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EFFECT OF BOUNCING AND PITCHING ON THE COUPLED NATURAL FREQUENCY OF AN AUTOMOBILE

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ABSTRACT
This paper describes a unique solution of bouncing and pitching interactions, in which it play an increasingly significant role in vehicle. While the behavior of vibrating system has motivated numerous experimental and numerical works, very few studies have been devoted to the case of vibration of car. This paper presents an investigation of bouncing – pitching effects, which is of strong interest for vibration of vehicle. The prediction of the natural frequencies in the car is thus a challenging task. This paper have a novel discussion of the dynamic behavior (natural frequencies and mode shapes) of vehicle and the relationships between the coupled natural frequency of the car. This work is theoretical and finite element method via ANSYS software study of dynamic performance of vehicle which is consists of two masses (body of car and tire) . The paper deals with obtaining the vibration characteristic of an automobile analyzed as system without damping due to the effect of bouncing and pitching. The focus of the vibration analysis is to obtain the eigenvectors, and the corresponding natural frequencies of the system. Good agreement is evident between the theoretical and finite element method in which the discrepancy between the two methods is about 0.05% for bouncing and 0.2% for pitching model. Also, it can be deduced that the join between car and tire, the result of the natural frequencies are differed from that of the natural frequency for the individual one, in which the higher frequency is increased and decreased the less one.
KEYWORDS: **coupled vibration, bouncing, pitching, Ansys**

تأثير الحركة العمودية والتأرجحية على الترددات الطبيعية المزدوجة للسيارة

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الخلاصة

ان هذا البحث يصف حلاً فريداً للحركة التارجحية والحركة العمودية التي تلعب دورا مهما في المركبات، حفز من خلاله العديد من التجارب والتحليلات العددية لدراسة سلوك نظام الاهتزازات في المركبة حيث انه يوجد عدد قليل من البحوث في هذا المجال حيث ان البحث الحالي يدرس تاثير الحركة التارجحية والعمودية والتي تؤدي الى اهتزازات قوية في المركبة ، وان التكهن للترددات الطبيعية التي تحدث في السيارة هو نوع من انواع التحدي حيث ان هذا البحث يتطرق الى تحليل نادر للسلوك الديناميكي (التردد الطبيعي وانماط الاشكال المقابلة له) حيث تضمن البحث دراسة نظرية ودراسة عددية باستخدام طريقة العناصر المحددة من خلال برنامج ANSYS للاداء الديناميكي للمركبة التي تتكون من كتلة السيارة والاطار حيث تم الحصول على تحليل المركبة بدون تخميد تحت تاثير الحركة العمودية والتارجحية. وقد تم التركيز على ايجاد الترددات الطبيعة وانماط الاشكال المقابلة لها. تطابق جيد للنتائج بين النظري والعناصر المحددة حيث ان نسبة الخطأ وصلت الى % في حالة الحركة العمودية و 0,2% في حالة الحركة التارجحية وتم الاستنتاج انه في حالة

الدمج بين جسمين مهتزين يكون التردد الطبيعي الناتج مختلفا عن التردد الطبيعي المفرد لكلتا الكتلتين حيث يتزايد التردد الطبيعي الاكثر قيمة ويقل التردد الطبيعي الاقل قيمة.

1-INTRODUCTION

Vibration, which occurs in most machines, structures, and mechanical components, can be desirable and undesirable. Vibrations on the strings of a harp are desirable because it produces a beautiful sound. However, vibration in a vehicle is very undesirable because it can cause discomfort to the passengers in the vehicle. Vibration is undesirable, not only because of the unpleasant motion, the noise and the dynamic stresses, which may lead to fatigue and failure of the structure, but also because of the energy losses and the reduction in performance which accompany the vibrations. Vibration analysis should be carried out as an inherent part of the design because of the devastating effects, which unwanted vibrations could have on machines and structures. Modifications can most easily be made in an effort to eliminate vibration, when necessary or to reduce it as much as possible. Bouncing and pitching effects play an increasingly significant role in a vehicle design. While the behavior of vibrating system has motivated numerous experimental and numerical works, very few studies have been devoted to the case of vibration of vehicle. Mimuro et al., 1990, showed in their research the parameter evaluation of vehicle dynamic performance . Loeb et al., 1990, examined on the relaxation length of a tire. The mechanical characteristics of the pneumatic tire have a large influence on the vehicle handling performance and directional response. Kinematic properties are those that occur when the tire is rolling on a surface. Mark and Shuguang, 2006, studied a mass-spring-damper model of a bouncing ball in which approximate expressions are derived for the model parameters as well as for the natural frequency and damping ratio, The results of an experimental test are used to provide predictions of the equivalent stiffness and damping, natural frequency and damping ratio, and coefficient of restitution for a bouncing ping pong ball. In our research an investigation of the coupled natural frequencies due to bouncing and pitching which is of strong interest for vehicle. This paper have a novel derivation of the dynamic behavior of vehicle where the dynamic characteristics of vehicle is studied using theoretical expression and finite element method via ANSYS software. The vehicle structure is discretized using mass and spring elements. The eigenvalues and eigenvectors are obtained. The modal analysis is presented as the deformed configuration of the vehicle. This study is identified the theoretical solution and finite element method via ANSYS software which is studied the modal analysis of vehicle.

2-THEORETICAL ANALYSIS

In vibration, a number of simplifying assumptions have to be made to model any real system. For example, a distributed mass may be considered as a lumped mass, or effect of damping in the system may be neglected particularly if only resonant frequencies are sought, or a non-linear spring may be considered linear, or certain elements may be neglected altogether if their effect is likely to be small. Furthermore, the directions of motion of the mass elements are usually restrained to those of immediate interest to the analyst.

A- Bouncing Model

A- 1 Undamped Bouncing Model

This model is analyzed using double masses, which represents tire and vehicle body masses as shown in **Fig.2.**

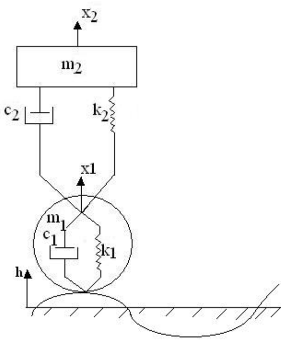


Fig.1.

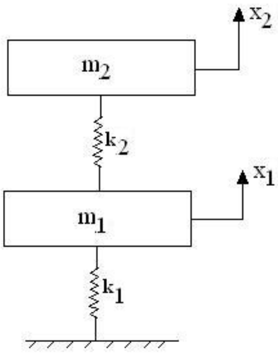


Fig.2.

Form **Fig.1.** it can be deduced the following equations:

$$m_2\ddot{x}_2 = -c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) \quad (1)$$

$$m_1\ddot{x}_1 = -c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) - c_1(\dot{x}_1 - \dot{h}) - k_1(x_1 - h) \quad (2)$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0 \quad (3)$$

$$m_1\ddot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) + c_1(\dot{x}_1 - \dot{h}) + k_1(x_1 - h) = 0 \quad (4)$$

To get coupled undamped natural frequency , $c_1=0$, $c_2=0$, $h=0$, $\dot{h} = 0$, as shown in **Fig.2.**

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \quad (5)$$

$$m_1\ddot{x}_1 + k_2(x_2 - x_1) + k_1x_1 = 0 \quad (6)$$

From harmonic motion, let $= e^{i\omega t}$, $\dot{x} = i\omega e^{i\omega t}$, $\ddot{x} = -\omega^2 e^{i\omega t}$

Thus $\ddot{x}_1 = -x_1\omega^2$ and $\ddot{x}_2 = -x_2\omega^2$

$$m_2(-x_2\omega^2) + k_2(x_2 - x_1) = 0 \quad (7)$$

$$m_1(-x_1\omega^2) + k_2(x_2 - x_1) + k_1x_1 = 0 \quad (8)$$

The above two equations is simplified to get :

$$x_1 - \left(-\omega^2 + \frac{k_1+k_2}{m_1}\right) - x_2\left(\frac{k_2}{m_1}\right) = 0 \quad (9)$$

$$-x_1\left(\frac{k_2}{m_2}\right) + x_2\left(-\omega^2 + \frac{k_2}{m_2}\right) = 0 \quad (10)$$

Solving Eqs. (9) and (10) , we get

$$x_1[(\omega^4 m_1 m_2) - \omega^2(k_2m_1 + k_1m_2 + k_2m_{21}) + k_2k_1] = 0 \quad (11)$$

Hence , the characteristic equation has the form :

$$\omega^4(m_1 m_2) - \omega^2(k_2 m_1 + k_1 m_2 + k_2 m_2) + k_2 k_1 = 0 \qquad (12)$$

Solving , we get

$$\omega_{1,2} = \sqrt{\frac{k_2 m_1 + k_1 m_2 + k_2 m_2}{2\, m_1 m_2} \mp \sqrt{\left(\frac{k_2 m_1 + k_1 m_2 + k_2 m_2}{2\, m_1 m_2}\right)^2 - \frac{k_2 k_1}{m_1 m_2}}}$$

(13)

Assume $= \frac{k_2 m_1 + k_1 m_2 + k_2 m_2}{2\, m_1 m_2}$, hence $\omega_{1,2} = \sqrt{B \mp \sqrt{B^2 - \frac{k_2 k_1}{m_1 m_2}}}$

(14)

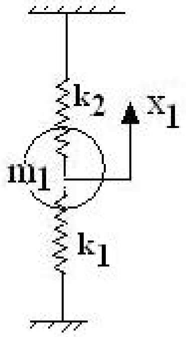


Fig.3.

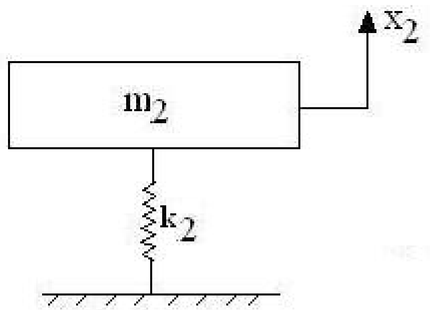


Fig.4.

Now the single natural frequency for the tire, can be deduced from **Fig.3.**

$$m_1 \ddot{x}_1 = -k_2 x_1 - k_1 x_1 \qquad (15)$$

Simplified the Eq.(15) with harmonic motion, we get

$$-\omega^2 + \left(\frac{k_1 + k_2}{m_1}\right) = 0 \qquad (16)$$

Thus, $\omega_I = \sqrt{\frac{k_1 + k_2}{m_1}}$ (17)

And the natural frequency for the car body only, can be deduced from **Fig.4.**

$$m_2 \ddot{x}_2 = -k_2 x_2 \qquad (18)$$

Simplified the Eq.(17) with harmonic motion, we get

$$-\omega^2 + \left(\frac{k_2}{m_2}\right) = 0 \qquad (19)$$

Thus, $\omega_{II} = \sqrt{\frac{k_2}{m_2}}$ (20)

Now the relationship between the couple and single natural frequencies , it can be get by substituting Eqs. (17) and (20) into Eq.(14), thus

$$\omega_{1,2} = \sqrt{\frac{1}{2}(\omega_{II}^2 + \omega_I^2) \mp \sqrt{\left(\frac{1}{2}(\omega_{II}^2 + \omega_I^2)\right)^2 - \frac{k_2 k_1}{m_1 m_2}}} \qquad (21)$$

From the above derivation , it can be deduced that when we join between two vibration bodies , the result of the natural frequency is differed from that of the natural frequency of the individual body. In which the higher frequency is increase and decrease the less one. Now, for the given data

$4m_2$= mass of the car body= 1000 kg , $4m_1$= mass of tires =100 kg , stiffness of spring, k_2 = 10000 N/m , and stiffness of tire, k_1= 10000 N/m, hence form Eq.(17) $\omega_I = 66.33$ rad/sec , and from Eq.(20) , $\omega_{II} = 6.32$ rad/sec, and the natural frequencies from Eq.(21) yield $\omega_1 = 66.35717$ rad/sec and $\omega_2 = 6.0279$ rad/sec. In this case the couple is presented due to differences in ω_1 ,ω_2 , ω_I, and ω_{II}. (i.e. $\omega_1 > \omega_I > \omega_{II} > \omega_2)$.

A-2 Damped Bouncing Model

This model is analyzed using double masses, which represents tire and vehicle body masses and (h) which is represented the excitation force as shown in **Fig.1.** In this case the response x_1 and x_2 for tire and car body are found respectively.

From Eqs. (1) and (2)

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0 \quad (22)$$

$$m_1\ddot{x}_1 - c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) + c_1(\dot{x}_1 - \dot{h}) + k_1(x_1 - h) = 0 \quad (23)$$

and we have, $\dot{x}_1 = i\omega x_1$, $\ddot{x}_1 = -\omega^2 x_1$, $\dot{h} = i\omega H, \dot{x}_2 = i\omega x_2$, $\ddot{x}_2 = -\omega^2 x_2$

$$-m_1\omega^2 x_1 - c_2 i\omega(x_2 - x_1) - k_2(x_2 - x_1) + i\omega k_1(x_1 - H) + c_1(x_1 - H) = 0 \quad (24)$$

$$-m_2\omega^2 x_2 + c_2 i\omega(x_2 - x_1) + k_2(x_2 - x_1) = 0 \quad (25)$$

Simplified

$$x_1(-m_1\omega^2 + c_2 i\omega + k_2 - i\omega c_1 + k_1) + x_2(-c_2 i\omega - k_2) = (i\omega c_1 + k_1)H \quad (26)$$

$$x_1(-c_2 i\omega - k_2) + x_2(-m_2\omega^2 + c_2 i\omega) = 0 \quad (27)$$

Rearranged, we get

$$x_1 I + x_2 II = H\,V \quad (28)$$

$$x_1 II + x_2 III = 0 \quad (29)$$

Where

I=$(-m_1\omega^2 + k_2 + k_1) + i(c_2\omega - \omega c_1)$

II= $-c_2 i\omega - k_2$

III=$-m_2\omega^2 + c_2 i\omega + k_2$

V= $i\omega c_1 + k_1$

Solving Eqs.(28) and (29) , we get

$x_1 = \frac{H\,V\,III}{I\,III - II^2}$ which represent the response of tire and $x_2 = \frac{-II\;H\,V}{I\,III - II^2}$ which represent the response of car body .

B- Pitching Model

B-1 Undamped Pitching Model

This model is analyzed using one masses, which represents the vehicle body masses and neglecting the tire mass, as shown in **Fig.5.**

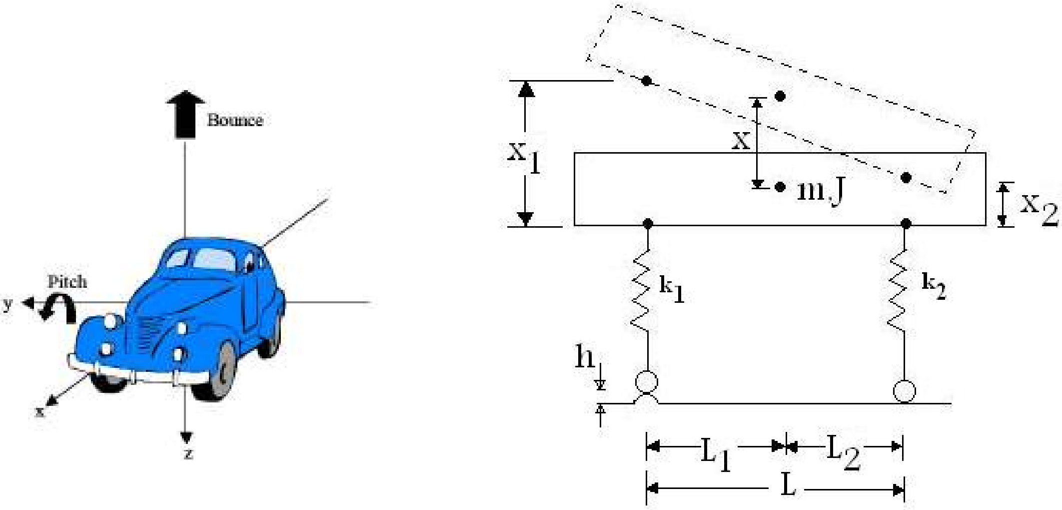


Fig.5. Pitching Model

From **Fig.5.** it can be deduced that

$$m\ddot{x} = -k_1(x_1 - h) - k_2x_2 \quad (30)$$

$$J\ddot{\alpha} = -k_1(x_1 - h)L_1 + k_2x_2L_2 \quad (31)$$

$$m\ddot{x} + k_1x_1 + k_2x_2 = k_1h \quad (32)$$

$$J\ddot{\alpha} + k_1x_1L_1 - k_2x_2L_2 = k_1hL_1 \quad (33)$$

we have $x_1 = x + L_1\alpha$, $x_2 = x + L_2\alpha$, $\ddot{x} = -\omega^2 x$, $\ddot{\alpha} = -\omega^2\alpha$, and $h=0$

Substituted into Eqs.(32) and (33) and rearranged we get

$$x(-m\omega^2 + k_1 + k_2) + \alpha(k_1L_1 - k_2L_2) = 0 \quad (34)$$

$$x(k_1L_1 - k_2L_2) + \alpha(-J\omega^2 + k_1L_1^2 + k_2L_2^2) = 0 \quad (35)$$

Solving we get

$$mJ\omega^4 - \omega^2[m(k_1L_1^2 + k_2L_2^2) + J(k_1 + k_2)] + k_1k_2L^2 = 0 \quad (36)$$

Solving we get

$$\omega_{3,4} = \frac{k_1+k_2}{2m} + \frac{k_1L_1^2+k_2L_2^2}{2J} \mp \sqrt{\left(\frac{m(k_1L_1^2+k_2L_2^2+J(k_1+k_2)}{2mJ}\right)^2 - \frac{k_1k_2L^2}{mJ}} \quad (37)$$

This coupled undamped natural frequency for the system in case of pitching model. Now in case of single natural frequency , first fixing the rotational movement i.e we have vertical movement only , as in **Fig.6.** from that , it can be deduced that

$$m_1\ddot{x} + k_2x + k_1x = 0$$

(38)

Simplified the Eq(38) with harmonic motion, we get

$$x\left[-\omega^2 + \left(\frac{k_1+k_2}{m}\right)\right] = 0 \tag{39}$$

Thus, $$\omega_m = \sqrt{\frac{k_1+k_2}{m}} \tag{40}$$

and secondly the vertical movement is fixed i.e. we have only rotational movement. From Fig.(7) , it can be deduced that

$$J_y\ddot{\alpha} = -k_1L_1\alpha L_1 - k_2L_2\alpha L_2$$

(41)

Simplified and rearranged, we get

$$\alpha\left[-\omega^2 + \frac{k_1L_1^2+k_2L_2^2}{J_y}\right] = 0 \tag{42}$$

$$\omega_f = \sqrt{\frac{k_1L_1^2+k_2L_2^2}{J_y}} \tag{43}$$

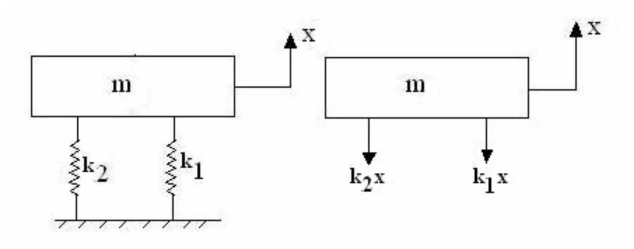


Fig.6.

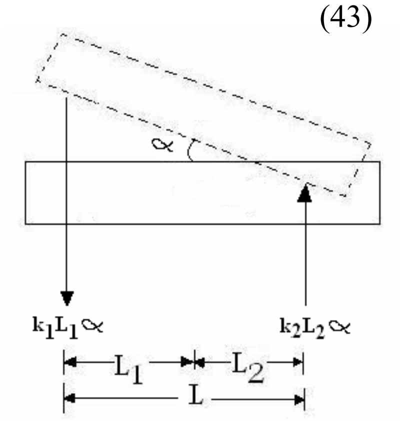


Fig.7.

The relationship between the couple and single natural frequencies , it can be get by substituting Eqs. (40) and (43) into Eq.(37), thus

$$\omega_{3,4} = \sqrt{\frac{1}{2}(\omega_m^2+\omega_f^2) \mp \sqrt{\left(\frac{1}{2}(\omega_m^2+\omega_f^2)\right)^2 - \frac{k_2k_1L^2}{m\,J_y}}} \tag{44}$$

In order to find the degree of coupling , we consider the body of car consists of three masses , concerned on the front, rear and in the center of gravity which are m_2,m_1 , and m_3 respectively.

From **Fig.8.** it can be deduced that

$$m_1 + m_2 + m_3 = m \tag{45}$$

$$m_1L_1 - m_2L_2 = 0 \tag{46}$$

$$m_1L_1^2 + m_2L_2^2 = mi^2 \tag{47}$$

Where i is the radius of gyration= $\sqrt{\frac{J}{m}}$

Solving Eqs. (45), (46) and (47) we get

$$m_3 = m\left[1 - \frac{i^2}{L_1L_2}\right] \tag{48}$$

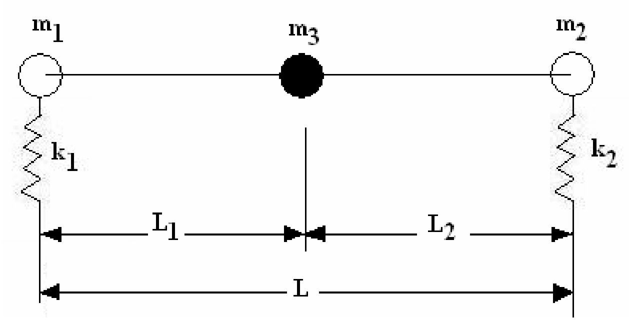


Fig.8.

When equal the natural frequencies in bouncing and pitching , the coupling equal zero, i.e. there is no coupling in natural frequencies. For the given data, m= mass of the car body= 1400 kg , J_y = pitching moment of inertia =3000 kg.m^2 , rear stiffness of spring, k_2 = 28000 N/m, front stiffness of spring, k_1= 20000 N/m , L_1=2 m and L_2 = 1.5 m

Hence, form Eq.(40) $\omega_m = 5.855$ rad/sec, and from Eq.(43) , $\omega_J = 6.90$ rad/sec, and the natural frequencies from Eq.(44) yield $\omega_3 = 6.893$ rad/sec and $\omega_4 = 5.8628$ rad/sec and the degree of coupling $m_3 = 405$ kg , i.e. $m_3>0$, the coupling exist.

3- FINITE ELEMENT EQUATION

The element equations of the dynamic system can be expressed in general form as:

$$[M_e]\{\ddot{x}_e\} + [K_e]\{x_e\} = \{F_{e(t)}\} \quad (49)$$

where : $[K_e] = \int_{vol} [B]^T \cdot [D] \cdot [B] \, dvol$ (50)

The analysis assembles all individual element equations to provide stiffness equations for the entire structure or mathematically

$$[M]\{\ddot{x}\} + [K]\{x\} = \{F_{(t)}\} \quad (51)$$

where: $[K] = \sum_{i=1}^{m} [K_e]$ (51)

$[M] = \sum_{i=1}^{m} [M_e]$ (52)

3.1 Eigenvalue Solution:

When finite element method is applied for the solution of eigenvalue problems, an algebraic eigenvalue problem is obtained as stated in Eq.(30). For most engineering problems, [K] and [M] will be symmetric matrices of order n (Erik, 1990).

$$[[D_o] - \lambda[I]]\{x\} = 0 \quad (53)$$

where $\lambda = \omega^2$ = eignvalue , $[D_o] = [M]^{-1}[K]$, [I] is a unit matrix and {x} is a eigenvector

hence the system equation can be written in the form :

$$[M]\{\ddot{x}\} = \{F\} - [K]\{x\} = \{F\} - \{F'\} = \{R\} \quad (54)$$

where $\{R\}$ is the residual force vector.

$$\{\ddot{x}\} = [M]^{-1}\{R\} \quad (55)$$

In practice, the above equation does not usually require solving of the matrix equation , since lumped masses are usually used which forms a diagonal mass matrix (Mareio Paz, 1990) .

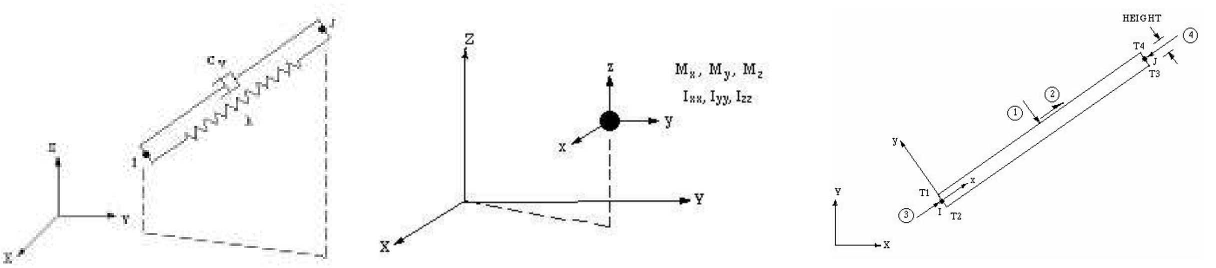
The solution to Eq.(55) is thus trivial, and the matrix equation is the set of independent equations for each degree of freedom j as follows:

$$\{x_j\} = \frac{\{R_j\}}{\{M_j\}} \quad (56)$$

4- MODEL GENERATION BY ANSYS

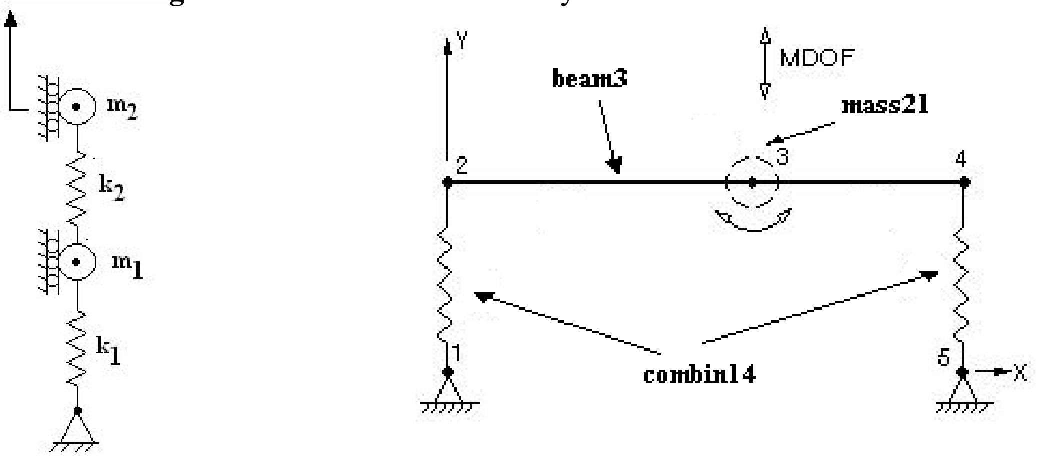
The ultimate purpose of a finite element analysis is to re-create mathematically the behavior of an actual engineering system(Saeed Mouveni, 1999). In other words, the analysis must be an accurate mathematical model of a physical prototype (Tim Langlais, 1999). In the broadest sense, the model comprises all the nodes, elements, material properties, real constants, boundary conditions and the other features that used to represent the physical system.

In ANSYS terminology, the term model generation usually takes on the narrower meaning of generating the nodes and elements that represent the special volume and connectivity of the actual system (ANSYS help, 1996 and Training Manual,1996). The method used in this research to generate a model is direct method with element (mass21) and element (combin14) for bouncing effect and (beam3) and (mass21) for pitching effect as shown in **Fig.9. Fig.10.** shows the finite element model for bouncing and pitching.



Combin14 mass21 beam3

Fig.9. Elements used in the analysis



Bouncing Pitching

Fig.10. Finite Element Model

5- RESULTS AND DISCUSSION

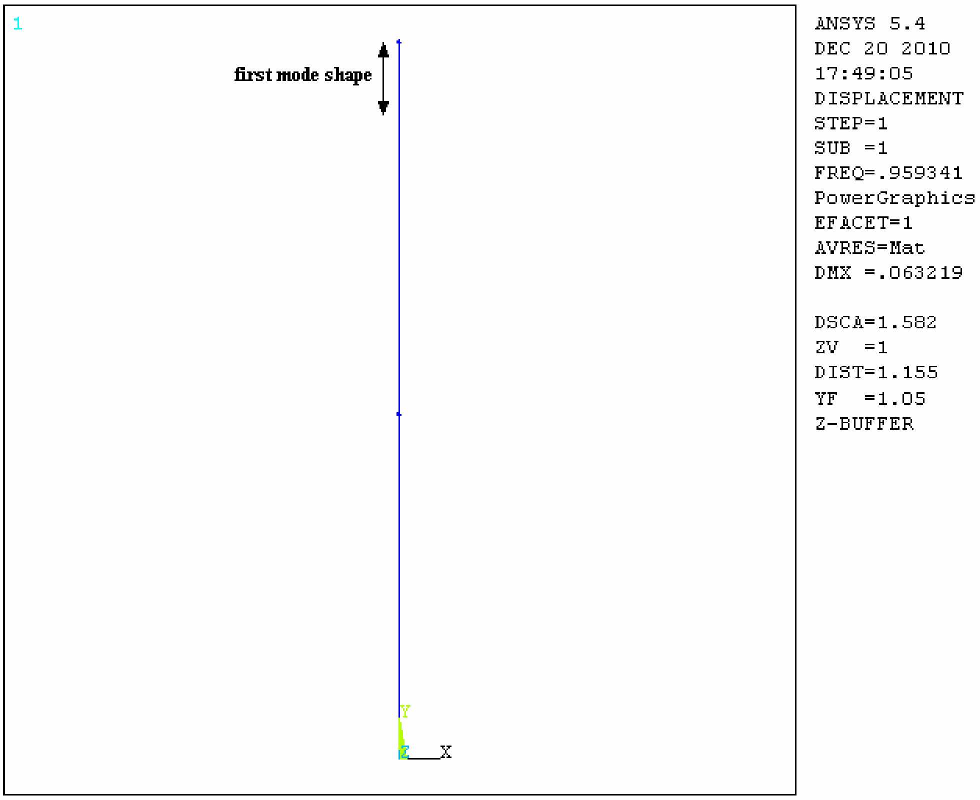
Free vibration analysis consists of studying the vibration characteristics of the system, such as natural frequencies and mode shapes. The natural frequencies and mode shapes of vehicle is very important parameter in the design system for dynamic loading conditions and minimization of machine failures. A detailed study is made using the formulation presented in this paper on the fundamental natural frequency and mode shape levels of vehicle. The free vibration characteristics have been investigated theoretically and by using ANSYS software.

The results reported two structural eigenvalues and eigenvectors which are based on the effect of bouncing and pitching. **Table 1** explained the natural frequencies of the vehicle with bouncing and pitching, respectively. It can be shown that the result of the natural frequency for mode no.1 and 2 in the theoretical and ANSYS analysis are good evident i.e. that the discrepancy for the bouncing model approximately to 0.05% and for pitching model is approximately 0.2%.

Table 1 Natural frequency of an automobile

Mode No.	**Natural freq. (rad/sec) (with bouncing effect)**			**Natural freq. (rad/sec) (with pitching effect)**		
	Theoretical Eq.(21)	*ANSYS*	*Discrepancy%*	*Theoretical Eq.(44)*	*ANSYS*	*Discrepancy%*
1	66.357	66.32	0.055	6.893	6.905	0.174
2	6.027	6.024	0.049	5.862	5.846	0.272

Fig.11. and **Fig.12.** show the mode shapes of the system with bouncing and pitching effects respectively .From **Fig.11.** it can be expressed that the maximum amplitude is 0.063219m for the first mode shape and 0.199916m for the second mode shape. And from **Fig.12.** it can be expressed that that the maximum amplitude is 0.029298m for the first mode shape and 0.034485m for the second mode shape.



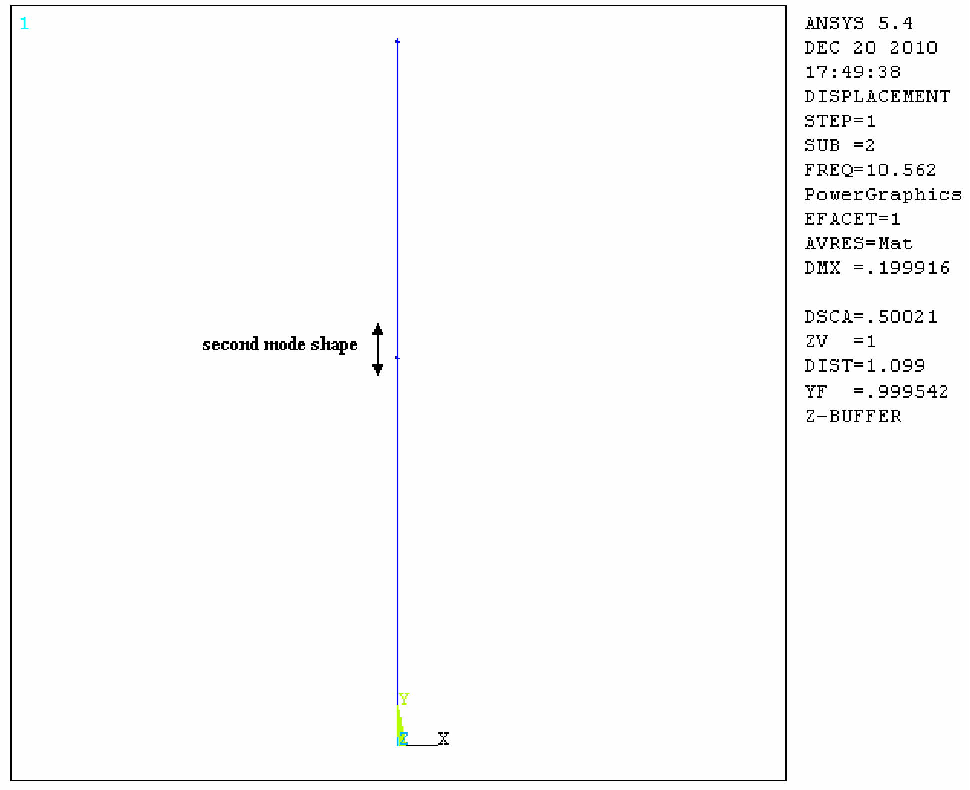
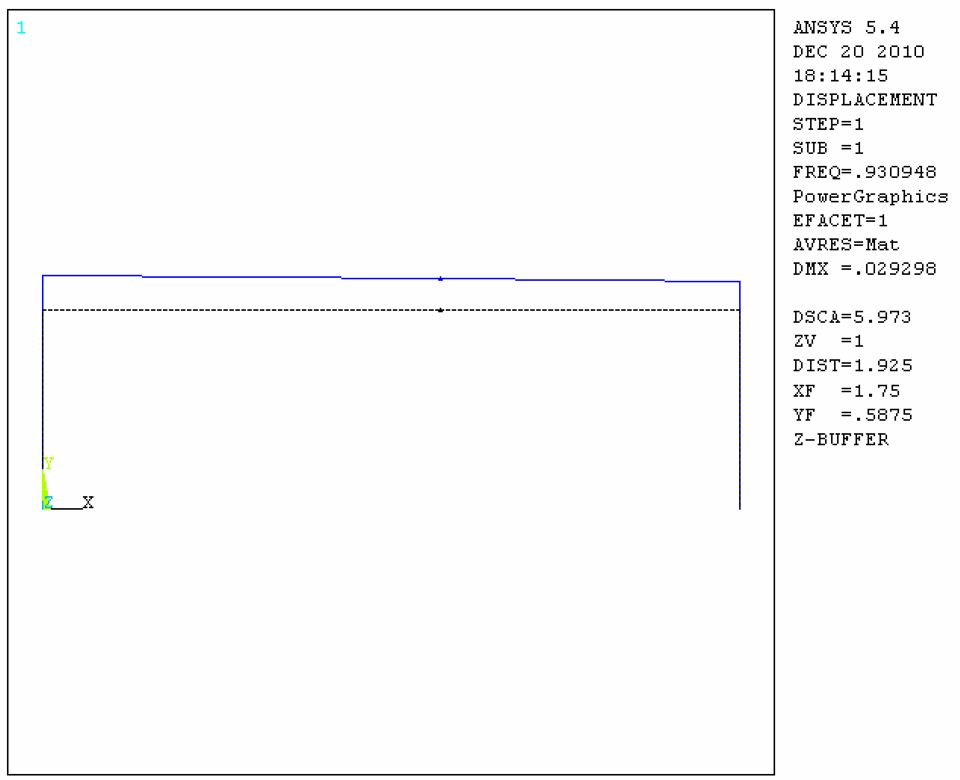


Fig.11. Mode shapes due to bouncing effect



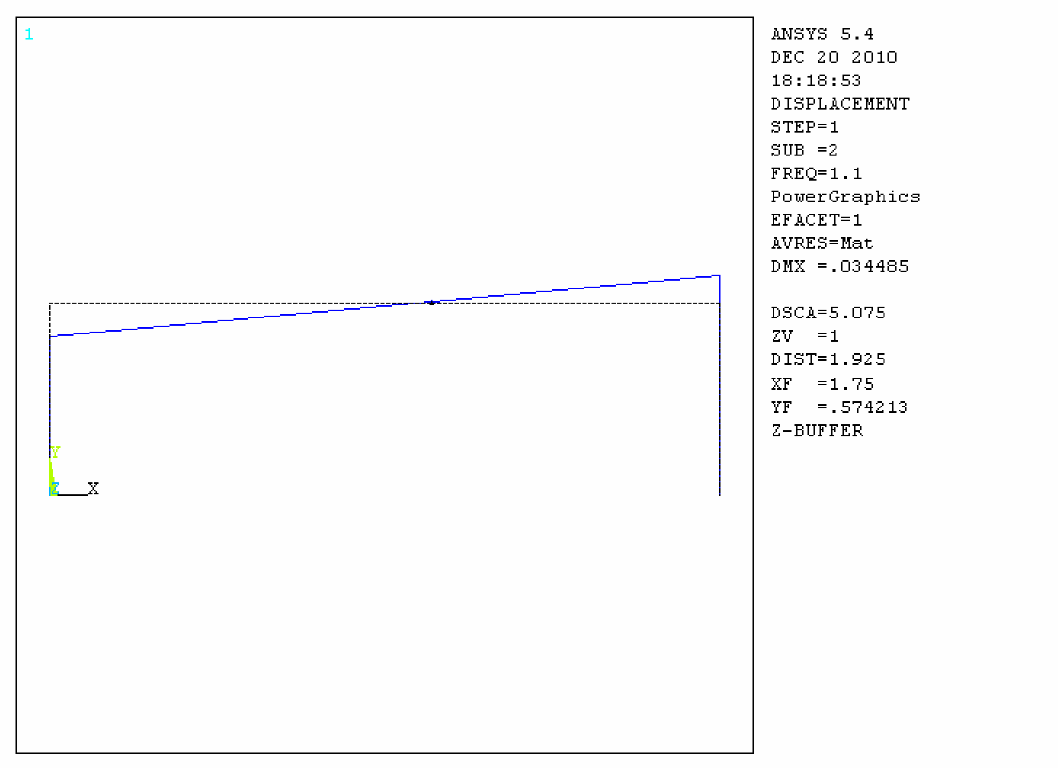


Fig.12. Mode shapes due to pitching effect

6-CONCLUSIONS

Vibration analysis is an inherent part of the design of any mechanical system. The work done in this paper was to develop equations of natural frequencies with the effects of bouncing and pitching of vibration in an automobile. Dynamic characteristics of the vehicle under the influence of the bouncing and pitching is studied theoretically and through finite element method. The results reported the two structural natural frequencies and mode shapes which are based on the behavior of vehicle, it can be concluded that the natural frequency of the vehicle decreasing with bouncing and pitching effect. Good agreement is evident between the theoretical and finite element method in which the discrepancy between the two methods is about 0.05% for bouncing and 0.2% for pitching model. Also, it can be deduced that the natural frequencies when join between the car body and tire is differed from that of the natural frequencies for the individual one, in which the higher frequency is increased and decreased the less one.

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SYMBOLS

m_1, m_2	discrete masses
k_1,k_2	spring constants
c_1,c_2	damping coefficient
h	excitation force
ω	natural frequency
[M]	mass matrix
[K[	stiffness matrix
{x}	displacement vector
$\ddot{\alpha}$	angular acceleration
J_y	pitching moment of inertia
m_3	degree