Applying Lie Group Symmetries to Nonlinear Partial Differential Equations

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Abstract:

In this paper an introduction to classical Lie point symmetry method is presented, review for of the role of this symmetries in solving partial differential equations . and then showing some recent results for the application of classical Lie point symmetry method to some nonlinear partial differential equations and determined the solvable Lie group generators of the point symmetries .

Introduction :

Sophus Lie pioneered the modern approach for studying and finding special solutions of systems of nonlinear partial differential equations (PDEs) at the end of the nineteenth century in1929. There's still not general theory for finding analytical solutions to a nonlinear PDEs such in the case of linear ones, the application of group of transformations to study nonlinear PDEs has to be one of the powerful methods that answer this difficult problem this method have been developed in the past few years by Ovsyannikov 1982, Ibragimov(1994 -1996) and P.Olver 1986 , and others further remarks on the historical development of this subject and its applications to nonlinear models occurring in different research can be found for example in (Ames 1965),Bluman and COLE 1969), Clarkson and Kruskal 1989), (Clarkson,1995),(Hydon 2000), (Stephani 1989).

The theory of continues group of transformations created by Lie become one of the most important tools for geometric and algebraic study of general nonlinear PDEs, and solve this PDEs by using their Lie group and continoues symmetry transformations and their invariance, and also the Symmetry analysis plays an important turn in the theory of differential equation. The original symmetry method for reduction of the order of ordinary differential equations (ODE) and reduction of the number of independent variable for both linear and nonlinear (PDE.) probably the most useful point transformation of PDEs are those which form acontinouse Lie point symmetry group(the classical Lie point symmetry method(CLS)) the method for determining the symmetry group of differential equation is straightforward and described in several books as in[1,2,3,5,6,7,8,9,10,13,16,17]

Although Lie point symmetries represent a very powerful tool, they can yield very lengthy calculations .in fact, interest in them and their generalizations has increased during the last twenty years because of the availability of symbolic computation packages.

In order to apply the algorithm of (CLS), the use of symbolic manipulation programs has become imperative, such that a verity of packages have been developed for many computer algebra system a survey of these programs can be found in Herman 1994. For the computer system APLE, the programs ESOLV (CarminatiandVu2000), SYMMETRY (Hickman, 2001) and RIF (Reidand Wittkopf, 2001) contain routines for generating classical symmetries. This method is very successfully used in several branches of physics such as quantum filed theory, classical mechanics and physical chemistry..

In this paper we will provide an introduction to symmetry analysis, a review of the role of symmetries in solving partial differential equations is presented. This paper began with some definitions and theorems selected in order.

Lie Point Symmetry Transformations:

Lie Group of Transformations

In this section we present the basics behind acontinouse transformations by only considering one parameter and n independent variable.[2,3,6,10,13,16]

If we have $\mathbf{x} = (x1, x2, \ldots, xn)$ in D R^n .the set of transformations

$\mathbf{\hat{x}} = \mathbf{X}(\mathbf{x},\varepsilon)$ (1)

Define for each $\mathbf{x}$ in D and parameter ε in S R, with the law of composition of parameters $\phi(\varepsilon,\delta)$ in S,forms *A one– parameter Lie group of transformations* G must be satisfy the following :

1- for each ε in S the transformations are one to one and onto *D*.

2- S with the law of composition ϕ forms a group *G* .

3- for each $\mathbf{x}$ in *D*, $\mathbf{\hat{x}} = \mathbf{x}$ when $\varepsilon = \varepsilon 0$ corresponds to the identity of *G*

4- if $\mathbf{\hat{x}} = \mathbf{X}(\mathbf{x},\varepsilon)$ and $\mathbf{\hat{x}} = \mathbf{X}(\mathbf{x},\delta)$ then $\mathbf{\hat{x}} = \mathbf{X}(\mathbf{x}, \phi(\varepsilon, \delta))$.

5- ε is a continues parameter, S is an interval in *R*, and $\varepsilon = 0$ corresponds to the identity element.

6- $\boldsymbol{X}$ is infinity differentiable with respect to $\mathbf{x}$ in D and an analytical of ε in S .

7- $\phi(\varepsilon, \delta)$ is an analytical function.

We expand a one- parameter Lie group of transformation around the identity $\varepsilon=0$

$$\hat{x} = x + \varepsilon \frac{\partial X(x,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} + O(\varepsilon^2) \qquad (2)$$

and let

$$\xi(x) = \frac{\partial X(x,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}$$

the transformation $\mathbf{\hat{x}} = \mathbf{x} + \varepsilon\, \xi(\mathbf{x})$ (3)

is called the infinitesimal transformation of the Lie group of transformations and the componts $\xi i(\mathbf{x})$ are called infinitesimals. The operator

$$X = X(x) = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i} \qquad (4)$$

Is called *the infinitesimal generator* of the one-parameter Lie group of transformationsand called Lie group generator.

The commentator, of two infinitesimal generator X1 and X2 is define as

[X1, X2] = X1X2- X2 X1

and the commentator of any two infinitesimal generator of an γ-parameter Lie group of transformations is also a

Lie group generator for one of its one – parameter subgroup.

Where the *Lie algebra* ℓ is a vector space over some field *f* with additional law of combination of elements in ℓ (the commentator) satisfying the properties .let $X\alpha$,$X\beta$,$X\gamma$ ε ℓ

1) $[X\alpha, X\beta] = -[X\beta, X\alpha]$

2) $[X\alpha,[X\beta,X\gamma]]+[X\beta,[X\gamma, X\alpha]]+[X\gamma,[X\alpha,X\beta]] = 0$

With most importantly closure with respect to commutation.

For example if we have the group of Rigid motion in R^2[15]

$x^* = x\cos\varepsilon 1 - y\sin\varepsilon 2 + \varepsilon 3$

$y^* = x\sin\varepsilon 1 + y\sin\varepsilon 2 + \varepsilon 3$

the corresponding infinitsimal generator are

$X1 = -y\partial x + x\partial y$, $X2 = \partial x$, $X3 = \partial y$

The commutator table of its Lie algebra follows:

	X1	*X2*	*X3*
X1	*0*	*-X3*	*X2*
X2	*X3*	*0*	*0*
X3	*-X2*	*0*	*0*

Lie Group of symmetry Transformations

The present section will present comprehensive method for differential equation via the use of symmetry groups. Symmetry group of differential equations transforms solutions of the system to other solutions.before attempting to determine the symmetry groups of systems of differential equations.

Consider the system of lines : x = cy + d and the one-parameter group of transformation,

Gc: (x,y) (x+cd, y+d); d € R ,c is constant. the lines are clearly Gc invariant ,and so Gc is asymmetry group of the system of lines.

The above example looks at symmetries of the solution of system of equatios.we can also examine the invariance of function, we said the surface ***F*(x)** = 0 is an *invariant surface* for a one- parameter Lie group of transformation if and only if

$F(\hat{x}) = 0$ when $F(x) = 0$ (5)

For example the function f(x,y) = x – cy is an invariant function since

f(x+cd,y+d) = f(x,y) , d € R

And we said the surface ***F*(x)** = 0 is an *invariant surface* for a one- parameter Lie group of transformation if and only if

$X F(x) = 0$ when $F(x) = 0$ (6)

Where ***X*** is the Lie group generator.

For the translation group Gc the infinitesimal generator is $X = c\,\partial x + \partial y$. we found that the function f(x,y) = x – cy is an invariant function .the same conclusion can easily derived from the condition of infinitesimal invariance since :

$Xf = c\,\partial x(x-cy) + \partial y(x-cy) = c - c = 0$

A one – parameter Lie group of transformation, which satisfies the invariance condition, given by(6) , is called *a one – parameter Lie group of symmetry transformations*.

These symmetry transformation can be used successfully when solving PDEs and the methods are discussed in [2,3].

Classical Lie Symmetries of PDEs:

Symmetry analysis plays an impotent role in the theory of differential equations.the original symmetry method for reduction of the order of ODEs and reduction of the number of independ and dependent variables for both linear and nonlinear PDEs is the classical Lie point symmetry method.

The final topic to be addressed before studying the symmetries of differential equations is the process of prolongation. The prolongation is vector function from the space of the independent variable to the space U^n , whose entries represent the values of f and all its derivatives up to order n. consider a system with two independent and one dependent variables then the space contains all partial derivatives of u (u,ux,uy,uxy,uxx,uyy).

As an example ,the Laplace `s equation in the plane :uxx + uyy = 0 the equation with coordinates (x,y,u,ux,uy,uxx,uxy,uyy).

Now let we consider the one- parameter Lie group of transformation:

$x^* = xi + \varepsilon\,\xi i(\mathbf{x}, u) + O(\varepsilon^2)$ (7)

$u^* = u + \varepsilon\,\eta(\mathbf{x},u) + O(\varepsilon^2)$

For i =1,….n acting on (**x**,u) – space

An operator

$$X = \sum_{i=1}^{n} \xi i(x,u)\frac{\partial}{\partial x i} + \eta(x,u)\frac{\partial}{\partial u} \quad , \; x \in R^{n} \qquad (8)$$

Is asymmetry of a PDE $F(\mathbf{x},u,ui,\dots) = 0$ of order p if

$$X^{[p]} F \big|_{F=0} = 0 \qquad (9)$$

Where $X^{[p]}$ is called the p th prolongation(extended) of the operator ***X*** **.**

Especial case for one depended variable and two independed variable

$xi^* = xi + \varepsilon\,\xi i(x1,x2, u) + O(\varepsilon^2)$

$u^* = u + \varepsilon\,\eta(x1,x2, u) + O(\varepsilon^2)$

$$ui^* = ui + \varepsilon\,\eta i^{(1)}(x1,x2, u, u1, u2, u11, u12, u22) + O(\varepsilon^2) \qquad (10)$$

$$uij^* = uij + \varepsilon\,\eta ij^{(2)}(x1,x2, u, u1, u2, u11, u12, u22) + O(\varepsilon^2)$$

(1) (2)

[2]

the extended ηi , ηij (i,j=1,2) are in [4,13] then $X^{[2]}$ has the form:

$$X^{[2]} = X + \eta 1^{(1)}\frac{\partial}{\partial u1} + \eta 2^{(1)}\frac{\partial}{\partial u2} + \eta 11^{(2)}\frac{\partial}{\partial u11} + \eta 12^{(2)}\frac{\partial}{\partial u12} + \eta 22^{(2)}\frac{\partial}{\partial u22} \quad , \; x \in R^{2}$$

$$\text{where} \quad ui = \frac{\partial u}{\partial xi} \quad , uij = \frac{\partial^2 u}{\partial xi\,\partial xj} \quad , \eta i^{(1)} = \frac{\partial \eta}{\partial xi} \quad , \eta ij^{(2)} = \frac{\partial^2 \eta}{\partial xi\,\partial xj}$$

the condition of invariance (the symmetry condition)

$$\xi 1\frac{\partial u}{\partial x1} + \xi 2\frac{\partial u}{\partial x2} + \dots + \xi n\frac{\partial u}{\partial xn} = \eta(x1, x2, \dots, xn, u) \qquad (11)$$

yields alinear system of PDEs in the functions { ξ1 , ξ 2 ,…, η } which is called the system of determining equations which is solved to obtained an infinitesimals Lie group of symmetries.

To find the similarity reduction and particular solutions to the PDE we solve the charistristic equation

$$\frac{dx1}{\xi 1} = \frac{dx2}{\xi 2} = \dots = \frac{dxn}{\xi n} = \frac{du}{\eta} \qquad (12)$$

and by several integrals we obtain the similarity solution which reduce the PDE to which is called the principle ODE.

Examples:

Classical Lie Group Symmetry of Kadomtsev – Petviashili(KP) equation.

The eq

$$u_{xt} + 3u_{yy} + 6u\,u_{xx} + 6u^2_x + u_{xxxx} = 0 \qquad (13)$$

Which is one the equations frequently examined in connection with solution procedures for non-linear models.this equation arises in several physical applications ranging from surface waves of rectangular canals to applications in plasma physics.

The point symmetries of the (KP)eq. Were examined in [18],

In expanded form Kp eq. In (10) has the operator

$$X = \xi 1(x,y,t,u)\frac{\partial}{\partial x} + \xi 2(x,y,t,u)\frac{\partial}{\partial y} + \xi 3(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u} \qquad (14)$$

and we can written the prolongation(extended) of the operator ***X*** of fourth prolongation ,

$$X^{[4]} = X + \eta^{(1)}_{100}\,\partial u_x + \eta^{(2)}_{200}\,\partial u_{xx} + \eta^{(2)}_{101}\,\partial u_{xt} + \eta^{(2)}_{020}\,\partial u_{yy} + \eta^{(4)}_{400}\,\partial u_{xxxx}$$

the symmetry condition of eq (13) yields alinear system of PDEs in the functions{ξ1,ξ 2 , ξ3 , η } which is called the system of determining equations whose solution is obtained in terms of arbitrary functions *f1(t),f2(t),f3(t),as:*

$$\xi 1 = f3(t) + \frac{1}{18}\left(6x\, f1'(t) - y(3f'(t) + y\, f1''(t))\right);$$

$$\xi 2 = f2(t) + \frac{2}{3}\, y\, f1'(t);$$

$$\xi 3 = f1(t);$$

$$\eta = \frac{1}{36}\left(-24\, u\, f1'(t) + 18 f2'(t) + 6x f1''(t) - 3y f1''(t) - y f1'''(t)\right)$$

hence we obtained an infinitesimal Lie group of symmetries.

To find similarity reduction and particular solutions to the KP eq we drive subgroup ,let

f1 (t)= k5 t +k6 t + k7,

f2(t)=k3t +k4

f3(t)=k1t+k2

yields aseven Lie algebra with basis

X ={2T,0,0,1} ,X ={1,0,0,0} , X={-Y,6T,0,0} ,

X={0,1,0,0} , X={6tx-y ,12ty,9t ,012tu+3x} ,

X= {x,2y,3t,-2u} , X={0,0,1,0}

Classical Lie Group Symmetry of the CMKdV-II Equation

The complex modified Korteweg-de Vries-II (CMKdV-II) [18]

$w_t - 6\,|w|^2 w_x + w_{xxx} = 0$ (15)

we first let w= u+iv and separate real and imaginary partsin (15) and obtain the system

$u_t - 6(u^2+v^2)u_x + u_{xxx} = 0$

$v_t - 6(u^2+v^2)v_x + v_{xxx} = 0$ (16)

we consider the one- parameter Lie group transformation of (*x,t,u,v*) given by

$x^* = x+\varepsilon\, \xi 1(x,t,u) + O(\varepsilon^2)$,

$t^* = t+\varepsilon\, \xi 2(x,t,u) + O(\varepsilon^2)$,

$u^* = u+\varepsilon\, \eta 1(x,t,u) + O(\varepsilon^2)$,

$v^* = v+\varepsilon\eta 2(x,t,u) + O(\varepsilon^2)$

the system in eq.(16)admits Lie group with generators

X1 = ∂x , *X2*= ∂t , *X3*= *v* ∂-u ∂v , *X4* = x ∂x+3*t* ∂t - u∂u - v ∂v

The nonvanishing commutators are *[X1,X4]=X1 ,[X2,X4]=3X*

Hence acommutator table can be formed as follows:

	X1	*X2*	*X3*	*X4*
X1	*0*	*0*	*0*	*X1*
X2	*0*	*0*	*0*	*3X2*
X3	*0*	*0*	*0*	*0*
X4	*-X1*	*-3X2*	*0*	*0*

Classical Lie Group Symmetry of Nonlinear Model of the Heat equation

The inhomogeneous nonlinear heat equation in the form [12]

$$f(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(g(x)u^n u_x^q\right), \qquad (17)$$

1- when $f(x)=x^p$ and $g(x) = x^m$, then eq.(17) become

$$x^p\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(x^m u^n u_x^q\right), \qquad (18)$$

Classical symmetries determines transformation, the corresponding generator of the it is written as

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u},$$

and the condition of invariance is

$$X^{(2)} = X + \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} \qquad (19)$$

is the second prolongation of the vector field *X* and η^x , η^t , η^{xx} are expressed in terms of ξ , τ , η and their derivatives. From eq.(19) and from the coefficient of the various monomials of u. we get the following set of determining equations

ξ = ξ((x) ,τ = τ (t) ,η= η (u)

$m\,\xi - mx\,\xi_x + q\,x^2\,\xi_{xx} = 0$ (20)

$n\eta - nu\eta_u - qu^2\eta_{uu} = 0$

$$\frac{(m-p)}{x}\xi + \frac{n}{u}\eta + \tau' - (q+1)\xi_x + (q-1)\eta_u = 0$$

and by solving eqs(20) we get :

ξ((x)= [2c2 +c1(1-n-q)]x/r , τ (t)= 2c2t +c3 , η (u)=-c1u

where c1 ,c2,c3 are arbitrary constants and r=p-m+q+1 . then we have vector field

$$X1 = \partial t, \quad X2 = \frac{1-n-q}{R}\, x\,\partial x - u\,\partial u, \quad X3 = 2t\,\partial t + \frac{2}{r}\, x\,\partial x$$

2-when *f(x)=a² = constant* and *g(x) =1* , then eq.(17) become

$$a^2\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^n u_x^q\right), \qquad (21)$$

the equations was in (21)has the solution in eqs.(20) itself at p=m=0 ,i.e.,

ξ((x)= [2c2 +c1(1-n-q)]x/(q+1) , τ (t)= 2c2t +c3 , η (u)=-c1u

and the vector field

$$X1 = \partial t\ ,\quad X2 = \frac{1-n-q}{q+1}\, x\, \partial x - u\, \partial u\ ,\quad X3 = 2t\, \partial t + \frac{2}{q+1}\, x \partial x$$

Classical Lie Group Symmetry ofa one dimensional Porous Medium Equation

We consider the one dimensional porous medium equation [11]

$$u_t = (u^N u_x)_x + \frac{\mu}{x} u^n u_x = 0 \qquad (22)$$

Lie Point Symmetry of the eq.(22) with the exception of the case where n=-1 , has been classified in [12]where the symmetry generator is

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}\ ,$$

The symmetry of eq.(22) summarized in the following table:

	n	μ	(ξ , τ , η)
X1 *X2* *X3*	≒10	arbitrary	(0,1,0) (*x,2t,0)* *(0,nt,-u)*
X 4	≒ -2,-1,0	$= \frac{3n+4}{n+2}$	-n/(n+2) -2(n+1)/(n+2) ((n+2)*x* , 0 ,-2 *x* *u*)
X5	=-1	=1	(*xlnx,0,2(lnx-1)u)*

The Lie Group Symmetry Algebra for Some Nonlinear PDEs

1- the Boussinesq equation in [17]

$u_{tt} + u\, u_{xx} + (u_x)^2 + u_{xxxx} = 0$

the Lie algebra are

$X1 = x\, \partial x + 2t\, \partial t - 2u\, \partial u$, $X2 = \partial x$, $X3 = \partial t$

2- the Burgers equation in [7]

$u_t + u\, u_x = u_{xx} = 0$

the Lie algebra are

$X1 = \partial t$, $X2 = \partial x$, $X3 = t\, \partial x + \partial u$, $X4 = 2t\, \partial t + x\, \partial x - u \partial u$, $X5 = t^2\, \partial t + x\, \partial x - u\, \partial u$

3- Kolmogorov-petrovskii-piskunov equation in [17]

$u_t = u_{xx} + u\,(1-u)(u-a)$, $-1 < a < 1$

the Lie point symmetries

$X1 = \partial t$, $X2 = \partial x$

4- Nonlinear Wave equation in [14]

$u_{tt} = u u_{xx}$

the Lie point symmetries

$X1 = \partial t$, $X2 = \partial x$, $X3 = t\, \partial t + x\, \partial x$, $X4 = t\, \partial t - 2u \partial u$

5-Zabolotskaya-Khokhlov equation in [15]

$u_{xt} - (u u_x)_x - u_{yy} = 0$

the Lie point symmetries

$X1 = \partial t$, $X2 = \partial x$, $X3 = \partial y$, $X4 = y\, \partial x + 2t \partial y$, $X5 = t\, \partial t + x\, \partial x + y\, \partial y$

$X6 = 4t\, \partial t + 2x\, \partial x + 3y \partial y - 2u \partial u$, $X7 = t\, \partial x - \partial u$

Discussion:

Clearly the method of symmetry analysis of differential equations allows one to rigorously constrain the solution set of a particular problem, thereby simplifying it and facilitating the search for solution. Using the method developed by Lie, the equations are seemingly forced to reveal their symmetries. Obviously much more can be done using symmetry analysis than was demonstrated in the paper, however hopefully this glimpse will whet the reader1s appetite for more.

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تطبيق تناظر زمرة لي على المعادلات التفاضلية الجزئية اللاخطية

مها فالح جاسم

قسم أنظمة الحاسبات، المعهد التقني، كركوك، جمهورية العراق

الملخص:

في هذا البحث مقدمه عن طريقة تماثل لي الاعتيادية قدمت ومراجعة لدور تماثل لي الاعتيادية في حل المعادلات التفاضلية الجزئية.وبعد ذلك عرضنا بَعْض النَتائِجِ الأخيرةِ لتطبيقِ طريقة تماثل لي الاعتيادية لبعض المعادلات التفاضلية الجزئية اللاخطية.