

Approximations of Entire Functions in Locally Global Norms

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Abstract

The purpose of this paper is to evaluate the error of the approximation of an entire function by some discrete operators in locally global quasi-norms ($L_{\delta,p}$ -space), we intend to establish new theorems concerning that Jackson polynomial and Valee-Poussin operator remain within the same bounds as bounded and periodic entire function in locally global norms ($L_{\delta,p}$), ($0 < p \leq 1$).

Key words : Entire functions, bounded measurable functions, quasi-normed space.

Introduction and Preliminaries

Al-Abdulla, A. [1], Al-Saidy, S.K. [2] and E.S.Bhayah [3] gave estimation for approximation of bounded measurable functions with some discrete operators in L_p -norm ($0 < p \leq 1$).

Here, we give an estimation for approximation of entire functions in $L_{\delta,p}$ -space.

Let $X = [-\pi, \pi]$ we denote the set of all 2π -periodic bounded measurable function with usual sup-norm by L_∞ , such that

$L_\infty(X) = \{f : f \text{ is } 2\pi\text{-periodic bounded measurable function}\}$ with norm

$$\|f\|_\infty = \sup \{|f(x)| : x \in X\} < \infty \quad \dots(1.1)$$

and the L_p -norm ($1 \leq p < \infty$) of $f \in L_p$ by $\|f\|_{L_p}$, such that

$$L_p(X) = \left\{ f : \|f\|_p = \left(\int_X |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}; \|f\|_{L_p(X)} = \|f\|_p \quad \dots(1.2)$$

Now let us consider the Dirichlet kernel of degree n , [4]

$$D_n(u) = \frac{1}{2} + \sum_{v=1}^n \cos(vu) \quad \exists u \in \mathbb{R}, n=0,1,\dots \quad \dots(1.3)$$

Let

$$K_n(u) = \frac{1}{n+1} [D_0(u) + D_1(u) + \dots + D_n(u)] \quad \dots(1.4)$$

be the Fejer kernel of degree not greater than n .

$$J_n(f, x) = \frac{2}{n+1} \sum_{k=0}^n f(x_{k,n}) K_n(X - X_{k,n}) \quad \dots(1.5)$$

where $X_{k,n} = \frac{2K\pi}{n+1}$, ($K = 0, 1, 2, \dots, n$), be the so called Jackson polynomial of function $f \in L_\infty$.

$$V_{2n}(t) = \frac{1}{n+1} [D_n(t) + D_{n+1}(t) + D_{2n}(t)] \quad \dots(1.6)$$

Let $X_j = \frac{2\pi j}{3n+1}$, $j=0,1,\dots,3n$. Then we define the following operator.

$$V_{2n,3n}(f, X) = \frac{2}{3n+1} \sum_{j=0}^{3n} f(X_j) V_{2n}(X - X_j) \quad \dots(1.7)$$

be the vee-poussin discrete operator of 2π -periodic bounded measurable function.

The unique linear trigonometric polynomial which is interpolating a given function $f \in L_p(X)$ at the point X_j is denote by $I_n(t)$ which has the representation:

$$I_n(f, X) = \frac{2}{2n+1} \sum_{j=0}^{2n} f(X_j) D_n(X - X_j) \quad \dots(1.8)$$

Now let B_n be the set of all entire functions, since the derivative of polynomial exists every where, then we get that every polynomial is an entire function [5], so we consider that $f \in B_n$ and $J_n(f) \in B_n$, $V_{2n,3n}(f) \in B_n$.

Let n, k be positive integers, $(0 < p \leq 1)$ and $(\delta > 0)$ are fixed numbers which will be used for the degree of approximating polynomial, for the rate order of modulus and for the space $L_{\delta,p}$ respectively.

We consider the locally global norm for $(\delta > 0)$, $(0 < p \leq \infty)$

$$\|f\|_{\delta,p} = \left(\int_X \sup \left\{ |f(y)|, y \in \left[x - \frac{\delta}{2}, x + \frac{\delta}{2} \right] \right\}^p dx \right)^{\frac{1}{p}}, \quad X \in [-\pi, \pi]. \quad \dots(1.9)$$

Now the k^{th} average modulus of smoothness for $f \in L_{\delta,p}$ are defined by the following respectively, [6], [7]

$$\left\{ \begin{array}{l} \tau_k(f, \frac{1}{n})_p = \left\| W_k(f, \cdot, \frac{1}{n}) \right\|_p \\ \tau_k(f, \frac{1}{n})_{\delta,p} = \left\| W_k(f, \cdot, \frac{1}{n}) \right\|_{\delta,p} \end{array} \right\} \quad \dots(1.10)$$

where the k^{th} modulus of smoothness for $f \in L_{\delta,p}$, $k \in \mathbb{N}$ is defined by

$$W_k(f, x, \frac{1}{n}) = \sup \left\{ \left| \Delta_h^k f(t) \right| : t, t + kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2} \right] \cap X \right\} \quad \dots(1.11)$$

$$\text{Now, we set } \Delta_h^k f(t) = \begin{cases} \sum_{m=0}^k (-1)^{k-m} \binom{k}{m} f(t + mh) & \text{if } t \text{ or } t + kh \in X \\ 0 & \text{otherwise} \end{cases}.$$

In the following we recall some theorems which are needed:-

Theorem 1.1: [6]

If $f \in B_n$, then for $(0 < p \leq 1)$ and $(\delta > 0)$, we have,

$$\|f\|_{\delta,p} \leq c(p) [(1 + n\delta)^{1-p} (ns)]^{\frac{1}{p}} \|f\|_p.$$

Theorem 1.2: [3]

If $f \in 2\pi$ -periodic bounded measurable functions, then for $(0 < p \leq 1)$

$$\|f - J_n(f)\|_p \leq C(p) \tau_1\left(f, \frac{-1}{n}\right)_p.$$

Theorem 1.3: [3]

If $f \in 2\pi$ -periodic bounded measurable function, then for $(0 < p \leq 1)$

$$\|f - V_{2n,3n}(f)\|_p \leq C(p, k, \ell) \tau_k\left(f, \frac{1}{2n}\right)_p, \text{ where } n=1,2,\dots \text{ and } (p,k,\ell) \text{ is a constant depends on } p, k \text{ and } \ell.$$

Theorem 1.4: [3]

Let f be 2π -periodic bounded measurable function, then for $(0 < p \leq 1)$, we have

$$\|f - I_n(f)\|_p \leq C(p, k, \ell) \tau_k\left(f, \frac{1}{n}\right)_p, \text{ where } p,k,\ell \text{ is a constant depends on } p, k \text{ and } \ell.$$

Main Results

We shall prove direct inequality to find the degree of approximation of 2π -periodic entire function by some discrete operators in $(L_{\delta,p})$ spaces, $(0 < p \leq 1)$.

Lemma 2.1:

Let f be 2π -periodic entire function, then for $(0 < p \leq 1)$, we have

$$\tau_k\left(f, \frac{1}{n}\right)_p \leq \tau_k\left(f, \frac{1}{n}\right)_{\delta,p}.$$

Proof:

$$\begin{aligned} \tau_k\left(f, \frac{1}{n}\right)_p &= \left\| W_k\left(f, \frac{1}{n}\right) \right\|_p \\ &= \left\| \sup \left\{ \left| \Delta_h^k f(t) \right|; t, t+kh \in \left[x - \frac{k}{2n}, x + \frac{k}{2n} \right] \cap X \right\} \right\|_p \\ &= \left\| \sup \left\{ \left| \sum_{i=0}^k (-1)^{i+k} \binom{k}{i} f(t+ih) \right|; t, t+kh \in \left[x - \frac{k}{2n}, x + \frac{k}{2n} \right] \cap X \right\} \right\|_p \\ &= \left(\int_X \sup \left\{ \left| \sum_{i=0}^k (-1)^{i+k} \binom{k}{i} f(t+ih) \right|^p; t, t+kh \in \left[x - \frac{k}{2n}, x + \frac{k}{2n} \right] \cap X \right\} dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_X \sup \left\{ \sup \left\{ \left| \sum_{i=0}^k (-1)^{i+k} \binom{k}{i} f(t+ih) \right|^p; t, t+kh \in \left[y - \frac{k}{2n}, y + \frac{k}{2n} \right] \cap X \right\} \right\} dy \right)^{\frac{1}{p}} \\ &= \left\| \sup \left\{ \left| \Delta_h^k f(t) \right|; t, t+kh \in \left[x - \frac{k}{2n}, x + \frac{k}{2n} \right] \cap X \right\} \right\|_{\delta,p} \\ &= \left\| W_k\left(f, \frac{1}{n}\right) \right\|_{\delta,p} \\ &= \tau_k\left(f, \frac{1}{n}\right)_{\delta,p} \end{aligned}$$

Theorem 2.2:

Let f be 2π -periodic bounded measurable entire function, $(f \in L_{\delta,p})$, $(0 < p \leq 1)$, we have

$\|f - J_n(f)\|_{\delta,p} \leq C(p) \tau_1(f, \frac{1}{n})_{\delta,p}$, where $C(p)$ is a constant depends only on p .

Proof:

By theorem (1.1), we get

$$\|f - J_n(f)\|_{\delta,p} \leq C(p) [1 + (1 - n\delta)^{1-p} (n\delta)^p]^{\frac{1}{p}} \|f - J_n(f)\|_p.$$

Now since $(\delta > 0)$, then

$$\|f - J_n(f)\|_{\delta,p} \leq C_1(p) \|f - J_n(f)\|_p.$$

Then by using theorem (1.2) and lemma (2.1), we get that

$$\begin{aligned} \|f - J_n(f)\|_{\delta,p} &\leq C_2(p) \tau_1(f, \frac{1}{n})_p \\ &\leq C(p) \tau_1(f, \frac{1}{n})_{\delta,p} \end{aligned}$$

Theorem 2.3:

Let f be 2π -periodic bounded measurable entire function, $(f \in L_{\delta,p})$, $(0 < p \leq 1)$, we have

$$\|f - V_{2n,3n}(f)\|_{\delta,p} \leq C(p, k, \ell) \tau_k(f, \frac{1}{2n})_{\delta,p}, \text{ where } p, k, \ell \text{ is a constant depends on } p, k \text{ and } \ell.$$

Proof:

By using theorem (1.1), we get

$$\|f - V_{2n,3n}(f)\|_{\delta,p} \leq C_1(p) [1 + (1 + n\delta)^{1-p} (n\delta)^p]^{\frac{1}{p}} \|f - V_{2n,3n}(f)\|_p.$$

Since $\delta = \frac{1}{n}$, then

$$\|f - V_{2n,3n}(f)\|_{\delta,p} \leq C_2(p) \|f - V_{2n,3n}(f)\|_p.$$

Now by using theorem (1.3) and lemma (2.1), we have

$$\begin{aligned} \|f - V_{2n,3n}(f)\|_{\delta,p} &\leq C(p, k, \ell) \tau_k(f, \frac{1}{2n})_p \\ &\leq C(p, k, \ell) \tau_k(f, \frac{1}{2n})_{\delta,p}. \end{aligned}$$

Theorem 2.4:

Let f be 2π -periodic bounded measurable entire function, $(f \in L_{\delta,p})$, $(0 < p \leq 1)$, we have

$$\|f - I_n(f)\|_{\delta,p} \leq C(p, k, \ell) \tau_k(f, \frac{1}{2n})_{\delta,p}, \text{ where } p, k, \ell \text{ is a constant depends on } p, k \text{ and } \ell.$$

Proof:

By using theorem (1.1), we get

$$\|f - I_n(f)\|_{\delta,p} \leq C_1(p) [1 + (1 + n\delta)^{1-p} (n\delta)^p y_p] \|f - I_n(f)\|_p.$$

Since $\delta = \frac{1}{n}$, then

$$\|f - I_n(f)\|_{\delta,p} \leq C_2(p) \|f - I_n(f)\|_p.$$

Then by using theorem (1.4) and lemma (2.1), we get

$$\begin{aligned}\|f - I_n(f)\|_{\delta,p} &\leq C(p,k,\ell) \tau_k\left(f, \frac{1}{n}\right)_p \\ &\leq C(p,k,\ell) \tau_k\left(f, \frac{1}{n}\right)_{\delta,p}.\end{aligned}$$

Conclusion

We found the degree of approximation of entire functions by using Jackson, Vallee Pousson and interpolation polynomials in locally quasi-norms $L_{\delta,p}$ ($0 < p < 1$).

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تقريب الدوال الداخلية بواسطة المتعددات المتقطعة في الفضاءات المحلية

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الخلاصة

الغرض من هذا البحث هو حساب مقدار الخطأ لتقريب الدوال الداخلية بواسطة بعض المؤثرات المتقطعة في

الفضاءات شبه المحلية باستعمال الوسيط $\tau_k(\delta, \frac{1}{n})_p$.

الكلمات المفتاحية : الدوال الداخلية ، الدوال محدودة القياس ، الفضاء شبه المعياري.