T(a,b;D_q)$ to give an extension for the Gasper integral (1.12) as follows:

Theorem 1.8 [4]. *Let $T(a,b;D_q)$ be defined as in* (1.17), *then*

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\rho e^{i\theta}/d,qde^{-i\theta}/\rho,\rho ce^{-i\theta},qe^{i\theta}/c\rho,abcdfe^{i\theta},ghe^{i\theta};q)_\infty}{(ae^{i\theta},be^{i\theta},ce^{-i\theta},de^{-i\theta},fe^{i\theta},he^{i\theta};q)_\infty}\\&\times{}_3\phi_2\left(\begin{matrix}g,ae^{i\theta},fe^{i\theta}\\abcdfe^{i\theta},ghe^{i\theta}\end{matrix};q,bcdh\right)d\theta\\&=\frac{(abcd,\rho c/d,dq/\rho c,\rho,q/\rho,acdf,bcdf,cgh;q)_\infty}{(q,ac,ad,bc,bd,cf,ch,df;q)_\infty}{}_3\phi_2\left(\begin{matrix}g,ac,fc\\cgh,acdf\end{matrix};q,dh\right).\end{aligned} \tag{1.18}$$

In 2016, Li and Tan [13] constructed the generalized q-exponential operator $\mathbb{T}\left[\begin{smallmatrix}u,v\\w\end{smallmatrix}\middle|q;cD_q\right]$ with three parameters as follows:

Definition 1.9 [13]. *The generalized q-exponential operator $\mathbb{T}\left[\begin{smallmatrix}u,v\\w\end{smallmatrix}\middle|q;cD_q\right]$ is defined by*

$$\mathbb{T}\left[\begin{matrix}u,v\\w\end{matrix}\middle|q;cD_q\right]=\sum_{n=0}^{\infty}\frac{(u,v;q)_n}{(q,w;q)_n}(cD_q)^n. \tag{1.19}$$

Li and Tan [13] used the operator $\mathbb{T}\left[\begin{smallmatrix}u,v\\w\end{smallmatrix}\middle|q;cD_q\right]$ to obtain the following generalization for the Askey-Wilson integral (1.13):

Theorem 1.10 [13]. *Let $\mathbb{T}\left[\begin{smallmatrix}u,v\\w\end{smallmatrix}\middle|q;cD_q\right]$ be defined as in* (1.19), *then*

$$\begin{aligned}&\int_0^{\pi}\frac{h(\cos2\theta;1)}{h(\cos\theta;a,b,c,d)}\sum_{n,k\geq0}\frac{(u,v;q)_{n+k}}{(q;q)_n(w;q)_{n+k}}\frac{(ae^{-i\theta},be^{-i\theta};q)_k}{(q,ab;q)_k}f^{n+k}e^{(k-n)i\theta}d\theta\\&=\frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}\sum_{n,k\geq0}\frac{(u,v;q)_{n+k}}{(q;q)_n(w;q)_{n+k}}\frac{(ad,bd)_k}{(q,abcd;q)_k}f^{n+k}d^nc^k.\end{aligned} \tag{1.20}$$

The following is how our paper is organized. In section 2, we build a generalized q-operator ${}_r\Phi_s$ and then determine its identities. In section 3, we employ the operator ${}_r\Phi_s$ to generalize the Andrews-Askey, Gasper, and Askey-Wilson integrals.

2. The General Operator ${}_r\Phi_s$ and its Identities

In this section, we introduce the general operator ${}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,cD_q\right)$. Then we give some of its identities.

Definition 2.1 *We define the general q-operator* ${}_r\Phi_s$ *as follows*:

$$ {}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,cD_q\right)=\sum_{n=0}^{\infty}\frac{W_n}{(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\left(cD_q\right)^n, \tag{2.1}$$

where $W_n=\frac{(a_1,\cdots,a_r;q)_n}{(b_1,\dots,b_s;q)_n}$.

In order to obtain many previously stated operators, several specific values may be assigned to the general q-operator ${}_r\Phi_s$ as follows:

- Setting $r=1$, $s=0$, $a_1=0$ and $c=b$, we get on the exponential operator $T(bD_q)$ defined by Chen and Liu [5] in 1997.
- If $r=1, s=0$, $a_1=a$ and $c=b$, we get on the Cauchy operator $T(a,b;D_q)$ defined by Chen and Gu [4] in 2008.
- When $r=2$, $s=1$, $a_1=u, a_2=v$ and $b_1=w$, we get on the generalized exponential operator with three parameters $\mathbb{T}\left[\begin{matrix}u,v\\ w\end{matrix}\middle|q;cD_q\right]$ constructed by Li and Tan [13] in 2016.

Other special values for the generalized q-operator ${}_r\Phi_s$ can be used to describe operators like those in [6,7.8, 14, 15, 16, 17,19].

The following operator identities will be derived using q-Liebniz formula (1.9).

Theorem 2.2 *Let* ${}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,cD_q\right)$ *be defined as in* (2.1), *then*

$$ {}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,cD_q\right)\left\{\frac{(av,au;q)_\infty}{(at,aw;q)_\infty}\right\}=\frac{(av,au;q)_\infty}{(at,aw;q)_\infty}$$

$$\times\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{W_{n+k}}{(q;q)_n}\frac{(v/t,aw;q)_k}{(q,av;q)_k}\frac{(u/w;q)_n}{(au;q)_{n+k}}\left[(-1)^{n+k}q^{\binom{n+k}{2}}\right]^{1+s-r}(cw)^n(ct)^k, \tag{2.2}$$

provided that $\max\{|at|,|aw|\}<1$.

Proof.

$$
{}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,cD_q\right)\left\{\frac{(av,au;q)_\infty}{(at,aw;q)_\infty}\right\}
$$

$$
=\sum_{n=0}^{\infty}\frac{W_n}{(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} c^n D_q^n\left\{\frac{(av;q)_\infty}{(at;q)_\infty}\frac{(au;q)_\infty}{(aw;q)_\infty}\right\} \qquad \text{(by using (2.1))}
$$

$$
=\sum_{n=0}^{\infty}\frac{W_n}{(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} c^n
$$

$$
\times\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix} q^{k^2-nk} D_q^k\left\{\frac{(av;q)_\infty}{(at;q)_\infty}\right\} D_q^{n-k}\left\{\frac{(auq^k;q)_\infty}{(awq^k;q)_\infty}\right\} \qquad \text{(by using (1.9))}
$$

$$
=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{W_n}{(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} c^n \begin{bmatrix}n\\k\end{bmatrix} q^{k^2-nk}
$$

$$
\times\, t^k\frac{(v/t;q)_k(avq^k;q)_\infty}{(at;q)_\infty}(wq^k)^{n-k}\frac{(u/w;q)_{n-k}(auq^n;q)_\infty}{(awq^k;q)_\infty} \qquad \text{(by using (1.10))} \qquad (2.3)
$$

$$
=\frac{(av,au;q)_\infty}{(at,aw;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{W_{n+k}}{(q;q)_n}\frac{(v/t,aw;q)_k}{(q,av;q)_k}\frac{(u/w;q)_n}{(au;q)_{n+k}}\left[(-1)^{n+k}q^{\binom{n+k}{2}}\right]^{1+s-r}(cw)^n(ct)^k.
$$

■

Setting $u=0$ in equation (2.2), we get the following corollary:

Corollary 2.1 *Let* ${}_r\Phi_s\left(\begin{smallmatrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{smallmatrix};q,cD_q\right)$ *be defined as in* (2.1), *then*

$$
{}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,cD_q\right)\left\{\frac{(av;q)_\infty}{(at,aw;q)_\infty}\right\}=\frac{(av;q)_\infty}{(at,aw;q)_\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{W_{n+k}}{(q;q)_n}\frac{(v/t,aw;q)_k}{(q,av;q)_k}
$$

$$
\times\left[(-1)^{n+k}q^{\binom{n+k}{2}}\right]^{1+s-r}(cw)^n(ct)^k, \quad \max\{|at|,|aw|\}<1. \qquad (2.4)
$$

Setting $u=0$ and then $w=0$ in equation (2.2), we get the following corollary:

Corollary 2.2 *Let* ${}_r\Phi_s\left(\begin{smallmatrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{smallmatrix};q,cD_q\right)$ *be defined as in* (2.1), *then*

$$
{}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,cD_q\right)\left\{\frac{(av;q)_\infty}{(at;q)_\infty}\right\}=\frac{(av;q)_\infty}{(at;q)_\infty}\sum_{k=0}^{\infty}\frac{W_k}{(q;q)_k}\frac{(v/t;q)_k}{(av;q)_k}
$$

$$\times\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(ct)^k, \qquad |at|<1. \qquad (2.5)$$

3. Applications in q-Integrals

In this section, we will use the general q-operator ${}_r\Phi_s$ to generalize various q-integrals, such as the Andrews-Askey integral, the Gasper integral, and the Askey-Wilson integral. We can use special values in q-integrals to achieve new findings or recover others.

3.1 Generalization of the Andrews-Askey Integral

Theorem 3.1**.** (Generalization of the Andrews-Askey integral). *Let* ${}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,cD_q\right)$ *be defined as in* (2.1), *then*

$$\int_c^d \frac{(qt/c,qt/d;q)_\infty}{(at,bt;q)_\infty}\sum_{k=0}^{\infty}\frac{W_k}{(q;q)_k}\frac{(f/t;q)_k}{(af;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(et)^k d_qt$$

$$=\frac{d(1-q)(q,dq/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty}\sum_{n,k\geq 0}\frac{W_{n+k}}{(q;q)_n}\frac{(f/d;q)_n}{(af;q)_{n+k}}\ (ed)^n$$

$$\times\left[(-1)^{n+k}q^{\binom{n+k}{2}}\right]^{1+s-r}\frac{(ad,bd;q)_k}{(q,abcd;q)_k}\ (ec)^k. \qquad (3.1)$$

Proof. Multiplying equation (1.11) by $(af;q)_\infty$, we have

$$\int_c^d \frac{(qt/c,qt/d;q)_\infty}{(bt;q)_\infty}\frac{(af;q)_\infty}{(at;q)_\infty}d_qt=\frac{d(1-q)(q,dq/c,c/d;q)_\infty}{(bc,bd;q)_\infty}\frac{(af,abcd;q)_\infty}{(ac,ad;q)_\infty}. \qquad (3.2)$$

Applying the operator ${}_r\Phi_s$ on both sides of (2.7), we get

$$\int_c^d \frac{(qt/c,qt/d;q)_\infty}{(bt;q)_\infty}\ {}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,eD_q\right)\left\{\frac{(af;q)_\infty}{(at;q)_\infty}\right\}d_qt$$

$$=\frac{d(1-q)(q,dq/c,c/d;q)_\infty}{(bc,bd;q)_\infty}\ {}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,eD_q\right)\left\{\frac{(af,abcd;q)_\infty}{(ac,ad;q)_\infty}\right\}. \qquad (3.3)$$

Using equation (2.5), we get

$${}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,eD_q\right)\left\{\frac{(af;q)_\infty}{(at;q)_\infty}\right\}d_qt$$

$$=\frac{(af;q)_\infty}{(at;q)_\infty}\sum_{k=0}^{\infty}\frac{W_k}{(q;q)_k}\frac{(f/t;q)_k}{(af;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}(et)^k. \qquad (3.4)$$

and using equation (2.2), we get

$$ {}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,eD_q\right)\left\{\frac{(af,abcd;q)_\infty}{(ac,ad;q)_\infty}\right\}=\frac{(af,abcd;q)_\infty}{(ac,ad;q)_\infty} $$

$$ \times\sum_{n,k\geq 0}\frac{W_{n+k}}{(q;q)_n}\frac{(f/d;q)_n}{(af;q)_{n+k}}(ed)^n\left[(-1)^{n+k}q^{\binom{n+k}{2}}\right]^{1+s-r}\frac{(ad,bd;q)_k}{(q,abcd;q)_k}(ec)^k. \qquad (3.5) $$

Substituting (3.4) and 3.5) into equation (3.3), we obtain the desired result. ■

- If $r=s=0$ and $f=0$ in equation (3.1), we obtain the following result:

Corollary 3.1.1 *We have*

$$ \int_c^d \frac{(qt/c,qt/d,et;q)_\infty}{(at,bt;q)_\infty}d_qt $$

$$ =\frac{d(1-q)(q,dq/c,c/d,abcd,ed;q)_\infty}{(ac,ad,bc,bd;q)_\infty}\,{}_2\Phi_2\left(\begin{matrix} ad,bd \\ ed,abcd;q,ec \end{matrix}\right). $$

- When $r=1$, $s=0$, $f=0$ and $a_1=abcd$ in equation (3.1), we recover Theorem 6.2. obtained by Chen and Liu [5] (equation (1.15)).

- If $r=2$, $s=1$, $a_1=q^{-N}$,$f=0$ and $e\to q/d$ in equation (3.1), by using q-Chu-Vandermond identity (1.4), and then equations (1.1), (1.3), we obtain the following result:

Corollary 3.1.2 *We have*

$$ \int_c^d \frac{(qt/c,qt/d;q)_\infty}{(at,bt;q)_\infty}\,{}_2\Phi_1\left(\begin{matrix} q^{-N},a_2 \\ b_1 \end{matrix};q,\frac{qt}{d}\right)d_qt $$

$$ =\frac{{a_2}^N d(1-q)(q,dq/c,c/d,abcd,b_1q^N,b_1/a_2;q)_\infty}{(ac,ad,bc,bd,b_1,b_1q^N/a_2;q)_\infty} $$

$$ \times\,{}_4\Phi_2\left(\begin{matrix} q^{-N},a_2,ad,bd \\ abcd,a_2q^{N-1}/b_1 \end{matrix};q,qc/b_1d\right). $$

3.2 Generalization of Gasper Integral

Theorem 3.2 (Generalization of Gasper integral). *Let* ${}_r\Phi_s\left(\begin{smallmatrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{smallmatrix};q,cD_q\right)$ *be defined as in* (2.1), *then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfe^{i\theta};q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta};q)_\infty}$$

$$\times\sum_{k,n\geq 0}\frac{W_{n+k}}{(q;q)_n}\frac{(abcd, bcdf;q)_k}{(q, abcdfe^{i\theta};q)_k}\ (ge^{i\theta})^k(gbcd)^n\left[(-1)^{n+k}q^{\binom{n+k}{2}}\right]^{1+s-r}d\theta$$

$$=\frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf;q)_\infty}{(q, ac, ad, bc, bd, cf, df;q)_\infty}$$

$$\times\sum_{k,n\geq 0}\frac{W_{n+k}}{(q;q)_n}\frac{(ad, fd;q)_k}{(q, acdf;q)_k}\left[(-1)^{n+k}q^{\binom{n+k}{2}}\right]^{1+s-r}(gc)^k(gd)^n. \qquad (3.6)$$

Proof. Rewrite (1.12) as:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, fe^{i\theta};q)_\infty}{(be^{i\theta}, ce^{-i\theta}, de^{-i\theta};q)_\infty}\frac{(abcdfe^{i\theta};q)_\infty}{(ae^{i\theta}, abcd;q)_\infty}d\theta$$

$$=\frac{(\rho c/d, dq/\rho c, \rho, q/\rho, bcdf;q)_\infty}{(q, bc, bd, cf, df;q)_\infty}\frac{(acdf;q)_\infty}{(ac, ad;q)_\infty}.$$

Applying the operator ${}_r\Phi_s$ on both sides of (3.7), we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, fe^{i\theta};q)_\infty}{(be^{i\theta}, ce^{-i\theta}, de^{-i\theta};q)_\infty}$$

$$\times\ {}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q, gD_q\right)\left\{\frac{(abcdfe^{i\theta};q)_\infty}{(ae^{i\theta}, abcd;q)_\infty}\right\}d\theta$$

$$=\frac{(\rho c/d, dq/\rho c, \rho, q/\rho, bcdf;q)_\infty}{(q, bc, bd, cf, df;q)_\infty}\ {}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q, gD_q\right)\left\{\frac{(acdf;q)_\infty}{(ac, ad;q)_\infty}\right\}. \qquad (3.8)$$

Using equation (2.4) on both sides of (3.8), we get

$${}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q, gD_q\right)\left\{\frac{(abcdfe^{i\theta};q)_\infty}{(ae^{i\theta}, abcd;q)_\infty}\right\}=\frac{(abcdfe^{i\theta};q)_\infty}{(ae^{i\theta}, abcd;q)_\infty}$$

$$\times\sum_{k,n\geq 0}\frac{W_{n+k}}{(q;q)_n}\frac{(abcd, bcdf;q)_k}{(q, abcdfe^{i\theta};q)_k}\left[(-1)^{n+k}q^{\binom{n+k}{2}}\right]^{1+s-r}(ge^{i\theta})^k(gbcd)^n. \qquad (3.9)$$

and

$${}_r\Phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q, gD_q\right)\left\{\frac{(acdf;q)_\infty}{(ac, ad;q)_\infty}\right\}$$

$$= \frac{(acdf;q)_\infty}{(ac,ad;q)_\infty} \sum_{k,n\geq 0} \frac{W_{n+k}}{(q;q)_n} \frac{(ad,fd;q)_k}{(q,acdf;q)_k} \left[(-1)^{n+k} q^{\binom{n+k}{2}}\right]^{1+s-r} (gc)^k (gd)^n. \qquad (3.10)$$

Substituting (3.9) and (3.10) into equation (3.8), we get the desired result. ■

- If $r = s = 0$ in equation (3.6), we get the following result:

Corollary 3.2.3 *We have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfe^{i\theta};q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta};q)_\infty}$$
$$\times\ {}_2\phi_2\left(\begin{matrix} abcd, bcdf \\ gbcd, abcdfe^{i\theta} \end{matrix}; q, ge^{i\theta}\right) d\theta$$
$$= \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf, gd;q)_\infty}{(q, ac, ad, bc, bd, cf, df, gbcd;q)_\infty}\ {}_2\phi_2\left(\begin{matrix} ad, fd \\ acdf, gd \end{matrix}; q, gc\right).$$

- If $r = 1$ and $s = 0$ in (3.6), we get the following equation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfe^{i\theta};q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta};q)_\infty}$$
$$\times \frac{(a_1gbcd;q)_\infty}{(gbcd;q)_\infty}\ {}_3\phi_2\left(\begin{matrix} a_1, bcdf, abcd \\ abcdfe^{i\theta}, a_1gbcd \end{matrix}; q, ge^{i\theta}\right) d\theta$$
$$= \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf, a_1gd;q)_\infty}{(q, ac, ad, bc, bd, cf, df, gd;q)_\infty}\ {}_3\phi_2\left(\begin{matrix} a_1, fd, ad \\ acdf, a_1gd \end{matrix}; q, gc\right). \qquad (3.11)$$

For ${}_3\phi_2$ in the left hand side of (3.11), replacing a, b, c, d, e by a_1, $bcdf$, $abcd$, $abcdfe^{i\theta}$, a_1gbcd, respectively, in transformation formula (1.6), and for ${}_3\phi_2$ in the right hand side of (3.11), replacing a, b, c, d, e by a_1, df, ad, $acdf$, a_1gd, respectively, in transformations formula (1.6), then substituting the result in equation (3.11), we get the following corollary:

Corollary 3.2.4 *We have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfe^{i\theta}, a_1ge^{i\theta};q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta}, ge^{i\theta};q)_\infty}$$
$$\times\ {}_3\phi_2\left(\begin{matrix} a_1, ae^{i\theta}, fe^{i\theta} \\ abcdfe^{i\theta}, a_1ge^{i\theta} \end{matrix}; q, gbcd\right) d\theta$$

$$= \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf, a_1cg; q)_\infty}{(q, ac, ad, bc, bd, cf, df, gc; q)_\infty} {}_3\Phi_2\left(\begin{matrix} a_1, ac, fc \\ acdf, a_1cg; q, gd \end{matrix}\right). \qquad (3.12)$$

* If $g = h$ and then $a_1 = g$ in equation (3.12), we reestablish Theorem 4.1. obtained by Chen and Gu [4] (equation (1.18)).

* Exchanging a and b and then replacing a_1 by $bcdf$ in equation (3.12), then using the q-Gauss sum (1.5), we recover Theorem 4.2. obtained by Zhang and Wang [18] (equation 1.16)).

• If $r = 2$, $s = 1$ in equation (3.6), we have the following result:

Corollary 3.2.5 *We have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfe^{i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, fe^{i\theta}; q)_\infty}$$
$$\times \sum_{k=0}^{\infty} \frac{(a_1, a_2, bcdf, abcd; q)_k}{(q, 0, b_1, abcdfe^{i\theta}; q)_k} (ge^{i\theta})^k \sum_{n=0}^{\infty} \frac{(a_1q^k, a_2q^k; q)_n}{(q, b_1q^k; q)_n} (gbcd)^n d\theta$$
$$= \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, acdf, bcdf; q)_\infty}{(q, ac, ad, bc, bd, cf, df; q)_\infty} \sum_{k=0}^{\infty} \frac{(a_1, a_2, ad, fd; q)_k}{(q, 0, b_1, acdf; q)_k} (gc)^k$$
$$\times \sum_{n=0}^{\infty} \frac{(a_1q^k, a_2q^k; q)_n}{(q, b_1q^k; q)_n} (gd)^n.$$

3.3 Generalization of the Askey-Wilson Integral

Theorem 3.3 (Generalization of the Askey-Wilson integral). *Let* ${}_r\Phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, cD_q\right)$ *be defined as in* (2.1), *then*

$$\int_0^{\pi} \frac{h(\cos 2\theta; 1)}{h(\cos\theta; a, b, c, d)} \sum_{n,k\geq 0} \frac{W_{n+k}}{(q;q)_n}\left[(-1)^{n+k} q^{\binom{n+k}{2}}\right]^{1+s-r} f^{n+k} e^{(k-n)i\theta} \frac{(ae^{-i\theta}, be^{-i\theta}; q)_k}{(q, ab; q)_k} d\theta$$
$$= \frac{2\pi (abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}$$

$$\times \sum_{n,k\geq 0} \frac{W_{n+k}}{(q;q)_n} \left[(-1)^{n+k} q^{\binom{n+k}{2}}\right]^{1+s-r} f^{n+k} d^n c^k \frac{(ad,bd)_k}{(q,abcd;q)_k}. \tag{3.13}$$

Proof. Rewrite equation (1.13) as:

$$\int_0^{\pi} \frac{h(\cos 2\theta;1)}{h(\cos 2\theta;b,c,d)} \frac{(ab;q)_\infty}{(ae^{i\theta},ae^{-i\theta};q)_\infty} d\theta = \frac{2\pi}{(q,bc,bd,cd;q)_\infty} \frac{(abcd;q)_\infty}{(ac,ad;q)_\infty}. \tag{3.14}$$

Applying the operator ${}_r\Phi_s$ on both sides of (3.14), we get:

$$\int_0^{\pi} \frac{h(\cos 2\theta;1)}{h(\cos 2\theta;b,c,d)} \, {}_r\Phi_s \left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, fD_q\right) \left\{\frac{(ab;q)_\infty}{(ae^{i\theta},ae^{-i\theta};q)_\infty}\right\} d\theta$$

$$= \frac{2\pi}{(q,bc,bd,cd;q)_\infty} \, {}_r\Phi_s \left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, fD_q\right) \left\{\frac{(abcd;q)_\infty}{(ac,ad;q)_\infty}\right\}. \tag{3.15}$$

Using equation (2.4), on both sides of (3.15), we have:

$${}_r\Phi_s \left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, fD_q\right) \left\{\frac{(ab;q)_\infty}{(ae^{i\theta},ae^{-i\theta};q)_\infty}\right\} d\theta = \frac{(ab;q)_\infty}{(ae^{i\theta},ae^{-i\theta};q)_\infty}$$

$$\times \sum_{n,k\geq 0} \frac{W_{n+k}}{(q;q)_n} \frac{(ae^{-i\theta},be^{-i\theta};q)_k}{(q,ab;q)_k} \left[(-1)^{n+k} q^{\binom{n+k}{2}}\right]^{1+s-r} f^{n+k} e^{(k-n)i\theta}. \tag{3.16}$$

and

$${}_r\Phi_s \left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}; q, fD_q\right) \left\{\frac{(abcd;q)_\infty}{(ac,ad;q)_\infty}\right\}$$

$$= \frac{(abcd;q)_\infty}{(ac,ad;q)_\infty} \sum_{n,k\geq 0} \frac{W_{n+k}}{(q;q)_n} \frac{(ad,bd)_k}{(q,abcd;q)_k} \left[(-1)^{n+k} q^{\binom{n+k}{2}}\right]^{1+s-r} f^{n+k} d^n c^k. \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15), the proof is completed. ■

• If $r = s = 0$ in equation (3.13), we obtain the following result:

Corollary 3.3.6 *We have*

$$\int_0^{\pi} \frac{h(\cos 2\theta;1)(fe^{-i\theta},q)_\infty}{h(\cos\theta;a,b,c,d)} \, {}_2\phi_2 \left(\begin{matrix} ae^{-i\theta}, be^{-i\theta} \\ ab, fe^{-i\theta} \end{matrix}; q, fe^{i\theta}\right) d\theta$$

$$= \frac{2\pi(abcd, fd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty} \, {}_2\Phi_2\left(\begin{matrix} ad, bd \\ abcd, fd \end{matrix} ; q, fc\right).$$

• If $r = 1$, $s = 0$ in equation (3.13), we get the following result:

Corollary 3.3.7 *We have*

$$\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos\theta; a, b, c, d)} \frac{(a_1 f e^{-i\theta}; q)_\infty}{(f e^{-i\theta}; q)_\infty} \, {}_3\phi_2\left(\begin{matrix} a_1, a e^{-i\theta}, b e^{-i\theta} \\ ab, a_1 f e^{-i\theta} \end{matrix} ; q, f e^{i\theta}\right) d\theta$$
$$= \frac{2\pi(abcd, a_1 fd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd, fd; q)_\infty} \, {}_3\Phi_2\left(\begin{matrix} a_1, ad, bd \\ abcd, a_1 fd \end{matrix} ; q, fc\right).$$

• If $r = 2$, $s = 1$ in equation (3.13), we regain Theorem 22 obtained by Li and Tan [13] (equation (1.20)).

4. Conclusions

Some well-known q-integrals, including the Andrews-Askey, Gasper, and Askey-Wilson integrals, can be successfully generalized using the generalized operator ${}_r\Phi_s$.

References

[1] G. E. Andrews and R. Askey, Another q-extension of the beta function, Proc. Amer. Math. Soc., 81 (1981) 97–100, https://doi.org/10.2307/2043995

[2] R. Askey, A q-extension of Cauchy's form of the beta integral, Q. J. Math., 32 (1981) 255–266, https://doi.org/10.1093/qmath/32.3.255

[3] R. Askey and J. A. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, Mem. Amer. Math. Soc., 54 (1985) 319, 1985, https://doi.org/10.1090/memo/0319

[4] V. Y. B. Chen and N. S. S. Gu, The Cauchy operator for basic hypergeometric series, Advances in Applied Mathematics, 41 (2008) 177–196, https://doi.org/10.1016/j.aam.2007.08.001

[5] W. Y. C. Chen and Z. G. Liu, Parameter augmenting for basic hypergeometric series, II,

J. Combin. Theory, Ser. A, 80 (1997) 175–195, https://doi.org/10.1006/jcta.1997.2801

[6] J-P. Fang, Extensions of q-Chu-Vandermonde's identity, J. Math. Anal. Appl., 339 (2008) 845-852, https://doi.org/10.1016/j.jmaa.2007.07.029

[7] J-P. Fang, Some applications of q-differential operator, J. Korean Math. Soc., 47 (2010) 223-233, https://doi.org/10.4134/JKMS.2010.47.2.223

[8] J-P. Fang, Applications of a generalized q-difference equation, Advances in Difference Equations, 2014 (2014) 1-19, https://doi.org/10.1186/1687-1847-2014-267

[9] G. Gasper, q-Extensions of Barnes', Cauchy's, and Euler's beta integrals, Topics Math. Anal., (1989) 294–314, https://doi.org/10.1142/9789814434201_0013

[10] G. Gasper and M. Rahman, Basic Hypergeometric Series, 2^{nd} ed., Cambridge University Press, Cambridge, MA, 2004, https://doi.org/10.1017/CBO9780511526251

[11] M. E. H. Ismail, D. Stanton and G. Viennot, The combinatorics of q-Hermite polynomials and the Askey-Wilson integral, European J. Combin., 8 (1987) 379–392, https://doi.org/10.1016/S0195-6698(87)80046-X

[12] M. E. H. Ismail and D. Stanton, On the Askey-Wilson and Rogers polynomials, Canad. J. Math., 40 (1988) 1025–1045, https://doi.org/10.1016/S0195-6698(87)80046-X

[13] N. N. Li, W. Tan, Two generalized q-exponential operators and their applications, Adv. Difference Equ., 53 (2016) 1–14, https://doi.org/10.1186/s13662-016-0741-6

[14] H.L. Saad, M.A. Abdlhusein, New application of the Cauchy operator on the homogeneous Rogers-Szego polynomials, The Ramanujan Journal, 56 (2021) 347–367, https://doi.org/10.1007/s11139-021-00432-9

[15] H. L. Saad, R. H. Jaber, Application of the Operator $\phi\begin{pmatrix} a,b,c \\ d,e \end{pmatrix};q,fD_q\Big)$ for the polynomials $Y_n(a,b,c;d,e;x,y|q)$, TWMS J. App. and Eng. Math., 12(2) (2022)691-702.

[16] H. L. Saad and A. A. Sukhi, The q-Exponential Operator, Appl. Math. Sci., 7 (2013) 6369–6380, http://dx.doi.org/10.12988/ams.2013.24287

[17] H. M. Srivastava and S. Arjika, Generating functions for some families of the generalized Al-Salam–Carlitz q-polynomials, Adv. Difference Equ., 498 (2020) 1–17, https://doi.org/10.1186/s13662-020-02963-9

[18] Z. Z. Zhang and J. Wang, Two operator identities and their applications to terminating basic hypergeometric series and q-integrals, J. Math. Anal. Appl., 312 (2005) 653–665, https://doi.org/10.1016/j.jmaa.2005.03.064

[19] Z. Z. Zhang and J. Z. Yang, Finite q-exponential operators with two parameters and their Applications, Acta Mathematica Sinica, Chinese Series, 53 (2010) 1007-1018, https://doi.org/10.12386/A20100112

تطبيقات المؤثر $_r\Phi_s$ في التكاملات-q

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المستخلص

أنشأنا مؤثر-q العام $_r\Phi_s$ بعد ذلك أوجدنا بعض معادلاته ، والتي نستخدمها لتعميم تكاملات-q المختلفة مثل تكامل Andrews-Askey ، تكامل Gasper ، وتكامل Askey-Wilson. في تكاملات-q ، نحدد قيمًا دقيقة لأيجاد نتائج جديدة معينة أو إعادة برهان نتائج أخرى.