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A new Kind of Discrete Topological Graphs with Some Properties

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Abstract

In this paper. A new definition of discrete topological graph is introduced. Some properties of this graph are proved. If $n > 2$ are evaluated G_τ has no pendant vertex , not tree, also the value of the diameter and the minimum degree of G_τ. If $n \geq 2$, G_τ has $(2n-3)$ complete bipartite induced subgraphs, G_τ is connected graph, simple graph, has no odd cycle, the clique number also proved, the value of the radius, the maximum degree and the chromatic number of G_τ have been studied.

Keywords: Discrete Topology, Topological Graph, clique number.

1-Introduction

This paper, concerned only with undirected simple graphs. All notations on graphs which are not defined here can be found in [13,15]. "Topological graph" is an important branch of graph theory studied the embedding graphs in a plain and surfaces [9]. "**A graph** G "is a pair (V, E), where $V = V(G)$ is a non-empty set whose elements are called vertices, $E = E(G)$ is a set of elements consists of unordered pairs, these elements are called edges or lines. A "**trivial** graph" is a graph with order $n = 1$. If $n > 1$ the graph is nontrivial. A vertex u is **incident** with edge e in G if e lies on it, also e is **incident** with u.Two vertices u and v of G are **adjacent** if there is an edge between them where e = $uv \in E$. The **adjacent edges** are two or more edges of G incident with a common vertex more than one edge joined two vertices in the graph. A "**degree**" of a vertex u is the number of edges that incident on it, denoted by $d(u)$ or $deg(u)$. The **minimum degree** of a graph G denoted by $\delta(G)$ is the smallest degree among all degrees of the vertices in G. The **maximum degree** of G denoted by $\Delta(G)$ is the largest degree among all degrees of the vertices in G. A"**pendant**" (end vertex or leaf) vertex is a vertex with degree one. A "**subgraph**" M of a graph G is a graph in which $V(M) \subseteq V(G)$ and $E(M) \subseteq E(G)$. An **induced** subgraph $G[M]$ is the subgraph of a graph G which is constructed by all vertices of $M \subseteq V(G)$ and every edge incident on two vertices of M. A **complete** graph K_n is a graph in which each vertex has a degree $n-1$. A "**null**" graph N_n is a graph without edges. A **path** graph P_n of order n, $(n \geq 1)$ and size $n-1$ is a sequence of n non-repeated vertices. A **cycle** graph C_n is a closed path with order and size n. A **"bipartite"** graph G is a graph with two disjoint vertices sets U_1 and U_2 such that any edge of G join one vertex from U_1 and one vertex from U_2. A **"complete bipartite"** graph $K_{n,m}$ of order $(n+m)$ and size nm is a bipartite graph with vertices sets U_1 of order n and U_2 of order m, in which each vertex of U_1 is adjacent with all vertices of U_2. The **"distance**" between two vertices v and u, is the length of a shortest $v-u$ path, denoted by $d(v,u)$. The **"eccentricity**" of a vertex v is the maximum distance from it to any other vertex, denoted by $e(v)$, where $e(v) = \max\{d(v,u), u \in V(G)\}$. The **diameter** of a connected graph G, is the maximum distance between any two vertices denoted by $\text{diam}(G)$. Also, the diameter is the maximum eccentricity among all vertices. The **radius** is the minimum eccentricity among all vertices of , denoted by $\text{rad}\, G$. The **"clique**" is complete induced subgraph of a graph G. The **clique number** is the order of the maximum clique in G, denoted by $\omega(G)$. Many authors studied the construction of graphs see $[1-7]$. If (X,τ) be any topological space, so the elements of τ are called **open sets**. If X be any non-empty set, and let τ be the collection of all subsets of X, where $\tau = P(X)$. Then τ is called the discrete topology on X. The topological space (X,τ) is called **discrete topological space**[8] .

2. Discrete Topological Graph

Many authors introduced a definition for discrete topological graph.

In [10] . Gave the following definition.

Definition 2.1: Let (X,τ) be a topological space. Define the graph $G_\tau=(V,E)$ such that

$V = \{u: u \in \tau, u \neq \emptyset, X\}$, $E = \{uv \in E(G_\tau)$ if $u \cap v \neq \emptyset, u \neq v$ and $u,v \in \tau\}$. They studied many properties of this graph.

In [16] . Introduced the following definition.

Definition2.2: Let X be not empty set, and τ be a discrete topology on X. The discrete topological graph referred to $G_τ$=(V, E) is a graph with the vertex set $V = \{A; A \in \tau,\ and\ A \neq \emptyset, X\}$, and the edge set

E =$\{AB; A \nsubseteq B\ and\ B \nsubseteq A\}$. They studied different properties of this graph.

In[11]. Also defined the discrete topological graph as follows:

Definition2.3: Let X be a nonempty set, and τ be a discrete topological space. The discrete topological graph referred to $G_τ$= (V, E) is a graph of vertices set, $V(G_\tau) = \tau - \{\emptyset, X\}$ and the edge set defined by

$E = \{AB; A \subset B\}$.

In this research we introduced a new definition of discrete topological graph, with some examples, and properties of this graph.

Definition 2.4: Let X be a nonempty set, and τ be a discrete topology on X. The discrete topological graph referred to $G_τ$ =(V, E) is a graph with vertex set $V = \{\ A\ :\ \in \tau,\ A \neq \emptyset\ \}$, and edge set

E =$\{\ A\,B: |\,A\,| = |\,B\,| - 1,\ B \in \tau\ \}$.

Example 2.1: Let X be not empty set with order n, and τ be discrete topology on X. we draw the discrete topological graphs $G_τ$ when $|\,X\,|$ = 2, 3, 4 and 5.

If $X = \{1,2\}$, then τ =$\{\emptyset,\ X, \{1\}, \{2\}\}$, and $V(\ G_τ)$ =$\{\{1\},\ \{2\}, \{1,2\}\}$. The discrete topological graph $G_τ$ is as in Figure 1.

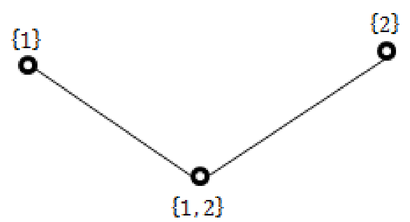

Figure 1. The **discrete topological graph $G_τ$ when $|\,X\,|$= 2.**

If $X = \{1,2,3\}$, then $\tau = \{\emptyset, X,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$, and $V(G_τ)$ = $\{\{1\},\{2\}, \{3\}, \{1,2\},\{1,3\}, \{2,3\}, \{1,2,3\}\}$. The discrete topological graph $G_τ$ is as in Figure 2.

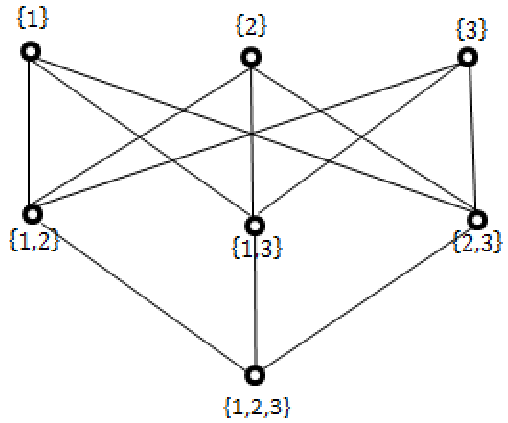

Figure 2. The discrete topological graph G_τ when $|X| = 3$.

If $X = \{1, 2, 3, 4\}$, then $\tau = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$ and $V(G_\tau) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}$. The discrete topological graph G_τ is as in Figure 3.

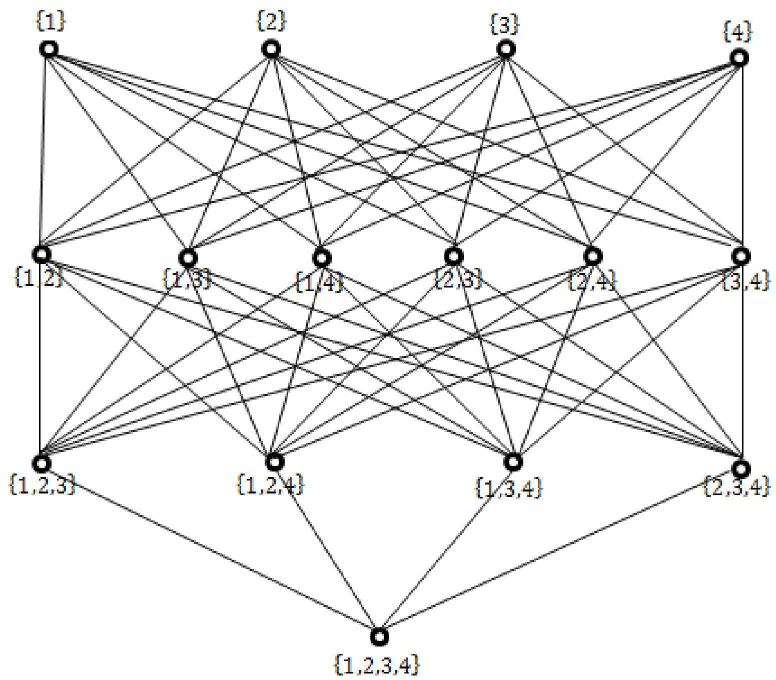

Figure 3. The discrete topological graph G_τ when $|X| = 4$.

If $X = \{1, 2, 3, 4, 5\}$, then, τ ={∅ , X,{1}, {2}, {3}, {4},{5}, {1,2}, {1,3}, {1,4}, {1,5}, {2,3}, {2,4}, {2,5},{3,4},{3,5},{4,5}, {1,2,3}, {1,2,4},{1,2,5},{1,3,4},{1,3,5} ,{1,4,5} , {2,3,4} , {2,3,5} , {3,4,5} , {1,2,3,4} , {1,2,3,5} , {1,2,4,5},{1,3,4,5},{2,3,4,5}}, and V(G_τ) ={{1},{2},{3},{4},{5},{1,2}. {1,3}, {1,4},{1,5},{2,3},{2,4},{2,5},{3,4},{3,5},{4,5},{1,2,3},{1,2,4},{1,2,5},{1,3,4}, {1.3.5},{1,4,5},{2,3,4},

{2,3,5} , {2,4,5} , {3,4,5} , {1,2,3,4} , {1,2,3,5} ,{1,2,4,5},{1,3,4,5} , {2,3,4,5} ,{1,2,3,4,5}}. The discrete topological graph G_τ is as in Figure 4.

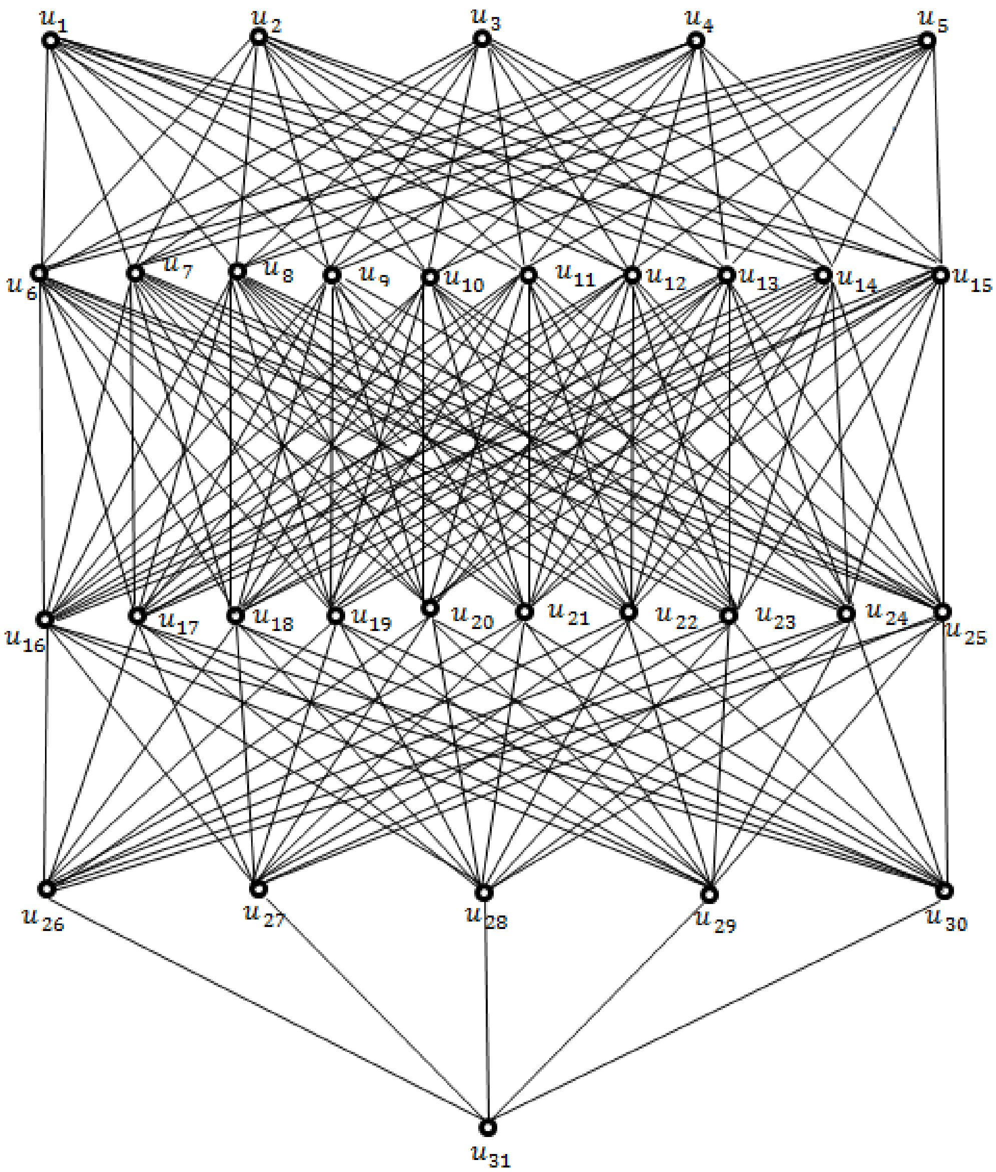

Figure 4. The discrete topological graph G_τ when | X |= 5.

Where u_1={1}, u_2={2}, u_3={3}, u_4={4}, u_5={5}, u_6={1,2}, u_7={1,3}, u_8={1,4}, u_9={1,5}, u_{10}={2,3}, u_{11}={2,4}, u_{12}={2,5}, u_{13}={3,4}, u_{14}={3,5}, u_{15}={4,5}, u_{16}={1,2,3}, u_{17}={1,2,4}, u_{18}={1,2,5},

u_{19}={1,3,4}, u_{20}={1,3,5}, u_{21}={1,4,5}, u_{22}={2,3,4}, u_{23}={2,3,5}, u_{24}={2,4,5}, u_{25}={3,4,5}, u_{26}={1,2,3,4}, u_{27}={1,2,3,5}, u_{28}={1,2,4,5}, u_{29}={1,3,4,5}, u_{30}={2,3,4,5}, u_{31}={1,2,3,4,5}.

3. Some Propeies of Discrete Topological Graph.

Here, some properties of discrete topological graph are proved.

Proposition 3.1: Let X be not empty set with order n ≥ 2 and τ be discrete topology on X.Then

the discrete topological graph $G_\tau \cong P_{F,H}$ where F is the order of the set of odd cardinality in G_τ, and H is the order of the set of even cardinality in G_τ, and each of m, h ≤ n.

$F = \binom{n}{1} + \binom{n}{3} + \ldots + \binom{n}{m}$, m is odd.

$H = \binom{n}{2} + \binom{n}{4} + \ldots + \binom{n}{h}$, h is even.

Proof: Let X = **{** 1, 2, … , n } be a set of order n ≥ 2, and τ be the discrete topology on X. Let $G_\tau = (V, E)$ be discrete topological graph on X. Then by Definition 2.4 $V = \{ \ : A \in \tau , A \neq \emptyset \}$.

Let F be the family of sets of odd cardinality in V, and H be the family of sets of even cardinality in V. By Definition 2.4, each edge in G_τ is join a vertex in a set of odd cardinality to a vertex in a set of even cardinality in V. No vertex in a set of odd cardinality join to a vertex in a set of odd cardinality, similarly no vertex in a set of even cardinality join to a vertex of even cardinality, That is the elements in the sets of odd cardinality F are disjoint, and the elements in the sets of even cardinality H are disjoint. Thus the vertices in G_τ can be partition into two subsets F and H such that each edge in G_τ join a vertex in F to a vertex in H, and $G_\tau \cong P_{F,H}$.

To explain proposition 3.1, we give the following example.

Example 3.1: If $|X|$= 4, then $V(G_\tau)$ = {{1}, {2}, {3} ,{4}, {1,2}, {1,3}, {1,4},{2,3}, {2,4}, {3,4}, {1,2,3}, {1,2,4}, {1,3,4}, {2,3,4},{1,2,3,4}}, and the sets of vertices of the bipartite graph $P_{F,H}$ are, F = {{1}, {2}, {3} ,{4},{1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}}, H ={{1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}, {1,2,3,4}}. The bipartite graph $P_{F,H}$ as in Figure 5.

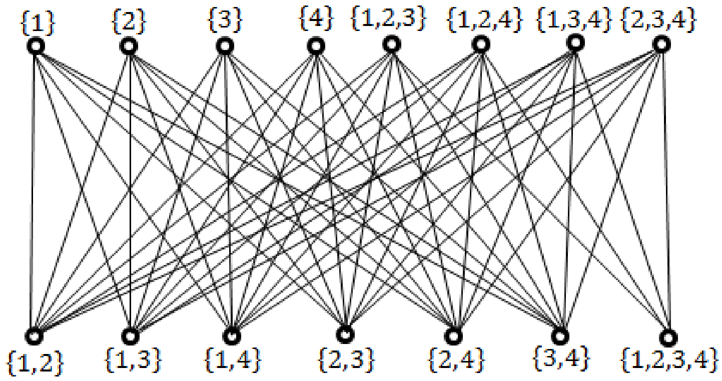

Figure 5. The bipartite graph $P_{F,H}$ when | X |= 4.

Proposition 3.2: Let X be not empty set of order n,($n \geq 2$) and τ be discrete topology on X. Then the size and order of discrete topological graph $G_\tau = (V, E)$ are :

$|E|$ = n $\binom{n}{2}$ + $\binom{n}{2}$ $\binom{n}{3}$ + … + $\binom{n}{n-1}$ $\binom{n}{n}$, and

$|V|$ = n + $\binom{n}{2}$ + $\binom{n}{3}$+ … + $\binom{n}{n-1}$ + 1

Proof: Let $|F_i|$= $\binom{n}{i}$ and $|H_j|$ = $\binom{n}{j}$ where i is odd and j is even. From Definition 2.4, each vertex in F_1 is adjacent to every vertex of H_2, and each vertex in H_2 is adjacent to every vertex of F_3 and so on up to each vertex in F_{n-1} is adjacent to vertex of H_n. That is , the number of edges which are joined F_1 with H_2 is n $\binom{n}{2}$ and the number of edges which are joined H_2 with F_3 is $\binom{n}{2}\binom{n}{3}$, and by repeating this process up to F_{n-1} and H_n are joined by $\binom{n}{n-1}\binom{n}{n}$ edges. Then the total number of edges in G_τ is $|E|$ = n $\binom{n}{2}$ + $\binom{n}{2}$ $\binom{n}{3}$ + … + $\binom{n}{n-1}$ $\binom{n}{n}$. As G_τ is discrete topological graph, then | V | = n + $\binom{n}{2}$ + $\binom{n}{3}$ + … + $\binom{n}{n-1}$ + 1.

Proposition 3.3: Let | X | = n, and G_τ be a discrete topological graph on X. Then G_τ has 2n−3 complete bipartite induced subgraphs.

Proof: Let $x_1, x_2, \ldots, x_n$ be the sets of vertices in G_τ of cardinality 1,2,… ,n respectively. To find the complete bipartite induced subgraphs in G_τ we have only two cases:

Case i: By Definition 2.4, each vertex in x_1 is adjacent to every vertex in x_2, and the vertices in x_1 are independent and the vertices in x_2 are independent. Thus the subgraph which induced by the sets of vertices x_1 and x_2 is complete bipartite subgraph $K_{|x_1|,|x_2|}$, similarly for subgraphs induced by $\{x_2, x_3\}$

,$\{x_3, x_4\}$, … , $\{x_{n-1}, x_n\}$. Hence the total subgraphs in this case are $n-1$ complete bipartite induced subgraphs.

Case ii: As each vertex in x_1 is adjacent to every vertex in x_2 and each vertex in x_3 is adjacent to every vertex in x_2. By Definition 2.4, no vertex in x_1 is adjacent to a vertex in x_3 , and the vertices in each of x_1, x_2, x_3 are disjoint. Then the induced subgraph induced by x_1, x_2, x_3 is complete bipartite subgraph. Similarly for the induced subgraphs induced by x_2, x_3 , x_4 and $x_3, x_4, x_5, \dots, x_{n-2}, x_{n-1}, x_n$. Then the total number of induced complete bipartite subgraphs in this case is $n-2$. Then the total number of induced complete bipartite subgraphs in the discrete topological graph G_τ is $(n-1) + (n-2) = 2n-3$.

Theorem 3.4[**14**]**:** A connected graph G is bipartite if and only if G has no odd cycle.

Theorem 3.5 [**14**]**:** Let G, be a graph and for each v ∈ G, d(v)≥ 2. Then G contains a cycle.

Proposition 3.6: Let | X | = n, (n ≥ 2) and G_τ be discrete topological graph on X. Then

(i) The discrete topological graph G_τ has no pendant vertex for n≥ 3.

(ii) G_τ is connected graph.

(iii) G_τ has no odd cycle

(iv) G_τ is not tree for $n > 2$.

(v) G_τ is simple graph.

Proof:

(i) If n = 2, then by Definition 2.4, $G_\tau \cong P_3$ and G_τ has two pendant vertices.

Suppose that n≥ 3. Then G_τ has n singleton elements. Let υ be a singleton element in G_τ . By Definition 2.4, υ is adjacent to $\binom{n}{2}$ elements. As n≥ 3, then H_2 has at least 3 elements, that is υ is adjacent to at least 3 elements, and $d(\upsilon)$ is at least 3. Hence no vertex with singleton element is pendent. Similarly let u be any set in G_τ with order | u | > 1. Then by Definition 2.4, each vertex in U is adjacent to every vertex in a set of order $|u|$+1and adjacent by every vertex in a set of order | u |−1. As $n \geq 3$, then the d(u) ≥ 3, and G_τ has no pendent vertex.

(ii) Follows from Definition 2.4.

(iii) From (ii) G_τ is connected, by proposition 3.1, G_τ is bipartite. Then by Theorem 3.4. G_τ has no odd cycle.

(iv) From (i) the minimum degree in the topological graph G_τ when $n > 2$ is greater than 2. Then by Theorem 3.5, G_τ contains a cycle. Hence G_τ is not tree

(v) Follows from Definition 2.4.

Proposition 3.7: Let | X | = n, n> 2 and G_τ be a discrete topological graph on X. Then

$$\Delta (G_\tau) = \begin{cases} \binom{n}{\frac{n}{2}-1} + \binom{n}{\frac{n}{2}+1} & \text{if n even} \\ \binom{n}{\lceil\frac{n}{2}\rceil-1} + \binom{n}{\lceil\frac{n}{2}\rceil+1} & \text{if n odd} \end{cases}$$

and $\delta (G_\tau) = \text{n}$

Proof: Let | X | = n, and $G_\tau = (V, E)$ be discrete topological graph on X. By Definition 2.4,

$V = \{ A : A \in \tau , A \neq \emptyset \}$.

Let A_1 be the family sets of V with singleton element;

A_2 be the family sets of V with two elements;

⋮

A_{n-1} be the family sets of V with $n-1$ elements.

A_n be the family sets of V with n elements.

Then the order of $A_1, A_2, \ldots, A_{n-1}, A_n$ is $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}, \binom{n}{n}$ respectively

If n = 2, then $G_\tau \cong P_3$ and $\Delta (G_\tau) = 2$, and $\delta (G_\tau) = 1$.

Now, if n is even, then the family sets $A_{\frac{n}{2}}$ has maximum order, and the order of the other family sets arranged in decreasing order from the right side of $A_{\frac{n}{2}}$. That is

$$\left.\begin{array}{l} |A_{\frac{n}{2}}| > |A_{\frac{n}{2}-1}| > \ldots > |A_2| > |A_1| \\ |A_{\frac{n}{2}}| > |A_{\frac{n}{2}+1}| > \ldots > |A_{n-1}| > |A_n| \end{array}\right\} \quad (1)$$

So the elements of $A_{\frac{n}{2}}$ has the maximum degrees, as each element in $A_{\frac{n}{2}}$ is adjacent by

$\binom{n}{\frac{n}{2}-1}$ elements in $A_{\frac{n}{2}-1}$ and adjacent to $\binom{n}{\frac{n}{2}+1}$ elements in $A_{\frac{n}{2}+1}$, Therefore the degree of any element in $A_{\frac{n}{2}}$ is equal to $\binom{n}{\frac{n}{2}-1}+\binom{n}{\frac{n}{2}+1}$ which is the maximum degree in G_τ.

If n is odd, then the family of sets $A_{\lceil\frac{n}{2}\rceil}$ and $A_{\lfloor\frac{n}{2}\rfloor}$ has the same order , and the order of the other family of sets arranged in non-decreasing order from the right side of $A_{\lfloor\frac{n}{2}\rfloor} = A_{\lceil\frac{n}{2}\rceil}$ the two families $A_{\lfloor\frac{n}{2}\rfloor}$ and $A_{\lceil\frac{n}{2}\rceil}$ that is

$$\left.\begin{array}{l} |A_{\lfloor\frac{n}{2}\rfloor}| = |A_{\lceil\frac{n}{2}\rceil}| > |A_{\lfloor\frac{n}{2}\rfloor-1}| > \dots > |A_2| \geq |A_1| \\ |A_{\lfloor\frac{n}{2}\rfloor}| = |A_{\lfloor\frac{n}{2}\rfloor}| > |A_{\lceil\frac{n}{2}\rceil+1}| > \dots > |A_{n-1}| > |A_n| \end{array}\right\} \quad (2)$$

So if we take the family $A_{\lfloor\frac{n}{2}\rfloor}$, the elements in $A_{\lceil\frac{n}{2}\rceil}$ has the maximum degree, as each element in $A_{\lfloor\frac{n}{2}\rfloor}$ is adjacent by $\binom{n}{\lceil\frac{n}{2}\rceil-1}$ and adjacent to $\binom{n}{\lceil\frac{n}{2}\rceil+1}$. Similarly if we take $A_{\lfloor\frac{n}{2}\rfloor}$. For the minimum degree in G_τ , from (1) and (2) we can see that the family set A_1 has only n singleton elements and each of them is adjacent to $\binom{n}{2}$ elements in A_2 , and A_n unique vertex with order n, and this vertex is adjacent by $\binom{n}{n-1}$ the elements of A_{n-1}. Now, we discuss with the following cases:

Case 1: if $|X| = 2$, $G_\tau \cong P_3$ and each element of A_1 has degree 1 which is the minimum degrees in G_τ .

Case 2: If $|X| = 3$, then A_1 has 3 elements each of them is adjacent to the 3 elements in A_2. That is the degree of each vertex in A_1 is 3, also A_n have one vertex only and it is adjacent by 3 elements in A_2 . That is the degree of the element of A_n is 3. Thus the minimum degree in G_τ when $|X| = 3$ lies in A_1 and A_n, and in each of them is equal to 3.

Case 3: If $|X| > 3$, in this case and by using the inequalities 1 and 2 above the vertex in A_n has the minimum degree of G_τ. As A_n has only one vertex which is adjacent by $\binom{n}{n-1}$ the elements of A_{n-1}, that is the degree of A_n is n.

Theorem 3.8 [13]: Let G be a graph. Then $\chi(G) = 2$ if and only if G is bipartite.

Proposition 3.9: Let $|X| = n$, $(n \geq 2)$, G_τ be a discrete topological graph on X. Then

(i) Rad $(G_\tau) = \left\lfloor\frac{n}{2}\right\rfloor$

(ii) Diam $(G_\tau) = n-1$ for $n > 2$.

(iii) The chromatic number $\chi(G_\tau) = 2$.

(iv) The clique number $\omega(G) = 2$.

Proof: Let $A_1, A_2, \ldots, A_n$ be the sets of V in G_τ. Then by Definition 2.4, we can see that the eccentricity of the elements of A_1 are equals. Similarly for the elements of $A_2, A_3, \ldots, A_n$.

Now we discuss two cases:

Case 1: If n is odd, then the eccentricity of any vertex in $A_{\lceil \frac{n}{2} \rceil}$ is $\left\lfloor \frac{n}{2} \right\rfloor$ and the eccentricity of the elements of G_τ is arranged in increasing order from the left and right sides of $\left\lfloor \frac{n}{2} \right\rfloor$ i.e.

$e(v \in A_1) > \cdots > e(v \in A_{\lceil \frac{n}{2} \rceil - 2}) > e(v \in A_{\lceil \frac{n}{2} \rceil - 1}) > e(v \in A_{\lceil \frac{n}{2} \rceil}) < e(v \in A_{\lceil \frac{n}{2} \rceil + 1}) < \cdots < e(v \in A_n)$.

Then the minimum eccentricity in G_τ is in the elements of $A_{\lceil \frac{n}{2} \rceil}$ and is equal to $\left\lfloor \frac{n}{2} \right\rfloor$. and the maximum eccentricity of G_τ is in the elements of A_1 or the vertex of A_n which is equal to $n-1$. Hence rad $(G_\tau) = \left\lfloor \frac{n}{2} \right\rfloor$ and diam $(G_\tau) = n-1$ in this case.

Case 2: If n is even, then the eccentricity of any elements in $A_{\frac{n}{2}}$ and $A_{\frac{n}{2}+1}$ is $\frac{n}{2}$, and the eccentricity of the other elements of G_τ is arranged in increasing order from the left and right sides of $A_{\frac{n}{2}} = A_{\frac{n}{2}+1}$; i.e. $e(v \in A_1) > \cdots > e(v \in A_{\frac{n}{2}-1}) > e(v \in A_{\frac{n}{2}}) = e(v \in A_{\frac{n}{2}+1}) < e(v \in A_{\frac{n}{2}+2}) < \ldots < e(v \in A_n)$.

Then the minimum eccentricity in G_τ is in the elements of $A_{\frac{n}{2}}$ or in the elements of $A_{\frac{n}{2}+1}$ which is equal to $\frac{n}{2}$. And the maximum eccentricity of G_τ in the elements of A_1 and the vertex of A_n which is equal to $n-1$. Hence rad $(G_\tau) = \frac{n}{2}$ and diam $(G_\tau) = n-1$ in this case, and **(i)**, **(ii)** are proved.

To prove **(iii)**. The proof is follows from Proposition 3.1 and Theorem 3.8.

(iv) Since each of the set $A_1, A_2, \ldots, A_n$ is independent set, then the prooph is follows.

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