Oscillations of First Order Linear Delay Differential Equations with positive and negative coefficients

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Abstract:

Oscillation criteria are obtained for all solutions of the first-order linear delay differential equations with positive and negative coefficients where we established some sufficient conditions so that every solution of (1.1) oscillate. This paper generalized the results in [11]. Some examples are considered to illustrate our main results.

Key word: Delay differential equations, oscillations.

Introduction:

The study of delay differential equation with positive (or negative) coefficients has been considered an attention of many researchers all over the world for the last several years see [1]-[3],[6],[8]-[10],and few of them investigated the case with positive and negative coefficients see [4]-[5],[7]. The authors in [11] investigated the first order neutral differential equations with positive and negative coefficients with constant delays. In this paper we generalized the result in [11] where we used variable delays. Consider the linear delay differential equation with positive and negative coefficients

$$\dot{x}(t)+P(t)x(\tau(t))-Q(t)x(\sigma(t))=0 \quad \dots (1)$$

Where $P,Q\in C([t_0,\infty),R^+)$, and τ,σ are continuous strictly increasing functions with $\lim_{t\to\infty}\tau(t)=\infty, \lim_{t\to\infty}\sigma(t)=\infty$, and

$$\tau(t)\le\sigma(t)<t \quad \dots (2)$$

By a solution of Eq.(1.1) we mean a function $x\in([t_x,\infty),R)$ such that x satisfies eq.(1.1), $t_x=\max\{\tau(t),\sigma(t)\}$.

A solution of eq.(1) is said to be oscillatory if it has arbitrarily large zeros , otherwise is said to be nonoscillatory .The purpose of this paper is to obtain sufficient conditions for the oscillation of all solutions of eq. (1).

1. Some Basic Lemmas:

The following lemmas will be useful in the proof of the main results:

Lemma 1 (theorem 2.1.1 [7]) . If $q(t)<t \quad \forall t\ge t_0$ is continuous function $\lim_{t\to\infty}q(t)=\infty$ and

$$\liminf_{t\to\infty}\int_{q(t)}^{t}P(s)ds>\frac{1}{e} \quad \dots. (1.1)$$

Then the following statements are true:

1. $\dot{x}(t)+P(t)x(q(t))\le 0$ has no eventually positive solutions.
2. $\dot{x}(t)+P(t)x(q(t))\ge 0$ has no eventually negative solutions.
3. $\dot{x}(t)-P(t)x(q(t))\ge 0$ has no eventually positive solutions.
4. $\dot{x}(t)-P(t)x(q(t))\le 0$ has no eventually negative solutions.

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Lemma 2 .(lemma 1.5.4 [5]) Let $a \in (-\infty, 0)$, $\tau \in (0, \infty)$, $t_0 \in R$ and suppose that a function $x \in [(t_0 - \tau, \infty), R]$ satisfies the inequality $x(t) \le a + \max_{t-\tau \le s \le t} x(s)$ for $t \ge t_0$. Then x cannot be a non-negative function.

Remark. We can generalize Lemma 2 by taking $\tau(t)$ to satisfy (2) and $x(t) \le a + \max_{\tau(t) \le s \le t} x(s)$, $t \ge t_0$.Then x is eventually negative function for $t \ge t_0$.

The following lemma improve lemma 2.6.1 given in [5]

Lemma 3.

Assume that (2) holds.

Let $x(t)$ be an eventually positive solution of (1) and set

$$z(t) = x(t) - \int_{\tau(t)}^{\sigma(t)} Q(\sigma^{-1}(s))x(s)ds, \qquad t \ge \sigma(\tau^{-1}(t_0))$$

… (1.2)

$$u(t) = x(t) - \int_{\tau(t)}^{\sigma(t)} P(\tau^{-1}(s))\, x(s)ds, \qquad t \ge \tau(\sigma^{-1}(t_0))$$

… (1.3)

then the following statements are true.

1- if $\int_{\tau(t)}^{\sigma(t)} Q(\sigma^{-1}(s))ds \le 1$, $P(t) \ge Q(\sigma^{-1}(\tau(t)))\dot{\tau}(t)$ and $\dot{\sigma}(t) \ge 1$ … (1.4)

then $z(t)$ is eventually positive and non increasing function.

2- if $\int_{\tau(t)}^{\sigma(t)} P(\tau^{-1}(s))ds \le 1, P(\tau^{-1}(\sigma(t)))\,\dot{\sigma}(t) \ge Q(t)$ and $\dot{\tau}(t) \le 1$ … (1.5)

then $u(t)$ is eventually positive and non-increasing function.

Proof.

Suppose that $x(t) > 0, x(\tau(t)) > 0$, and $x(\sigma(t)) > 0, t \ge t_0$

1. Differentiate (1.2) and use (1) we get

$$\dot{z}(t) = -[P(t) - Q(\sigma^{-1}(\tau(t)))\dot{\tau}(t)]$$
$$x(\tau(t)) + Q(t)x(\sigma(t))(1 - \dot{\sigma}(t))$$

Using (1.4) this yields

$$\dot{z}(t) \le -[P(t) - Q(\sigma^{-1}(\tau(t)))\,\dot{\tau}(t)]\,x(\tau(t)) \le 0$$

…(1.6)

to show that $z(t)$ is eventually positive , suppose that $z(t) \le 0$ since $z(t)$ is not equivalent to 0 for $t \ge t_1 \ge t_0$ then there exists $t_2 \ge t_1$ such that $z(t_2) < 0$ then $z(t) \le z(t_2)$ *for* $t \ge t_2$ then (1.2) will be

$$x(t) = z(t) + \int_{\tau(t)}^{\sigma(t)} Q(\sigma^{-1}(s))x(s)\,ds \le z(t_2) + \int_{\tau(t)}^{\sigma(t)} Q(\sigma^{-1}(s))x(s)\,ds$$
$$\le z(t_2) + \int_{\tau(t)}^{\sigma(t)} Q(\sigma^{-1}(s))\,ds\,(\max_{\tau(t) \le s \le \sigma(t)} x(s))$$

$$x(t) \le z(t_2) + \max_{\tau(t) \le s \le t} x(s) \qquad t \ge t_2$$

Then by lemma 2 we see that $x(t) < 0$ for $t \ge t_2$, this is a contradiction.

2. Differential (1.3) and use (1) we get

$$\dot{u}(t) = [Q(t) - P(\tau^{-1}(\sigma(t)))\dot{\sigma}(t)]x(\sigma(t))$$
$$+ P(t)x(\tau(t))(\dot{\tau}(t) - 1)$$

Using (1.4) this yields

$$\dot{u}(t) \le [Q(t) - P(\tau^{-1}(\sigma(t)))\dot{\sigma}(t)]x(\sigma(t)) \le 0$$

… (1.7)

To show that $u(t)$ eventually positive, suppose that $u(t) \le 0, t \ge t_0$,

since $u(t)$ is not equivalent to 0 for $t \ge t_1 \ge t_0$ then there exists $t_2 \ge t_1$ such that $u(t_2) < 0$ then $u(t) \le u(t_2)$ *for* $t \ge t_2$ then (1.3) will be

$$x(t) = u(t) +$$
$$\int_{\tau(t)}^{\sigma(t)} P(\tau^{-1}(s))x(s)ds \le u(t_2) + \int_{\tau(t)}^{\sigma(t)} P(\tau^{-1}(s))ds[\max_{\tau(t) \le s \le \sigma(t)} x(s)]$$

$$x(t) \le u(t_2) + \max_{\tau(t) \ge s \le t} x(s) \quad \textit{for} \quad t \ge t_2$$

thus by Lemma 2 we see that $x(t) < 0$, $t \geq t_2$.
This is a contradiction, the proof of lemma is complete. ■

2.Main results:

The next result provid a sufficient conditions for the oscillation of all solutions of eq. (1)

Theorem1.

Assume that (2),(1.4) hold and that

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} [P(s) - Q(\sigma^{-1}(\tau(s)))\dot{\tau}(s)]\,ds > \frac{1}{e} \quad \dots (2.1)$$

Then every solution of (1) oscillates.

Proof. Assume for the sake of contradiction that equation (1) has an eventually positive solution $x(t)$,by Lemma 3 (1) it follows that $z(t)$ which is defined by (1.2) is an eventually positive and non increasing function and $z(t) \leq x(t)$ also from (1.6) we see that eventually

$$\dot{z}(t) + [P(t) - Q(\sigma^{-1}(\tau(t)))\dot{\tau}(t)]x\,(\tau(t)) \leq 0$$

Or

$$\dot{z}(t) + [P(t) - Q(\sigma^{-1}(\tau(t)))\dot{\tau}(t)]\,z(\tau(t)) \leq 0$$

but in view of (2.1) it follow from Lemma 1(1) that the last inequality cannot have an eventually positive solutions. Which is a contradiction since $z(t)$ is eventually positive function. ■

Theorem 2.

Assume that (2),(1.6) hold and that

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} [P(\tau^{-1}(\sigma(s)))\dot{\sigma}(s) - Q(s)]\,ds > \frac{1}{e} \quad \dots (2.2)$$

Then every solution of equation (1) oscillates.

Proof: Assume for the sake of contradiction that (1) has eventually positive solution $x(t)$.By Lemma 2 (2) it follows that $u(t)$ which is defined by (1.3) is eventually positive and monotone decreasing function and $u(t) \leq x(t)$. Also from (1.7) we see that eventually

$$\dot{u}(t) + [P(\tau^{-1}(\sigma(t)))\,\dot{\sigma}(t) - Q(t)]\,x(\sigma(t)) \leq 0$$

$$\dot{u}(t) + [P(\tau^{-1}(\sigma(t)))\,\dot{\sigma}(t) - Q(t)]\,u(\sigma(t)) \leq 0$$

But in view of (2.2) it follow from Lemma 1(2) that the last inequality cannot eventually have a positive solution. Which is a contradiction since $u(t)$ is eventually positive function. ■

Example 1.

Consider the delay differential equation;

$$\dot{x}(t) + \frac{t^2}{(t-\frac{5\pi}{2})^2}x(t-\frac{5\pi}{2}) - \frac{2t}{(t-2\pi)^2}x(t-2\pi) = 0 \qquad t > \frac{5\pi}{2}$$

(E1)

One can find that conditions (1.4) and (2.1) are met so according to theorem 1 every solution of equation (E1) oscillate for instance the solution $x(t) = t^2 \sin t$ is oscillatory solution. ■

Example 2.

Consider the delay differential equation;

$$\dot{x}(t) + \frac{t-2\pi}{t^2}x(t-2\pi) - \frac{t-\frac{3\pi}{2}}{t}x(t-\frac{3\pi}{2}) = 0 \qquad t > 0$$

(E2)

One can find that conditions (1.5) and (2.2) are met so according to theorem 2 every solution of equation (E2) oscillate for instance the solution $x(t) = \frac{1}{t}\sin t$ is oscillatory solution. ■

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تذبذب المعادلات التفاضلية التباطوئية الخطية من الرتبة الأولى ذات المعاملات الموجبة والسالبة

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الخلاصة:

في هذا البحث تمت دراسة المعادلات التباطؤية الخطية ذات المعاملات الموجبة و السالبة من الرتبة الأولى . حيث تم إيجاد شروط ضرورية وكافية لضمان تذبذب كافة حلول المعادلة $x'(t) + P(t)\,x(\tau(t)) - Q(t)\,x(\sigma(t)) = 0$, وشروط كافية أخرى للحلول غير المتذبذبة كي تكون متقاربة الى الصفر,وقد أعطينا بعض الأمثلة لتوضيح هذه النتائج .