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Hermite Polynomials for Solving Volterra-Fredholm Integro-Differential Equations

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Abstract:

In this paper, Hermite polynomials (HPs) are introduced to solve the 2nd kind Volterra-Fredholm integro-differential equations (VFIDEs) of the first and second order. This technique is based on replacing the unknown function "infinite series" by truncated series of that is well know by Hermite expansion of functions. The presented method converts the equation into matrix form or a system of algebraic equations with Hermite coefficients which they must be determined. The existence and uniqueness of the solution are proved.The convergence analysis of the method are studied.Some examples for the first, and second orders of 2nd kind VFIDEs are given to demonstrate the effectiveness and the precision of the proposed method.

Keywords: Numerical Solution, Hermite Polynomials Method, Matrix equation, Volterra-Fredholm Integro-Differential Equations.

1. Introduction

The integro-differential equation(IDE) plays an significant part in numerous linear functional analytic subfields and its applications in engineering, mechanics, physics, chemistry, biology, and economics concepts and electro-stations. It's known this type of IDEs are typically challenging to resolve analytically, therefore approximation techniques are necessary to find the solution to such types of equations [1]. In the survey for finding the solution of the integro-differential equations, several methods have been developed in the recent years, for example, in 2019, the Adomain decomposition method applied to solve IDEs [2], this method is one of the few approximation methods which work without needing to a computer program unlike many other numerical methods which they need of utilizing the computer programs in order to get the solution[3]. In 2020 the Homotopy analysis method used to solve VFIDEs [4] .In 2021 the Monic Chebyshev polynomials [5],the mixed Euler polynomials with least-squares method[6]. In 2022, the moving least-squares method [7], the power series and shifted chebyshev polynomials[8]. next and in 2023, this problem have been studied by using successive approximation series [9].

This paper is organized as follows.In section two the definition of Hermite polynomials is given.In section three a matrix formulation for Hermite polynomials is stated. In section four solution of the VFIDE with HPs is obtained. The existence and uniqueness of the solution are proved in section five.In section six the convergence analysis of the method are studied. Some examples for the first, and second orders VFIDEs in section seven for confirming the efficiency and the accuracy of the proposed method. lastly, .section 5 contains conclusions of the search.

2. Hermite polynomials

The general form of the HPs [10] of n^{th} degree over the interval $[a, b]$ are

defined by

$$\mathcal{H}_n(\tau) = (-1)^n e^{\tau^2} \frac{d^n}{d\tau^n}\left(e^{-\tau^2}\right), \quad n = 0,1,\ldots$$

The first sixth terms of the Hermite polynomials are given as

$$\mathcal{H}_0(\tau) = 1$$

$$\mathcal{H}_1(\tau) = 2\tau$$

$$\mathcal{H}_2(\tau) = -2 + 4\tau^2$$

$$\mathcal{H}_3(\tau) = -(12\tau - 8\tau^3)$$

$$\mathcal{H}_4(\tau) = (12 - 48\tau^2 + 16\tau^4)$$

$$\mathcal{H}_5(\tau) = (120\tau - 160\tau^2 + 32\tau^5)$$

$$\mathcal{H}_6(\tau) = -(120 - 72\tau^2 + 480\tau^4 - 64\tau^6)$$

3. Matrix Formulation for Hermite Polynomials

The solution "function" $y(x)$ of any equation can be expressed by HPs as follows

$$y(x) = \alpha_0\, \mathcal{H}_0(x) + \alpha_1\, \mathcal{H}_1(x) + \cdots + \alpha_n\, \mathcal{H}_n(x) \qquad (1)$$

where α_i $(i = 0,1,\ldots,n)$ are unknown coefficients to be determined.

Rewriting eq. (1) as a dot product of the following two vectors

$$y(x) = [\mathcal{H}_0(x)\ \mathcal{H}_1(x) \ldots\ \mathcal{H}_n(x)] \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \qquad (2)$$

This equation can be converted to the form

$$y(x) = [1\ x \ldots\ x^n] \begin{bmatrix} h_{00} & h_{01} & \cdots & h_{0n} \\ 0 & h_{12} & \cdots & h_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{nn} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

where h_{ij} $(i,j = 0,1,\ldots,n)$ are the power basis coefficients that are used to calculate the HPs. It notes that in this instance, the matrix is upper triangular. Hence, the matrices which are obtained from the first and the second derivatives of HPs are

$$y(x)' = [0\ 1 \ldots\ nx^{n-1}] \begin{bmatrix} h_{00} & h_{01} & \cdots & h_{0n} \\ 0 & h_{12} & \cdots & h_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{nn} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

And

$$y(x)'' = [0\ 0 \ldots\ n(n-1)x^{n-2}] \begin{bmatrix} h_{00} & h_{01} & \cdots & h_{0n} \\ 0 & h_{12} & \cdots & h_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{nn} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

4. Solution of the VFIDE with HPs

Consider the following m^{th}-order VFIDEs:

$$y(x)^{(m)} = f(x) + \lambda_1 \int_a^x k_1(x,t)y(t)\,dt + \lambda_2 \int_c^d k_2(x,t)y(t)\,dt \qquad (3)$$

where $a \le x \le b, c \le x \le d$; $\lambda_1$and λ_2 are constants $f(x), k_1(x,t)$ and $k_2(x,t)$ are continuous functions and $y(x)$ is the unknown function to be determined.

To determine a numerical solution(NS) of (3). Firstly; we suppose the function $\hat{y}(x)$ defined in $[a,d]$ may be represented by the infinite series as follows:

$$\hat{y}(x) = \sum_{i=0}^{\infty} \alpha_i\, \mathcal{H}_i(x) \qquad (4)$$

Then (4) can be truncated to a finite series as:

$$\hat{y}(x) = \alpha.\mathcal{H}(x) = \sum_{i=0}^{n} \alpha_i\, \mathcal{H}_i(x) \qquad (5)$$

with their m derivatives

$$\hat{y}(x)^{(m)} = \sum_{i=0}^{n} \alpha_i\, \mathcal{H}^{(i)}(x) \qquad (6)$$

where $\mathcal{H}(x) = [\mathcal{H}_0(x)\,\mathcal{H}_1(x) \dots \mathcal{H}_n(x)], \alpha = [\alpha_0\ \alpha_1 \dots \alpha_n]^T$.

In this point, the aim is to determine the Hermite coefficients α_i, that is the vector α, through substitute the Hermite nodes $x_i = a + ih;\ \ i = 0,1,\dots,n$, into (5) and (6), .i.e.

$$\hat{y}^{(m)} - K_1 - K_2 = f \qquad (7)$$

where

$$K_1 = \lambda_1 \int_a^x k_1(x,t) \sum_{i=0}^{n} \alpha_i\, \mathcal{H}_i(t)\,dt \quad \text{and}$$

$$K_2 = \lambda_2 \int_a^x k_2(x,t) \sum_{i=0}^{n} \alpha_i\, \mathcal{H}_i(t)\,dt$$

and then eq.(7) can be rearranged to the following “matrix” form described by the following linear algebraic equations in (n+1) unknown coefficients to determine α:

$$A\alpha = B \qquad (8)$$

System (8) is solved using the software(Matlab R2010b) to acquire the unidentified vector α , which are subsequently replaced in to (5) to get the

NS of (1). The procedures for locating the NS for are outlined in the following algorithm LVFIDE of the second kind.

4.1 Algorithm (HP-LVFIDE)

Input: $f(x), k_1(x,t), k_2(x,t), \lambda_1, \lambda_2, y(x), a, c, d, x.$

Step 1: Choose n, the of HP, $\mathcal{H}_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$

Step 2: Substitute the HPs in the LIDE of second kind

$$\sum_{i=0}^{n} \alpha_i\, \mathcal{H}^{(i)}(x) = f(x) + \lambda_1 \int_a^x k_1(x,t) \sum_{i=0}^{n} \alpha_i\, \mathcal{H}_i(t)\, dt + \lambda_2 \int_c^d k_2(x,t) \sum_{i=0}^{n} \alpha_i\, \mathcal{H}_i(t)\, dt$$

Step 3: Compute the integration

$$\int_a^x k_1(x,t)\,[1\ t \ldots\ t^n] \begin{bmatrix} h_{00} & h_{01} & \cdots & h_{0n} \\ 0 & h_{12} & \cdots & h_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{nn} \end{bmatrix} dx$$

Step 4: Compute the integration

$$\int_c^d k_2(x,t)\,[1\ t \ldots\ t^n] \begin{bmatrix} h_{00} & h_{01} & \cdots & h_{0n} \\ 0 & h_{12} & \cdots & h_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{nn} \end{bmatrix} dx$$

Step 5: Choose x_i where

$$x_i = a + ih, and\ h = (b-a)/n; \quad i = 0,1,\ldots,n, x_i \in [a,d].$$

Step 6: Solve the algebraic system to find the unknown coefficients $\alpha_i \ (i = 0,1,\ldots,n)$.

Step 7: Substitute $\alpha_i \ (i = 0,1,\ldots,n)$ in $\hat{y}(x) = \sum_{i=0}^{n} \alpha_i \, \mathcal{H}_i(x), \mathcal{H}_i$ NS $\hat{y}(x)$

is obtained.
Output: Polynomials of degree n.

End.

5. Existence and Uniqueness Solution of LVFIDEs

Theorem(1) (Existence and Uniqueness of Solution)

The LVFIDEs in the equation(3) has a unique solution when $0 < \beta < 1$.

Proof: let $y(x)$ and $y^*(x)$ be two different solutions of the equation(3),then

$$|y(x) - y^*(x)| = \left|\lambda_1 \int_a^x k_1(x,t)y(t)dt + \lambda_2 \int_c^d k_2(x,t)y(t)dt - \lambda_1 \int_a^x k_1(x,t)y^*(t)dt - \lambda_2 \int_c^d k_2(x,t)y^*(t)dt\right| \quad (9)$$

$$\leq \left|\lambda_1 \int_a^x k_1(x,t)[y(t) - y^*(t)]\,dt + \lambda_2 \int_c^d k_2(x,t)[y(t) - y^*(t)]dt\right| \quad (10)$$

$$\leq \lambda_1 \int_a^x |k_1(x,t)|\,|y(t) - y^*(t)|\,dt + \lambda_2 \int_c^d |k_2(x,t)|\,|y(t) - y^*(t)|\,dt \quad (11)$$

$$\leq \lambda_1 \rho_1 |y(x) - y^*(x)|(b-a) + \lambda_2 \rho_2 |y(x) - y^*(x)|(d-c) \quad (12)$$

$$\leq |y(x) - y^*(x)|(\lambda_1 \rho_1 (b-a) + \lambda_2 \rho_2 (d-c)) \quad (13)$$

$$= \beta |y(x) - y^*(x)| \quad (14)$$

Thus,

$$|y(x) - y^*(x)| \leq \beta |y(x) - y^*(x)| \quad (15)$$

Then, we obtain
$(1-\beta)\left|y(x)-y^*(x)\right| \leq 0$. Since $0<\beta<1$, therefore

$\left|y(x)-y^*(x)\right| = 0$

So, $y(x)=y^*(x)$.

6 .Convergence of the HPs

In this section, we prove that the NS $\hat{y}(x)$ is convergent to the ES $y(x)$ of the problem(3).

Theorem(2): let $\hat{y}_n(x)$ be Ps of degree n that their numerical coefficients are created by solving the linear system(8).there exists an integer N such that, for all $n \geq N$, these Ps converge to the ES of the LVFIDEs(3).

Proof: we explain the technique's convergence

$$\hat{y}^{(m)}(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)\hat{y}(t)dt + \lambda_2 \int_c^d k_2(x,t)\hat{y}(t)dt \qquad (16)$$

Subtracting(16) from(3),to get

$$E_n(x) = y^{(m)}(x) - \hat{y}^{(m)}(x) \qquad (17)$$

Now

$$|E_n(x)| = \left|y^{(m)}(x) - \hat{y}^{(m)}(x)\right| \leq \Big| f(x) + \lambda_1 \int_a^x k_1(x,t)y(t)dt + \lambda_2 \int_c^d k_2(x,t)y(t)dt - f(x) - \lambda_1 \int_a^x k_1(x,t)\hat{y}^{(m)}(t)dt - \lambda_2 \int_c^d k_2(x,t)\hat{y}^{(m)}(t)dt \Big| \qquad (18)$$

$$|E_n(x)| \leq \left| \lambda_1 \int_a^x k_1(x,t)E_n(t)dt + \lambda_2 \int_c^d k_2(x,t)E_n(t)dt \right| \qquad (19)$$

Then

$$\frac{\|E_n(x_i)\|_\infty}{\|E_n(t)\|_\infty} \leq \left| \lambda_1 \int_a^{x_i} k_1(x,t)E_n(t)dt + \lambda_2 \int_c^d k_2(x,t)E_n(t)dt \right| \qquad (20)$$

Thus, this method is convergence.

7. Illustrations Examples:-

In this section two examples are considered to demonstrate the effectiveness and accuracy for the presented method. The Hermite nodes "points of collocation" are defined by $x_l = a + l\frac{(b-a)}{n}$; $l = 0,1,\dots,n$.

Example 1: Consider the following VFIDE

$$y' = 6 - 2x - \frac{1}{2}x^2 - x^3 + \int_0^x (x-t)y(t)\,dt + \int_{-1}^{1} x\,y(t)\,dt, \;\; y(0) = 1$$

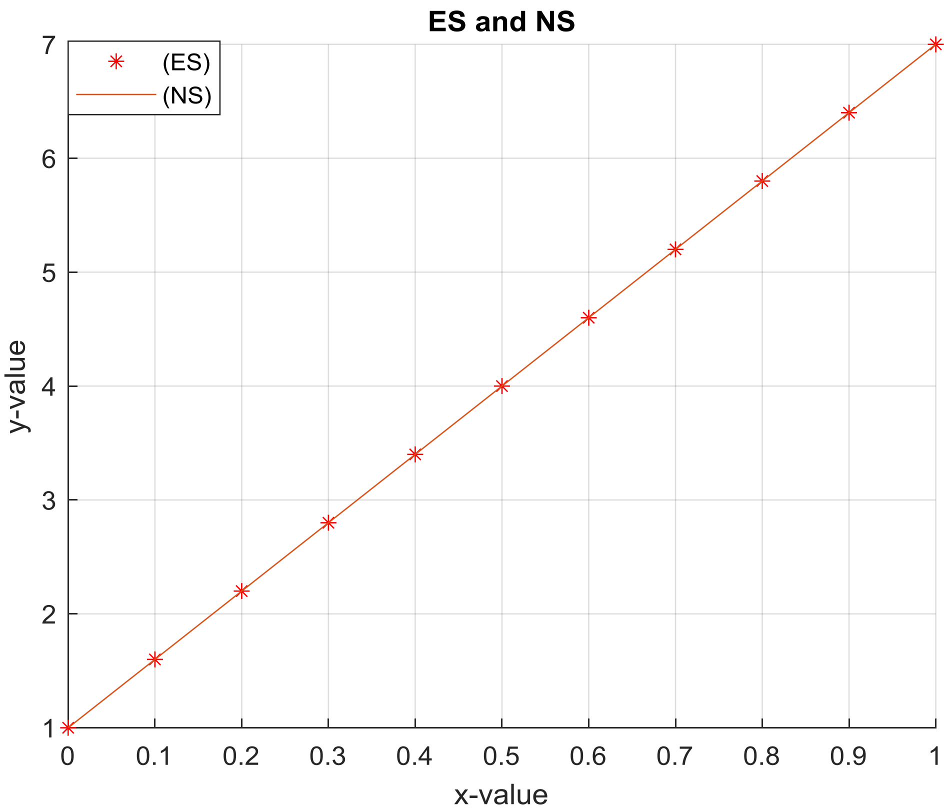
With an exact solve(ES) $y(x) = 1 + 6x$ [11].

Now, using the algorithm described above with choosing the degree of HP $n = 2$. The obtained values of the unknowns are $\alpha_0 = 1$, $\alpha_1 = 3$, $\alpha_2 = 0$,

and then the NS of VFIDE is : $\hat{y}(x) = \alpha_0 + 2\alpha_1 x + \alpha_2(-2 + 4x^2)$.

Table(1): Comparison between the ES and NS

x	$y(ES)$	$\hat{y}(NS)$	$\lvert y - \hat{y}\rvert$
0	1.000000000000000	1.000000000000000	0.000000000000000
0.1	1.600000000000000	1.600000000000000	0.000000000000000
0.2	2.200000000000000	2.200000000000000	0.000000000000000
0.3	2.800000000000000	2.800000000000000	0.000000000000000
0.4	3.400000000000000	3.400000000000000	0.000000000000000
0.5	4.000000000000000	4.000000000000000	0.000000000000000
0.6	4.600000000000000	4.600000000000000	0.000000000000000
0.7	5.199999999999999	5.199999999999999	0.000000000000000
0.8	5.800000000000001	5.800000000000001	0.000000000000000
0.9	6.400000000000000	6.400000000000000	0.000000000000000
1	7.000000000000000	7.000000000000000	0.000000000000000



Figure(1): The ES and NS when(n=2)

Example 2: Consider the following VFIDE

$y'' = -8 + 6x - 3x^2 + x^3 + \int_0^x y(t)\,dt + \int_{-1}^{1} (1 - 2xt)\, y(t)\, dt,\ y(0) = 2, y'(0) = 6$

With an exact solve $y(x) = 2 + 6x - 3x^2$ [12].

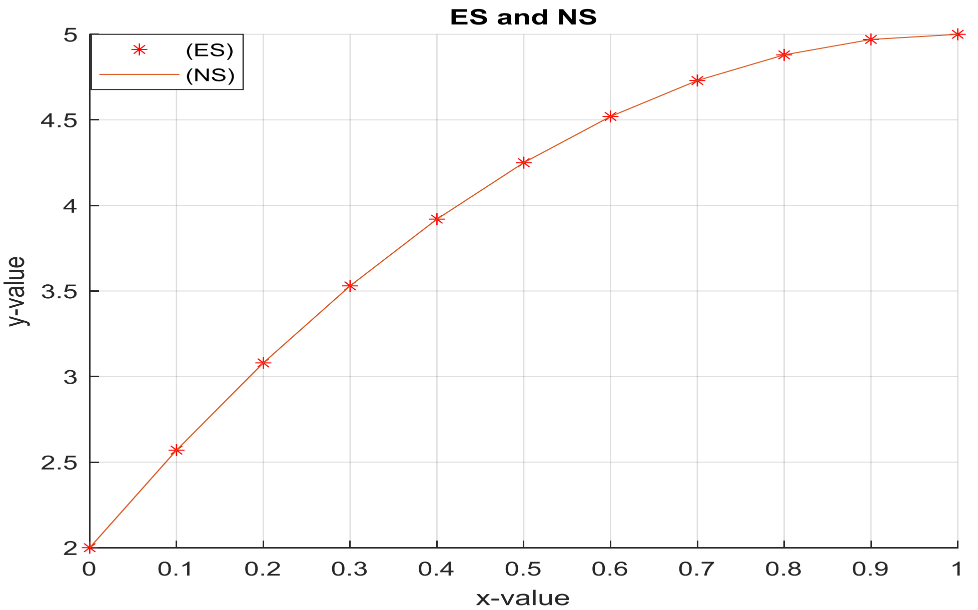
Now, using the algorithm described above with choosing the degree of HP $n = 2$. The obtained values of the unknowns are

$\alpha_0 = 0.5000,\ \alpha_1 = 3.0000,\ \alpha_2 = -0.7500$, and then the NS of VFIDE is

$\hat{y}(x) = \alpha_0 + 2\alpha_1 x + \alpha_2(-2 + 4x^2)$.

Table(2): Comparison between the ES and NS

x	$y(ES)$	$\hat{y}(NS)$	$\lvert y-\hat{y}\rvert$
0	2.000000000000000	2.000000000000000	0.000000000000000
0.1	2.570000000000000	2.570000000000000	0.000000000000000
0.2	3.080000000000000	3.080000000000000	0.000000000000000
0.3	3.530000000000000	3.530000000000000	0.000000000000000
0.4	3.920000000000000	3.920000000000000	0.000000000000000
0.5	4.250000000000000	4.250000000000000	0.000000000000000
0.6	4.520000000000000	4.520000000000000	0.000000000000000
0.7	4.730000000000000	4.730000000000000	0.000000000000000
0.8	4.880000000000001	4.880000000000001	0.000000000000000
0.9	4.970000000000001	4.970000000000001	0.000000000000000
1	5.000000000000000	5.000000000000000	0.000000000000000



Figure(2): The ES and NS when(n=2)

8. Conclusion

According to the results were obtained from the two examples, it is concluding HPs is a powerful and effective technique for finding an approximate solution of the 2nd kind VFIDE of the first and the second order. The presented method is a very simple and straight forward. The results demonstrate that the method is good and closely agrees with the exact solutions even at lower values of n.

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متعددات حدود هيرميت لحل معادلات فولتيرا-فريدهولم التكاملية التفاضلية

مستخلص البحث:
في هذا البحث، تم تقديم متعددات حدود هيرميت (HPs) لحل معادلات فولتيرا-فريدهولم التكاملية التفاضلية من النوع الثاني (VFIDEs) ومن الرتبة الأولى والثانية. تعتمد هذه التقنية على استبدال الدالة المجهولة "المتسلسلة اللانهائية" بمتسلسلة مقطوعة والتي تعرف بتوسع هيرمت للدوال. تقوم الطريقة المقدمة بتحويل المعادلة إلى شكل مصفوفة أو نظام من المعادلات الجبرية ذات معاملات هيرميت والتي يجب تحديدها. تمت إثبات الوجود ووحدانية الحل لهذه المسألة. وتم دراسة تحليل التقارب للطريقة. وتم إعطاء بعض الأمثلة من الرتبة الاولى والثانية ومن النوع الثاني لإثبات فعالية ودقة الطريقة المقترحة.

الكلمات المفتاحية: الحل العددي، طريقة متعددات حدود هيرميت، معادلة المصفوفة، معادلات فولتيرا-فريدهولم التكاملية التفاضلية.