

(σ, τ) - Strongly Derivations Pairs on Rings

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Abstract

Let R be an associative ring. In this paper we present the definition of (σ, τ) - Strongly derivation pair and Jordan (σ, τ) - strongly derivation pair on a ring R , and study the relation between them. Also, we study prime rings, semiprime rings, and rings that have commutator left nonzero divisor with (σ, τ) - strongly derivation pair, to obtain a (σ, τ) - derivation. Where $\sigma, \tau: R \rightarrow R$ are two mappings of R .

Keywords

Prime ring, semiprime ring, (σ, τ) -derivation, (σ, τ) -Strongly derivation pair, Jordan (σ, τ) -Strongly derivation pair.

§₁ Basic Concepts

Deinition 1.1: [1]

A nonempty set R is said to be associative ring if in R there are defined two operations, denoted by $+$ and $\cdot$ respectively, such that for all a, b, c in R :

- 1- $a + b$ is in R
- 2- $a + b = b + a$
- 3- $(a+b) + c = a + (b+c)$
- 4- There is an element 0 in R such that $a+0 = a$ (for every a in R)
- 5- There exists an element $-a$ in R such that $a + (-a)=0$.
- 6- $a \cdot b$ is in R .
- 7- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- 8- $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot a + c \cdot a$

Deinition 1.2: [1]

A ring R is called prime ring if for any $a, b \in R$, $a R b = \{0\}$, implies that either $a=0$ or $b=0$.

Definition 1.3:[1]

A ring R is called semiprime ring if for any $a \in R$, $aRa = \{0\}$, implies that $a=0$.

Remark 1.4:[1]

Every prime ring is semiprime ring, but the converse in general is not true. The following example justifies this remark.

Example 1.5: [1]

$R = Z_6$ is a semiprime ring but is not prime.

Let $a \in R$ such that $aRa = \{0\}$, implies that $a^2 = 0$, hence $a=0$, therefore R is a semiprime ring. But R is not prime, since $2 \neq 0$ and $3 \neq 0$ implies that $2R3 = \{0\}$.

Definition 1.6:[2]

A ring R is said to be n -torsion free, where $n \neq 0$ is an integer if whenever $na=0$, with $a \in R$, then $a=0$.

Definition 1.7:[2]

Let R be a ring. A Lie product $[\cdot, \cdot]$ on R is defined as $[x, y] = xy - yx$, for all $x, y \in R$.

Definition 1.8:[2]

Let R be a ring. An additive mapping $d: R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$ and we say that d is a Jordan derivation if $d(x^2) = d(x)x + xd(x)$, for all $x \in R$.

Definition 1.9:[3]

Let R be a ring. An additive mapping $d: R \rightarrow R$ is called a (σ, τ) -derivation, where $\sigma, \tau: R \rightarrow R$ are two mappings of R , if

$d(xy) = d(x)\sigma(y) + \tau(x)d(y)$, for all $x, y \in R$, and we say that d is a Jordan (σ, τ) -derivation if $d(x^2) = d(x)\sigma(x) + \tau(x)d(x)$, for all $x \in R$.

Definition 1.10:[4]

Let R be a ring, additive mappings $d, g: R \rightarrow R$ is called S-derivation pair (d, g) if satisfies the following equations:

$$d(xy) = d(x)y + xg(y), \text{ for all } x, y \in R.$$

$$g(xy) = g(x)y + xd(y), \text{ for all } x, y \in R.$$

And is called Jordan S-derivation pair if:

$$d(x^2) = d(x)x + xg(x), \text{ for all } x \in R.$$

$$g(x^2) = g(x)x + xd(x), \text{ for all } x \in R.$$

Example 1.11:[4]

Let R be a non commutative ring and let $a, b \in R$, such that $xa = xb = 0$, for all $x \in R$.

Define $d, g: R \rightarrow R$, as follows:

$$d(x) = ax, g(x) = bx$$

Then (d, g) is a S-derivation pair of R .

Remark 1.12:[4]

Every S-derivation pair is a Jordan S-derivation pair, but the converse is in general not true. The following example illustrates this remark.

Example 1.13:[4]

Let R be a 2-torsion free non commutative ring, and let $a \in R$, such that $xax = 0$, for all $x \in R$, but $xay \neq 0$, for some $(x \neq y) \in R$.

An additive pair $d, g : R \rightarrow R$ is defined as

$$d(x) = xa + ax, \quad g(x) = [x, a]$$

Then (d, g) is Jordan S-derivation pair, but not a S-derivation pair.

Definition 1.14:[5]

A ring R is said to be a commutator right (resp. left) nonzero divisor, if there exists elements a and b of R , such that $c[a, b] = 0$ (resp. $[a, b]c = 0$) implies $c = 0$, for every $c \in R$.

§₂ (σ, τ) -S-Derivation pairs

In this section, we will introduce the definition of (σ, τ) -Strongly derivation pair, and we denoted by (σ, τ) -S-derivation pair, and Jordan (σ, τ) -Strongly derivation pair and we denoted by Jordan (σ, τ) -S-derivation pair, also we will give the relation between them.

Where $\sigma, \tau : R \rightarrow R$ are two mappings on R .

Now, in this section we introduce the principle definition.

Definition 2.1

Let R be a ring, additive mappings $d, g : R \rightarrow R$ is called (σ, τ) -S-derivation pair (d, g) where $\sigma, \tau : R \rightarrow R$ are two mappings of R , if satisfy the following equations:

$$d(xy) = d(x)\sigma(y) + \tau(x)g(y), \text{ for all } x, y \in R.$$

$$g(xy) = g(x)\sigma(y) + \tau(x)d(y), \text{ for all } x, y \in R.$$

And is called Jordan (σ, τ) -S-derivation pair if:

$$d(x^2) = d(x)\sigma(x) + \tau(x)g(x), \text{ for all } x \in R.$$

$$g(x^2) = g(x)\sigma(x) + \tau(x)d(x), \text{ for all } x \in R.$$

The following example explains the principle definition:

Example 2.2

Let R be a non commutative ring and let $a, b \in R$, such that

$$\tau(x)a = \tau(x)b = 0, \text{ for all } x \in R.$$

Define $d, g : R \rightarrow R$ as follows:

$$d(x) = a\sigma(x), \quad g(x) = b\sigma(x), \text{ for all } x \in R$$

where $\sigma, \tau : R \rightarrow R$ are two endomorphism mappings.

Then (d, g) is a (σ, τ) -S-derivation pair of R .

Let $x, y \in R$, so:

$$\begin{aligned} d(xy) &= a\sigma(xy) \\ &= a\sigma(x)\sigma(y) \\ &= a\sigma(x)\sigma(y) + \tau(x)b\sigma(y) \\ &= d(x)\sigma(y) + \tau(x)g(y) \end{aligned}$$

Also:

$$\begin{aligned} g(xy) &= b\sigma(xy) \\ &= b\sigma(x)\sigma(y) \\ &= b\sigma(x)\sigma(y) + \tau(x)a\sigma(y) \\ &= g(x)\sigma(y) + \tau(x)d(y) \end{aligned}$$

Hence (d, g) is a (σ, τ) -S-derivation pair.

Remark 2.3

Every (σ, τ) -S-derivation pair is a Jordan (σ, τ) -S-derivation pair, but the Converse is in general not true.

The following example illustrates this:

Example 2.4

Let R be a 2-torsion free non commutative ring, and let $a \in R$, such that $\tau(x)a\sigma(x) = 0$, for all $x \in R$, but $\tau(x)a\sigma(y) \neq 0$, for some $(x \neq y) \in R$.

Define an additive pair $d, g: R \rightarrow R$, as follows:

$$d(x) = \tau(x)a + a\sigma(x), \quad g(x) = \tau(x)a - a\sigma(x), \quad \text{for all } x \in R.$$

where $\sigma, \tau: R \rightarrow R$ are two endomorphism mappings.

Then (d, g) is a Jordan (σ, τ) -S-derivation pair, but not a (σ, τ) -S-derivation pair.

Let $x, y \in R$, so:

$$\begin{aligned} d(x^2) &= \tau(x^2)a + a\sigma(x^2) \\ d(x)\sigma(x) + \tau(x)g(x) &= (\tau(x)a + a\sigma(x))\sigma(x) + \tau(x)(\tau(x)a - a\sigma(x)) \\ &= \tau(x)a\sigma(x) + a\sigma(x)\sigma(x) + \tau(x)\tau(x)a - \tau(x)a\sigma(x) \\ &= \tau(x^2)a + a\sigma(x^2) \end{aligned}$$

$$\text{Hence } d(x^2) = d(x)\sigma(x) + \tau(x)g(x)$$

Also:

$$g(x^2) = \tau(x^2)a - a\sigma(x^2) = g(x)\sigma(x) + \tau(x)d(x)$$

Thus, (d, g) is Jordan (σ, τ) -S-derivation pair.

Now, we show that (d, g) is not (σ, τ) -S-derivation pair.

$$\begin{aligned} d(xy) &= \tau(xy)a + a\sigma(xy) \\ d(x)\sigma(y) + \tau(x)g(y) &= (\tau(x)a + a\sigma(x))\sigma(y) + \tau(x)(\tau(y)a - a\sigma(y)) \\ &= \tau(x)a\sigma(y) + a\sigma(x)\sigma(y) + \tau(x)\tau(y)a - \tau(x)a\sigma(y) \\ &= \tau(xy)a + a\sigma(xy) \end{aligned}$$

$$\text{Hence } d(xy) = d(x)\sigma(y) + \tau(x)g(y)$$

But:

$$\begin{aligned} g(xy) &= g(x)\sigma(y) + \tau(x)d(y) \\ &= (\tau(x)a - a\sigma(x))\sigma(y) + \tau(x)(\tau(y)a + a\sigma(y)) \\ &= \tau(x)a\sigma(y) - a\sigma(x)\sigma(y) + \tau(x)\tau(y)a + \tau(x)a\sigma(y) \\ &= \tau(xy)a - a\sigma(xy) + 2\tau(x)a\sigma(y) \end{aligned}$$

On the other hand:

$$g(xy) = \tau(xy)a - a\sigma(xy)$$

Since $\tau(x)a\sigma(y) \neq 0$, for some $x \neq y \in R$, the two expressions are not equal, hence we get (d, g) is not (σ, τ) -S-derivation pair.

Proposition 2.5

Let R be a semiprime ring. Suppose that σ, τ are automorphisms of R . If R admits a (σ, τ) -S-derivation pair (d, g) , such that $d(x)g(y)=0$ (resp. $g(x)d(y)=0$), for all $x, y \in R$, then $d=0$ (resp. $g=0$).

Proof

We have

$$d(x)g(y)=0, \text{ for all } x, y \in R \quad (1)$$

Replacing yx for y in (1) and using (1), we have:

$$d(x)g(yx)=0, \text{ for all } x, y \in R.$$

$$d(x)(g(y)\sigma(x) + \tau(y)d(x))=0, \text{ for all } x, y \in R.$$

$$d(x)g(y)\sigma(x) + d(x)\tau(y)d(x)=0, \text{ for all } x, y \in R.$$

$$d(x)\tau(y)d(x)=0, \text{ for all } x, y \in R. \quad (2)$$

By semiprimeness of R , (2) gives:

$$d(x)=0, \text{ for all } x \in R.$$

If we have

$$g(x)d(y)=0, \text{ for all } x, y \in R \quad (3)$$

Replacing yx for y in (3) and using (3), we have:

$$g(x)d(yx)=0, \text{ for all } x, y \in R.$$

$$g(x)(d(y)\sigma(x) + \tau(y)g(x))=0, \text{ for all } x, y \in R.$$

$$g(x)d(y)\sigma(x) + g(x)\tau(y)g(x)=0, \text{ for all } x, y \in R.$$

$$g(x)\tau(y)g(x)=0, \text{ for all } x, y \in R \quad (4)$$

By semiprimeness of R , (4) gives:

$$g(x)=0, \text{ for all } x \in R.$$

Proposition 2.6

Let R be a semiprime ring. Suppose that σ, τ are automorphisms of R . If R admits a (σ, τ) -S-derivation pair (d, g) , such that $d(x)=\pm \sigma(x)$ (resp. $g(x)=\pm \sigma(x)$), for all $x \in R$, then $g=0$ (resp. $d=0$).

Proof

We have

$$d(x)=\sigma(x), \text{ for all } x \in R \quad (1)$$

Replacing x by xy in (1) and using (1), we get:

$$d(xy) = \sigma(xy), \text{ for all } x, y \in R.$$

$$d(x)\sigma(y) + \tau(x)g(y)=\sigma(xy), \text{ for all } x, y \in R.$$

$$\sigma(x)\sigma(y)+\tau(x)g(y)=\sigma(x)\sigma(y), \text{ for all } x, y \in R.$$

$$\tau(x)g(y)=0, \text{ for all } x, y \in R \quad (2)$$

Left multiplication of (2) by $g(y)$, leads to:

$$g(y)\tau(x)g(y)=0, \text{ for all } x, y \in R \quad (3)$$

By semiprimeness of R , (3) gives:

$$g(y)=0, \text{ for all } y \in R.$$

Similarly, we can show if $d(x) = -\sigma(x)$, for all $x \in R$, then $g=0$
 In the same way, if $g(x) = \pm \sigma(x)$, for all $x \in R$, then $d=0$.

Proposition 2.7

Let R be any ring and σ, τ are two mappings on R . Then

- 1- If (d, g) is a (σ, τ) -S-derivation pair on R , then $d+g$ is a (σ, τ) -derivation.
- 2- If (d, g) is a Jordan (σ, τ) -S-derivation pair on R , then $d+g$ is a Jordan (σ, τ) -derivation.

Proof

- 1- We have

(d, g) is a (σ, τ) -S-derivation pair, so

$$d(xy) = d(x)\sigma(y) + \tau(x)g(y), \text{ for all } x, y \in R \quad (1)$$

$$g(xy) = g(x)\sigma(y) + \tau(x)d(y), \text{ for all } x, y \in R \quad (2)$$

By adding (1) and (2), we get

$$(d+g)(xy) = (d+g)(x)\sigma(y) + \tau(x)(d+g)(y)$$

Hence $d+g$ is a (σ, τ) -derivation

- 2- We have

(d, g) is a Jordan (σ, τ) -S-derivation pair, so

$$d(x^2) = d(x)\sigma(x) + \tau(x)g(x), \text{ for all } x \in R \quad (3)$$

$$g(x^2) = g(x)\sigma(x) + \tau(x)d(x), \text{ for all } x \in R \quad (4)$$

By adding (3) and (4), we get

$$(d+g)(x^2) = (d+g)(x)\sigma(x) + \tau(x)(d+g)(x), \text{ for all } x \in R.$$

Hence $d+g$ is a Jordan (σ, τ) -derivation.

§3 Relation Between (σ, τ) -S-Derivation pairs and (σ, τ) -Derivations

In this section, we study prime rings, semiprime rings, and rings that have a commutator left nonzero divisor with (σ, τ) -S-derivation pair, to obtain a (σ, τ) -derivation.

Theorem 3.1

Let R be a 2-torsion free semiprime ring, and (d, g) be a (σ, τ) -S-derivation pair on R , then d and g are (σ, τ) -derivations. Where σ, τ are automorphisms of R .

Proof

Suppose that (d, g) is (σ, τ) -S-derivation pair. Then:

$$d(xy) = d(x)\sigma(y) + \tau(x)g(y), \text{ for all } x, y \in R \quad (1)$$

That is:

$$d(xy) = d(x)\sigma(y) + \tau(x)g(y)\sigma(x) + \tau(x)\tau(y)d(x), \text{ for all } x, y \in R \quad (2)$$

Also:

$$d(xy) = d((xy)x) = d(xy)\sigma(x) + \tau(xy)g(x), \text{ for all } x, y \in R \quad (3)$$

That is:

$$d(xy) = d(x)\sigma(y)\sigma(x) + \tau(x)g(y)\sigma(x) + \tau(xy)g(x), \text{ for all } x, y \in R \quad (4)$$

From (2) and (4), we get:

$$\tau(xy)(d(x) - g(x)) = 0, \text{ for all } x, y \in R \quad (5)$$

Replace $\tau(y)$ by $(d(x) - g(x))\tau(y)\tau(x)$ in (5), we get:

$$\tau(x)(d(x) - g(x))\tau(y)\tau(x)(d(x) - g(x)) = 0, \text{ for all } x, y \in R \quad (6)$$

Since R is semiprime, we get:

$$\tau(x)d(x) = \tau(x)g(x), \text{ for all } x \in R \quad (7)$$

It follows that:

$$d(x^2) = d(x)\sigma(x) + \tau(x)d(x), \text{ for all } x \in R \quad (8)$$

And:

$$g(x^2) = g(x)\sigma(x) + \tau(x)g(x), \text{ for all } x \in R \quad (9)$$

Thus, by using [3, Theorem 2.3.7], we obtain that d and g are (σ, τ) -derivations on R .

Theorem 3.2

Let R be a prime, and (d, g) be a (σ, τ) -S-derivation pair on R , then d and g are (σ, τ) -derivations. Where σ, τ are automorphisms of R .

Proof

Since (d, g) is (σ, τ) -S-derivation pair, we have (see how relation (5) was obtained from relation (1) in the proof of Theorem 3.1)

$$\tau(xy)(d(x) - g(x)) = 0, \text{ for all } x, y \in R \quad (1)$$

And, by primeness of R , we get:

$$d(x) = g(x), \text{ for all } x \in R \quad (2)$$

And hence d and g are (σ, τ) -derivations on R .

Theorem 3.3

Let R be a ring which has a commutator left nonzero divisor and (d, g) be a (σ, τ) -S-derivation pair on R , then d and g are (σ, τ) -derivations. Where σ, τ are automorphisms of R .

Proof

1. That is We have:

$$2. d(yx^2) = d(y)\sigma(x^2) + \tau(y)g(x^2), \text{ for all } x, y \in R \quad (1)$$

3. That is:

$$4. d(yx^2) = d(y)\sigma(x^2) + \tau(y)g(x)\sigma(x) + \tau(y)\tau(x)d(x), \text{ for all } x, y \in R \quad (2)$$

5. On the other hand:

$$6. d(yx^2) = d(yx)\sigma(x) + \tau(yx)g(x), \text{ for all } x, y \in R \quad (3)$$

7.

$$8. d(yx^2) = d(y)\sigma(x^2) + \tau(y)g(x)\sigma(x) + \tau(y)\tau(x)g(x), \text{ for all } x, y \in R \quad (4)$$

9. From (2) and (4), we obtain:

$$\tau(y)(\tau(x)d(x)-\tau(x)g(x))=0, \text{ for all } x,y \in R \text{ (5)}$$

Replacing y by yr in (5), to get:

$$\tau(yr)(\tau(x)d(x)-\tau(x)g(x))=0, \text{ for all } x,y,r \in R \text{ (6)}$$

Again, left multiplying of (5) by $\tau(r)$, to get:

$$\tau(r)\tau(y)(\tau(x)d(x)-\tau(x)g(x))=0, \text{ for all } x,y,r \in R \text{ (7)}$$

Subtracting (7) from (6), we get:

$$[\tau(y),\tau(r)](\tau(x)d(x)-\tau(x)g(x))=0, \text{ for all } x,y,r \in R \text{ (8)}$$

Since R has a commutator left nonzero divisor, we get:

$$\tau(x)d(x)=\tau(x)g(x), \text{ for all } x \in R \text{ (9)}$$

Linearizing (9), we get:

$$\tau(x)d(y) + \tau(y)d(x) = \tau(x)g(y) + \tau(y)g(x), \text{ for all } x,y \in R \text{ (10)}$$

That is:

$$\tau(x)(d-g)(y) + \tau(y)(d-g)(x)=0, \text{ for all } x,y \in R \text{ (11)}$$

Replacing y by ry in (11), to get:

$$\tau(x)(d-g)(ry) + \tau(ry)(d-g)(x)=0, \text{ for all } x,y,r \in R \text{ (12)}$$

Again, left multiplying of (11) by $\tau(r)$, to get:

$$\tau(r)\tau(x)(d-g)(y) + \tau(r)\tau(y)(d-g)(x)=0, \text{ for all } x,y,r \in R \text{ (13)}$$

Subtracting (12) from (13), we get:

$$\tau(rx)(d-g)(y) - \tau(x)(d-g)(ry)=0, \text{ for all } x,y,r \in R \text{ (14)}$$

Replacing x by sx in (14), to get:

$$\tau(rsx)(d-g)(y) - \tau(sx)(d-g)(ry)=0, \text{ for all } x,y,r,s \in R \text{ (15)}$$

Also, left multiplying of (14) by $\tau(s)$, to get:

$$\tau(srx)(d-g)(y) - \tau(sx)(d-g)(ry)=0, \text{ for all } x,y,r,s \in R \text{ (16)}$$

Subtracting (16) from (15), we get:

$$[\tau(r),\tau(s)]\tau(x)(d-g)(y)=0, \text{ for all } x,y,r,s \in R \text{ (17)}$$

Since R has a commutator left nonzero divisor, we get:

$$\tau(x)(d-g)(y)=0, \text{ for all } x,y \in R \text{ (18)}$$

That is:

$$\tau(x)d(y)=\tau(x)g(y), \text{ for all } x,y \in R \text{ (19)}$$

Hence d and g are (σ,τ) -derivations.

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الاشتقاقات المزدوجة القوية (σ, τ) على الحلقات

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الخلاصة

لتكن R حلقة تجميعية. في هذا البحث قدمنا تعريف الاشتقاق المزدوج القوي (σ, τ) واشتقاق جوردان المزدوج القوي (σ, τ) في الحلقة R ، ودراسة العلاقة بينهم. كذلك، ندرس الحلقات الأولية، الحلقات شبه الأولية، والحلقات التي لها مبدل قاسم غير صفري أبسر مع الاشتقاق المزدوج القوي (σ, τ) للحصول على الاشتقاق (σ, τ) . أي ان $\sigma, \tau: R \rightarrow R$ هما دالتين على الحلقة R .

الكلمات المفتاحية :

حلقة أولية، حلقة شبه أولية، مشتقة (σ, τ) ، الاشتقاق المزدوج القوي (σ, τ) ، اشتقاق جوردان المزدوج القوي (σ, τ) .

