Neural Networks of the Rational r-th Powers of the Multivariate Bernstein Operators

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Doi:10.29072/basjs.20220201

ARTICLE INFO

Keywords

The neural network of multivariate rational Bernstein operators, Sigmoidal functions, Pointwise and uniform approximation theorems, Lipschitz space.

ABSTRACT

In this study, a novel neural network for the multivariance Bernstein operators' rational powers was developed. A positive integer is required by these networks. In the space of all real-valued continuous functions, the pointwise and uniform approximation theorems are introduced and examined first. After that, the Lipschitz space is used to study two key theorems. Additionally, some numerical examples are provided to demonstrate how well these neural networks approximate two test functions. The numerical outcomes demonstrate that as input grows, the neural network provides a better approximation. Finally, the graphs used to represent these neural network approximations show the average error between the approximation and the test function.

Received 1 May 2022; Received in revised form3 Aug 2022; Accepted 15 Aug 2022, Published 31 Aug 2022

1. Introduction

Neural networks (NNs) with a single layer from the type of feed-forward are studied in many papers [1, 2, 3, 4, 5, 6, 7, 8], with σ as the activation function $\sigma\colon \mathbb{R} \to \mathbb{R}$, defined as:

$$N_\sigma(\mathbf{x}) = \sum_{i=0}^{n} c_i \sigma(\boldsymbol{\alpha}_i \cdot \mathbf{x} + \beta_i), \qquad n \in \mathbb{N}^+, \tag{1}$$

were, $\mathbf{x} \in \mathbb{R}^s, s \in \mathbb{N}^+, 0 \le i \le n,$ the β_i, c_i in $\mathbb{R}$, threshold values, coefficients, $\boldsymbol{\alpha}_i$, $\boldsymbol{\alpha}_i \cdot \mathbf{x}$ in $\mathbb{R}^s$ denote to weights, inner product and for σ is the activation function $\sigma\colon \mathbb{R} \to \mathbb{R}$.

In 2013 [9,10], Costarelli and Spigler introduced neural networks in the univariate and the multivariate Bernstein and studied the behavior of these neural networks, defined as:

for $f\colon \mathcal{R} \to \mathbb{R}$ bounded function, and $\mathbf{x}$ in $\mathcal{R} \coloneqq [a_1, b_1] \times \dots \times [a_s, b_s]$.

$$F_n(f;\mathbf{x}) = \frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \cdots \sum_{k_s=\lceil na_s\rceil}^{\lfloor nb_s\rfloor} f\left(\frac{\mathbf{k}}{n}\right) \Psi_\sigma(n\mathbf{x}-\mathbf{k})}{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \cdots \sum_{k_s=\lceil na_s\rceil}^{\lfloor nb_s\rfloor} \Psi_\sigma(n\mathbf{x}-\mathbf{k})}, \qquad n \in \mathbb{N}^+. \tag{2}$$

where Ψ_σ is a density function that is built from a sigmoidal function σ, $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{Z}^+$, As usual, the symbols $\lfloor . \rfloor$ and $\lceil . \rceil$ denote taking the "floor" and the "ceiling" of a given number, respectively. Costarelli and Spigler extended Eq. 2 using the Kantorovich type to define and study approximation theorems to this neural network, defined as [11]:

for $f\colon \mathcal{R} \to \mathbb{R}$, and $\mathbf{x} \in \mathcal{R}$, a locally integrable

$$K_n(f;\mathbf{x}) = \frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \cdots \sum_{k_s=\lceil na_s\rceil}^{\lfloor nb_s\rfloor} \left[n^s \int_{R_{\mathbf{k}}^n} f(\mathbf{u})d\mathbf{u}\right] \Psi_\sigma(n\mathbf{x}-\mathbf{k})}{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \cdots \sum_{k_s=\lceil na_s\rceil}^{\lfloor nb_s\rfloor} \Psi_\sigma(n\mathbf{x}-\mathbf{k})}. \tag{3}$$

Furthermore, Costarelli and Vinti give two formulas similar to the neural networks in form Eq. 2 by using max-product and studying convergence theorems, defined as [12]:

for J_n represents the set of indexes $\mathbf{k}$, where $f\colon \mathcal{R} \to \mathbb{R}$, and $\mathbf{x} \in \mathcal{R}$, a bounded function,

$$M_n(f,\mathbf{x}) = \frac{\bigvee_{\mathbf{k}\in J_n} f\left(\frac{\mathbf{k}}{n}\right) \Psi_\sigma(n\mathbf{x}-\mathbf{k})}{\bigvee_{\mathbf{k}\in J_n} \Psi_\sigma(n\mathbf{x}-\mathbf{k})}. \tag{4}$$

where the notation $\vee$ defines as: $\bigvee_{k\in J} A_k = \sup\{A_k : k \in J\}$, for $f\colon \mathbb{R}^s \to \mathbb{R}$, and $\mathbf{x} \in \mathbb{R}^s$, a bounded function,

$$MQ_n(f,\mathbf{x}) = \frac{\bigvee_{\mathbf{k}\in\mathbb{Z}^s} f\left(\frac{\mathbf{k}}{n}\right)\Psi_\sigma(n\mathbf{x}-\mathbf{k})}{\bigvee_{\mathbf{k}\in\mathbb{Z}^s}\Psi_\sigma(n\mathbf{x}-\mathbf{k})}. \quad (5)$$

In one-dimensional and multi-dimensional scenarios, Shivam and Kumar created and investigated a neural network of the exponential type and examined their behavior [13]. Additionally, Costarelli and coworkers [14] developed the multivariate max-product NN utilizing the Kantorovich type, which is described as:

for $f:\mathcal{R}\to\mathbb{R}$, and $\mathbf{x}\in\mathcal{R}$, a bounded and locally integrable function

$$K_n^M(f,\mathbf{x}) = \frac{\bigvee_{\mathbf{k}\in J_n}\left[n^s\int_{R_{\mathbf{k}}^n} f(\mathbf{u})d\mathbf{u}\right]\Psi_\sigma(n\mathbf{x}-\mathbf{k})}{\bigvee_{\mathbf{k}\in J_n}\Psi_\sigma(n\mathbf{x}-\mathbf{k})}. \quad (6)$$

In 2021 [15], Mohammad and Abdul Samad introduced and study the sequence of linear and positive operators of r-th power of the rational Bernstein polynomials, defined as:

for $f\in C[0,1]$ and $r\in\mathbb{N}:=\{1,2,\dots\}$.

$$B_{n,r}(f;x) = \frac{\sum_{k=0}^{n} b_{n,k}^r(x) f\left(\frac{k}{n}\right)}{\sum_{k=0}^{n} b_{n,k}^r(x)}. \quad (7)$$

where $b_{n,k}^r(x) = \left(b_{n,k}(x)\right)^r$, $x\in[0,1]$, $f\in C[0,1]$.

Also, can see the family of operators that was established in [16-19], to be used to build new neural networks. The current study developed a novel neural network based on the multivariance Bernstein operators' rational r-the powers. First, the pointwise and uniform approximation theorems are introduced and studied in the space of all real-valued continuous functions on $\mathcal{R}$. Then, the two theorems above are studied in the Lipschitz space on v. Also, some numerical examples are given to show the approximation of these neural networks to two test functions.

2. Construction of neural network and Preliminary Results

Several preliminary results are recalled in this part. The measurable is called a sigmoidal function if satisfying the conditions $\lim_{x\to-\infty}\sigma(x)=0$ and $\lim_{x\to+\infty}\sigma(x)=1$, such a function:

i) Logistic function $\sigma_l(x) = (1+e^{-x})^{-1}$;

ii) Gompertz function $\sigma_{\alpha\beta}(x) = e^{-\alpha e^{-\beta x}}$, $x\in\mathbb{R}$, $\alpha,\beta>0$;

iii) Hyperbolic tangent $\sigma_h(x) = \frac{1}{2}[\tanh(x)+1]$.

Also, the function $\Phi_\sigma(x)$ is defined as

$$\Phi_\sigma(x) = \frac{1}{2}[\sigma(x+1) - \sigma(x-1)], x \in \mathbb{R}. \quad (8)$$

Furthermore, as shown in [10], any non-decreasing function and $\sigma(2) > \sigma(0)$ "which is only a technical condition" satisfy various assumptions, of which we list a few:

i) the function $\sigma \in C^2(\mathbb{R})$is concave for $x \geq 0$;

ii) $g_\sigma(x) = \sigma(x) - 1/2$, is an odd function;

iii) $\sigma(x) = \mathcal{O}(|x|^{-1-\alpha})$ as $x \to -\infty$, for some $\alpha > 0$.

We need to give the following definitions:

Definition 2.1. [10]

Any measurable function with the condition $\lim_{x\to-\infty} \zeta(x) = 0, \lim_{x\to+\infty} \zeta(x) = 1$ it's known as a sigmoidal function.

Definition 2.2 [10]

Lipschitz classes are defined as:

$$\mathrm{Lip}(v) = \{f \in C^0(\mathcal{R}): \exists\, \gamma > 0, C > 0 \text{ so that } \forall\, \mathbf{x} \in \mathcal{R}, |f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x})| \leq C\|\mathbf{y}\|_2^v,$$
$$\forall\, \|\mathbf{y}\|_2 \leq \gamma \text{ with} (\mathbf{x}+\mathbf{y}) \in \mathcal{R}, 0 < v \leq 1\}.$$

The following lemmas stated some properties of the functions Φ_σ and Ψ_σ.

3. Main Results

In the following definition, we will define and investigate multivariate NN operators $Q_r(f;\mathbf{x})$:

Definition 3.1

For a bounded and continuous function $f: \mathcal{R} \to \mathbb{R}$, the neural network of rational r-th powers of the multivariate Bernstein $Q_r(f;\mathbf{x})$, activated by the sigmoidal function σ acting on f, $r \in \mathbb{N} := \{1,2,...\}$, defined by:

$$Q_r(f;\mathbf{x}) = \frac{\sum_{\mathbf{k}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k}) f(\mathbf{k}/n)}{\sum_{\mathbf{k}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k})} \quad (9)$$

$$\sum_{\mathbf{k}} = \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_s=\lceil na_s \rceil}^{\lfloor nb_s \rfloor},$$

observe that $Q_r(1;\mathbf{x}) = 1$, and $Q_1(.;\mathbf{x}) = F_n(.;\mathbf{x})$ for every $\mathbf{x} \in \mathcal{R}$ and n tends to infinity, the multivariate for the Φ_σ^r define a function $\Psi_\sigma^r(\mathbf{x}) = \Phi_\sigma^r(x_1) \cdot \Phi_\sigma^r(x_2) \cdot ... \cdot \Phi_\sigma^r(x_s)$.

Definition 3.2

For $v > 0$, the discrete absolutely moment of the function Φ_σ^r of order v is defined as

$$m_v(\Phi_\sigma^r) = \sup_{x\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \Phi_\sigma^r(x-k)|x-k|^v. \quad (10)$$

The properties of the functions Φ_σ and Ψ_σ in [9] and [10] are needed to give and prove the following Lammas (3.1-3.3) directly.

Lemma 3.1

Some properties of the function $\Phi_\sigma^r(x)$ defined in $\mathbb{R}$:

i) $\Phi_\sigma^r(x) \geq 0, \forall x \in \mathbb{R}$ and $\lim_{x\to\pm\infty} \Phi_\sigma^r(x) = 0$;

ii) the function $\Phi_\sigma^r(x)$ is even;

iii) $\forall x \in \mathbb{R}$, we have $\sum_{k\in\mathbb{Z}} \Phi_\sigma^r(x-k) \leq 1/2^{2(r-1)}$, for $r = 1,2,3,\ldots$;

iv) $\Phi_\sigma^r(x) = \mathcal{O}(|x|^{-r(1+\alpha)})$ as $x \to \pm\infty$.

Proof.

One can easily prove this lemma by direct computation and the proving of properties of the function $\Phi_\sigma$in [9]. □

The next lemma gives some properties for the function $\Psi_\sigma^r(\mathbf{x}-\mathbf{k})$.

Lemma 3.2

Some properties of the function $\Psi_\sigma^r(\mathbf{x}-\mathbf{k})$ are defined in $\mathbb{R}^s$:

i) $\forall \mathbf{x} \in \mathbb{R}^s$, we have $\sum_{\mathbf{k}} \Psi_\sigma^r(\mathbf{x}-\mathbf{k}) \leq 1/2^{2s(r-1)}$, for $r = 1,2,3,\ldots$;

ii) on the compact set of $\mathbb{R}^s$, the series $\sum_{\mathbf{k}} \Psi_\sigma^r(\mathbf{x}-\mathbf{k})$ converges uniformly;

iii) $\lim_{n\to\infty} \sum_{\|\mathbf{x}-\mathbf{k}\|>\gamma n} \Psi_\sigma^r(\mathbf{x}-\mathbf{k}) = 0$ are converges uniformly to $\mathbf{x} \in \mathbb{R}^s$;

and $\sum_{\|\mathbf{x}-\mathbf{k}\|>\gamma n} \Psi_\sigma^r(\mathbf{x}-\mathbf{k}) = \mathcal{O}(n^{-v})$ in particularly for $0 < v < \alpha$, where $\gamma, \alpha > 0$, α is a constant and $\|\mathbf{x}\|_\infty = \max\{|x_i|, i = 1,\ldots,s\}$.

Proof.

One can easily prove this lemma by direct computation and the proof of properties of the Lemma (2.4, 2.5) in [10]. □

Lemma 3.3

i) for $x \in [a,b] \subset \mathbb{R}$, then

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_\sigma^r(nx-k)} \le \frac{1}{\Phi_\sigma^r(1)};$$

ii) for $\mathbf{x} \in \mathcal{R} \subset \mathbb{R}^s$, then

$$\frac{1}{\prod_{i=1}^{s} \sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_\sigma^r(nx_i-k_i)} \le \frac{1}{[\Phi_\sigma^r(1)]^s}.$$

Proof.

One can easily prove this lemma by direct computation and using the proof of lemma 2.7 in [10]. □

The following theorem studies the pointwise and the uniform convergence for the NN, $Q_r(f;\mathbf{x})$.

Theorem 3.1

For $f: \mathcal{R} \to \mathbb{R}$ bounded and continuous function, $\lim_{n\to\infty} Q_r(f;\mathbf{x}) = f(\mathbf{x})$ where f is continuous at each point $\mathbf{x} \in \mathcal{R}$. If $f \in C^0(\mathcal{R})$, then

$$\lim_{n\to\infty} \sup_{\mathbf{x}\in\mathcal{R}} |Q_r(f;\mathbf{x}) - f(\mathbf{x})| = \lim_{n\to\infty} \|Q_r(f;.) - f(.)\|_\infty = 0.$$

Proof.

Suppose $\mathbf{x} \in \mathcal{R}$ is a point of continuity of f we have

$$|Q_r(f;\mathbf{x}) - f(\mathbf{x})| = \left| \frac{\sum_{\mathbf{k}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k}) f(\mathbf{k}/n)}{\sum_{\mathbf{k}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k})} - f(\mathbf{x}) \right|$$

and by using Lemma 3.3, we get:

$$|Q_r(f;\mathbf{x}) - f(\mathbf{x})| = \left| \frac{\sum_{\mathbf{k}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k}) \left(f\left(\frac{\mathbf{k}}{n}\right) - f(\mathbf{x}) \right)}{\sum_{\mathbf{k}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k})} \right|$$

$$\le \frac{1}{[\Phi_\sigma^r(1)]^s} \sum_{\mathbf{k}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k}) |f(\mathbf{k}/n) - f(\mathbf{x})|$$

$\forall n \to \infty, n \in N^+, \mathbf{x} \in \mathbb{R}^s$ are arbitrary but fixed. Suppose for a fixed $\varepsilon > 0$, and from the continuity of f at $\mathbf{x}$, $\exists \gamma > 0 : |f(\mathbf{y}) - f(\mathbf{x})| < \varepsilon$, $\forall \mathbf{y} \in \mathcal{R}$ with $\|\mathbf{y} - \mathbf{x}\|_2 < \gamma$, the symbol $\|.\|_2$ denote to Euclidean norm, then

$$|Q_r(f;\mathbf{x}) - f(\mathbf{x})| \le \frac{1}{[\Phi_\sigma^r(1)]^s} \left\{ \sum_{\|\mathbf{k}/n - \mathbf{x}\| < \frac{\gamma}{\sqrt{s}}} \Psi_\sigma^r(n\mathbf{x} - \mathbf{k}) |f(\mathbf{k}/n) - f(\mathbf{x})| \right.$$

$$\left. + \sum_{\|\mathbf{k}/n - \mathbf{x}\| \ge \frac{\gamma}{\sqrt{s}}} \Psi_\sigma^r(n\mathbf{x} - \mathbf{k}) |f(\mathbf{k}/n) - f(\mathbf{x})| \right\}$$

$$:= \frac{1}{[\Phi_\sigma^r(1)]^s} (I_1 + I_2).$$

Now using the continuity of f and Lemma 3.2, we get that $\|\mathbf{k}/n - \mathbf{x}\|_2 \le \sqrt{s}\|\mathbf{k}/n - \mathbf{x}\| \le \gamma$. So estimation I_1 is,

$$I_1 < \varepsilon \sum_{\|\mathbf{k}/n - \mathbf{x}\| \le \frac{\gamma}{\sqrt{s}}} \Psi_\sigma^r(n\mathbf{x} - \mathbf{k}) \le \varepsilon.$$

And from the boundedness of f and Lemma 3.2, for sufficiently large n, then

$$I_2 \le 2\|f\|_\infty \sum_{\|\mathbf{k}/n - \mathbf{x}\| > \frac{\gamma}{\sqrt{s}}} \Psi_\sigma^r(n\mathbf{x} - \mathbf{k}) < 2\|f\|_\infty \varepsilon,$$

uniformly for all $\mathbf{x} \in \mathbb{R}^s$. Since ε arbitrarily, the first side of the theorem holds. When $f \in C^0(\mathcal{R})$, the proof in the reverse direction is easily followed by removing $\gamma > 0$ with the parameter of the uniform continuity of f on $\mathcal{R}$. □

Now, in the following, study the order of approximation of (NN) operators in $C^0(\mathcal{R})$.

Theorem 3.2

Suppose $f \in Lip(v)$ for some v, at $0 < v \le 1$.and let sigmoidal function σ satisfy the condition (iii) for some $\alpha > 1$. Then,

$$\|Q_r(f;\mathbf{x}) - f(.)\|_\infty = \mathcal{O}(n^{-v}) \text{ as } n \to \infty.$$

Proof.

Let $f \in Lip(v)$, $\forall \mathbf{x} \in \mathcal{R}$, for some $v \in (0,-1]$, by using Lemma 3.3, we obtain

$$|Q_r(f;\mathbf{x}) - f(\mathbf{x})| = \left|\frac{\sum_{\mathbf{k}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k}) f(\mathbf{k}/n)}{\sum_{\mathbf{k}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k})} - f(\mathbf{x})\right|$$

$$= \left|\frac{\sum_{\mathbf{k}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k})\left(f\left(\frac{\mathbf{k}}{n}\right) - f(\mathbf{x})\right)}{\sum_{\mathbf{k}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k})}\right|$$

$$\leq \frac{1}{[\Phi_\sigma^r(1)]^s} \sum_{\mathbf{k}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k}) |f(\mathbf{k}/n) - f(\mathbf{x})|$$

Now, using the $Lip(v)$ formulation, where, $\gamma, C > 0$ are constants with respect to$f$, then

$$|Q_r(f;\mathbf{x}) - f(\mathbf{x})| \leq \frac{1}{[\Phi_\sigma^r(1)]^s} \left\{ \sum_{\|\mathbf{k}/n-\mathbf{x}\| \leq \frac{\gamma}{\sqrt{s}}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k}) |f(\mathbf{k}/n) - f(\mathbf{x})| \right.$$

$$\left. + \sum_{\|\mathbf{k}/n-\mathbf{x}\| > \frac{\gamma}{\sqrt{s}}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k}) |f(\mathbf{k}/n) - f(\mathbf{x})| \right\}$$

$$:= \frac{1}{[\Phi_\sigma^r(1)]^s}(J_1 + J_2)$$

since $f \in Lip(v)$, then for

$$\|\mathbf{k}/n - \mathbf{x}\|_2 \leq \sqrt{s}\|\mathbf{k}/n - \mathbf{x}\| \leq \gamma,$$

and hence

$$|f(\mathbf{k}/n) - f(\mathbf{x})| < C\|\mathbf{k}/n - \mathbf{x}\|_2^v \leq Cs^{\frac{v}{2}}\|\mathbf{k}/n - \mathbf{x}\|^v.$$

$$J_1 \leq n^{-v} C s^{\frac{v}{2}} \sum_{\|\mathbf{k}/n-\mathbf{x}\| \leq \frac{\gamma}{\sqrt{s}}} \Psi_\sigma^r(n\mathbf{x}-\mathbf{k}) \|n\mathbf{x}-\mathbf{k}\|^v$$

for fixed $0 < v_i < \alpha$, by using Lemma 3.2, for a compact subset $K \subset \mathbb{R}^s$. $\forall \mathbf{x} \in \mathbb{R}^s$, if $n \to \infty$ one can write the flowing:

$$J_1 \leq n^{-v} C s^{\frac{v}{2}} \sum_{j=1}^{s} \left\{ \sum_{k_j \in \mathbb{Z}} \Phi_\sigma^r (n x_j - k_j) |n x_j - k_j|^v \left[\sum_{\mathbf{k}_{[j]} \in \mathbb{Z}^{s-1}} \Psi_\sigma^{r[j]} (n \mathbf{x}_{[j]} - \mathbf{k}_{[j]}) \right] \right\}$$

Where

$\Psi_\sigma^{r[j]} (n \mathbf{x}_{[j]} - \mathbf{k}_{[j]}) =$

$$\Phi_\sigma^r (n x_1 - k_1) \cdot \ldots \cdot \Phi_\sigma^r (n x_{j-1} - k_{j-1}) \cdot \Phi_\sigma^r (n x_{j+1} - k_{j+1}) \cdot \ldots \cdot \Phi_\sigma^r (n x_s - k_s),$$

Notice for every $j = 1, \ldots, s$ $\mathbf{x}_{[j]} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_s) \in \mathbb{R}^{s-1}, \mathbf{k}_{[j]} = (k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_s) \in \mathbb{Z}^{s-1}$. Now let $k_{[j]} \subset \mathbb{R}$ the set of the j-th projection of a compact set K for all elements. By using Lemma 3.2, and for all sufficiently large $N \in \mathbb{N}^+$, then

$$J_1 \leq \frac{n^{-v} C s^{\frac{v}{2}}}{2^{2(s-1)(r-1)}} \sum_{j=1}^{s} \left\{ \sum_{k_j \in \mathbb{Z}} \Phi_\sigma^r (n x_j - k_j) |n x_j - k_j|^v \right\}$$

$$\leq \frac{n^{-v} C s^{1+\frac{v}{2}}}{2^{2(s-1)(r-1)}} m_v(\Phi_\sigma^r).$$

note that $m_v(\Phi_\sigma^r) < \infty$, where $m_v(\Phi_\sigma^r)$ give in Definition 3.2 since $v < \alpha$ then:

$$J_1 = \mathcal{O}(n^{-v}), n \to \infty.$$

Now, we estimate J_2 by using the other direction of Lemma 3.2,

$$J_2 \leq 2\|f\|_\infty \sum_{\|\mathbf{k}/n - \mathbf{x}\| > \frac{\gamma}{\sqrt{s}}} \Psi_\sigma^r (n\mathbf{x} - \mathbf{k}) = \mathcal{O}(n^{-v}) \text{, as } n \to \infty. \qquad \square$$

Now, even when $\alpha \in (0,1]$, an approximation order can be obtained, as the following:

Theorem 3.3

Let the function σ for some $\alpha \in (0,1]$ satisfy the condition (iii), and let $f \in Lip(v)$ for some $v \in (0,1]$, Then,

i) If $v < \alpha$,

$$\|Q_r(f;.)-f(.)\|_\infty = \mathcal{O}(n^{-v}),\text{ as } n\to\infty;$$

ii) If $\alpha \le v < 1$, then

$$\|Q_r(f;.)-f(.)\|_\infty = \mathcal{O}(n^{-\alpha+\varepsilon}),\text{ as } n\to\infty;$$

$\forall\ 0<\varepsilon<\alpha.$

Proof.

i) for function $f\in Lip(v)$ at $0<v<\alpha$,

$$\|Q_r(f;.)-f(.)\|_\infty = \mathcal{O}(n^{-v}),\text{ as } n\to\infty$$

Then the proving by using the same step of the Theorem 3.2.

ii) As a special case for all $f\in Lip(v)$ with $\alpha\le v\le 1$, with ε is fixed but arbitrary choose $\beta:=\alpha-\varepsilon$, and get $0<\beta<\alpha$, by based on part (i) then,

$$\|Q_r(f;.)-f(.)\|_\infty = \mathcal{O}\left(n^{-\beta}\right) = \mathcal{O}(n^{-\alpha+\varepsilon}),\text{ as } n\to\infty$$

for function $f\in Lip(\beta)$, at $0<\varepsilon<\alpha$. □

4. Numerical Example

For the values of $n=10,30, r=2,3,4$, and the test functions $f(x,y)=\cos(9xy)+2\sin(x+y)$ and $h(x,y)=\frac{1}{4}(x+y)^3,(x,y)\in[0,1]^2$, we will give numerical examples for the NN operators $Q_r(.;x,y)$ with the NN operators $F_n(.;x,y)$, We make a comparison of the results in the graphics of examples of NN operator convergence $Q_r(.;x,y), F_n(.;x,y)$ of the results the test functions $f(x,y), h(x,y)$. We also provide the following table for the average error function for $Q_r(.;x,y), F_n(.;x,y)$:

Example 4.1

The convergence of NN operators $Q_r(f;x,y), F_n(f;x,y)$ to test function $f(x,y)$can be given in the following figures for $n=10,30$, $r=2,3,4$.

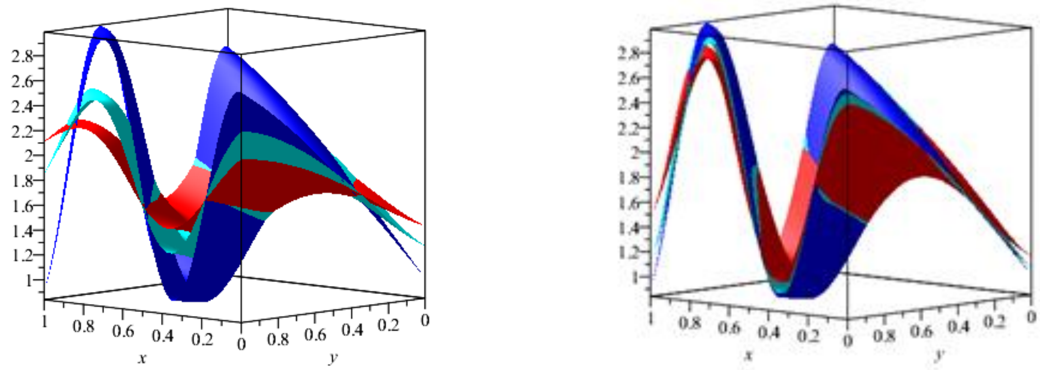

Figure1: For $n = 10{,}30$, $r = 2$ the numerical convergence of NN operators $F_n(f;x,y)$ (red) and $Q_r(f;x,y)$ "Cyan" to $f(x,y)$ (blue).

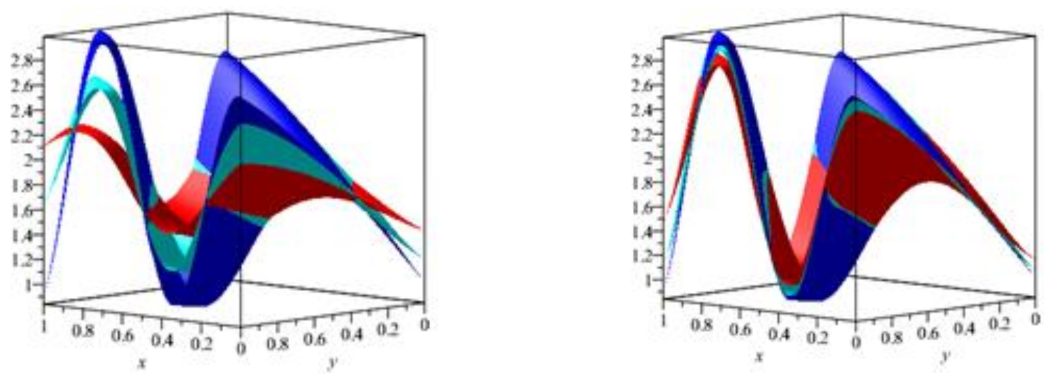

Figure 2: For $n = 10{,}30$, $r = 3$ the numerical convergence of NN operators $F_n(f;x,y)$ (red) and $Q_r(f;x,y)$ "Cyan" to $f(x,y)$ (blue).

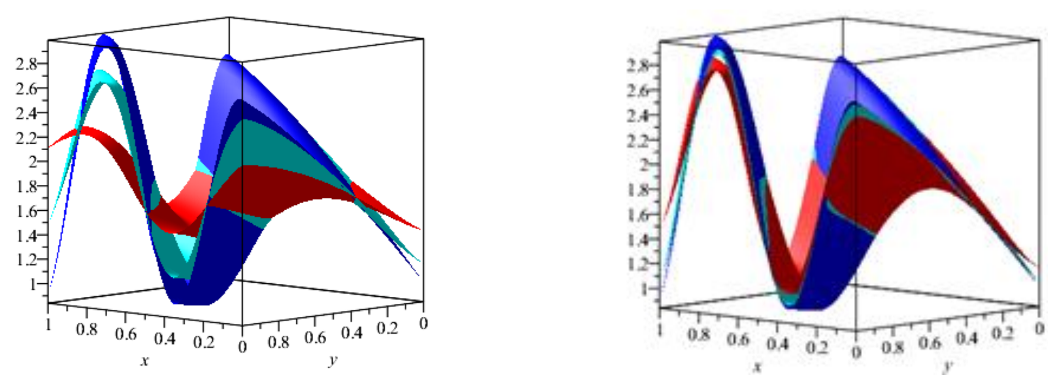

Figure 3: For $n = 10,30$, $r = 4$ the numerical convergence of NN operators $F_n(f; x, y)$ (red) and $Q_r(f; x, y)$ "Cyan" to $f(x, y)$ (blue).

Example 4.2

The convergence of NN operators $Q_r(h; x, y), F_n(h; x, y)$ to test function $h(x, y)$can be given in the following figures for $n = 10,30$, $r = 2,3,4$.

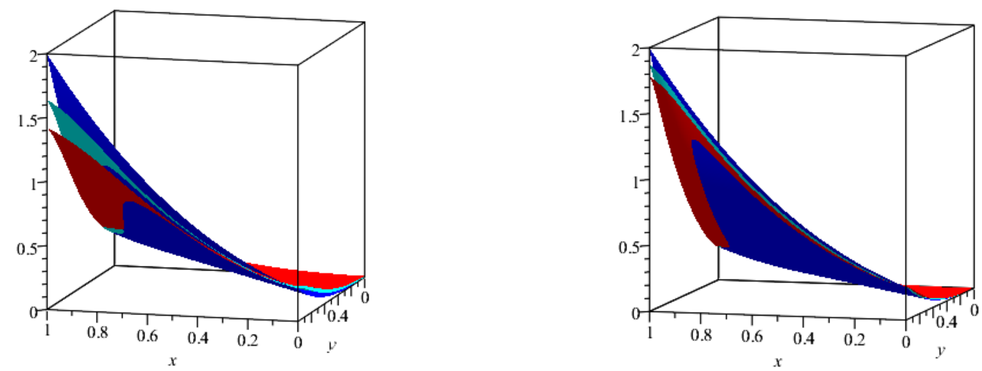

Figure 4: For $n = 10,30$, $r = 2$ the numerical convergence of NN operators $F_n(h; x, y)$ (red) and $Q_r(h; x, y)$ "Cyan" to $h(x, y)$ (blue).

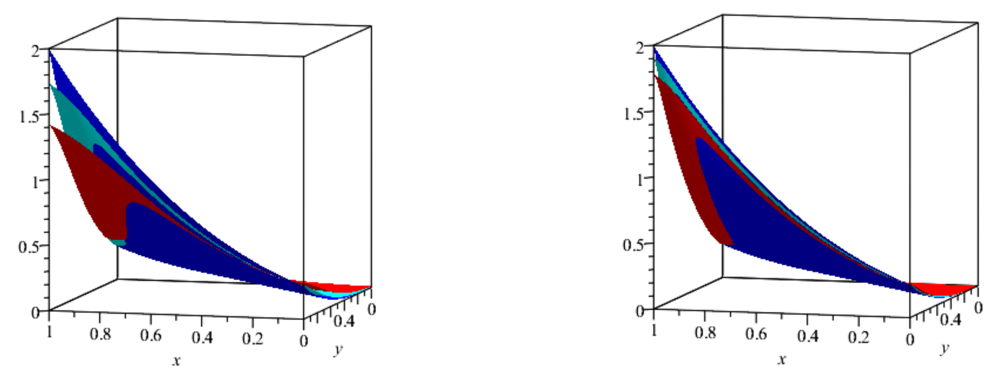

Figure 5: For $n = 10,30$, $r = 3$ the numerical convergence of NN operators $F_n(h; x, y)$ (red) and $Q_r(h; x, y)$ "Cyan" to $h(x, y)$ (blue).

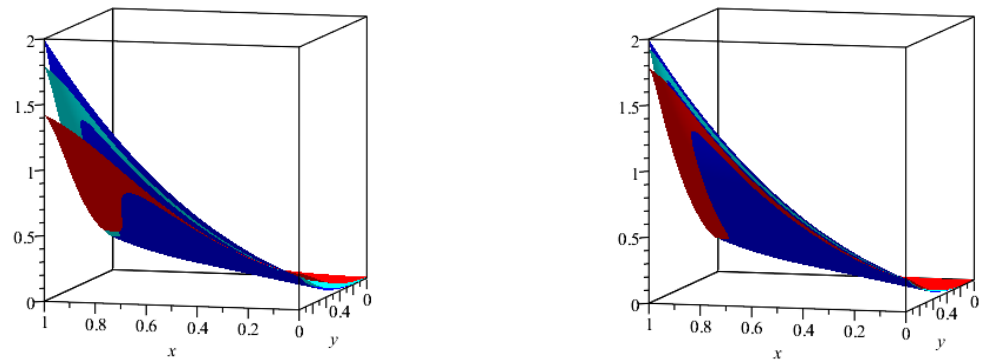

Figure 6: For $n = 10,30$, $r = 4$ the numerical convergence of NN operators $F_n(h; x, y)$ (red) and $Q_r(h; x, y)$ "Cyan" to $h(x, y)$ (blue).

Now, as shown in the Table 1, the values of the average error between the test function and NN in two dimensions are calculated as follows:

$$A(x, y) = \frac{\sum_{k1=0}^{n} \sum_{k2=0}^{n} |U_n(g; x, y) - g(x, y)|}{(n + 1)^2}, \forall (x, y) \in \mathbb{R}^2$$

Using the following two test functions, $f(x, y)$ and $h(x, y)$:

Table1: Average Error

Neural Network	n	$r = 1$	$r = 2$	$r = 3$	$r = 4$
$Q_r(f; x, y)$	10	1.203503767	0.9412620573	0.7206772999	0.5709241359
	30	0.5585328529	0.2993268089	0.2055316359	0.1556577549
$Q_r(h; x, y)$	10	0.5774244320	0.3583003550	0.2648725190	0.2096147940
	30	0.2181245850	0.1291939990	0.0939463700	0.0737079440

Conclusions

When r gets bigger, the NN operators $Q_r(.; x, y)$get closer numerically to the test functions $f(x, y)$ and $h(x, y)$ than the classical NN operators $F_n(.; x, y)$.

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الشبكات العصبية للقوى الرائية لمؤثر متعدد متغيرات برنستاين الكسرية

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المستخلص

قدم هذا البحث شبكة عصبية جديدة على القوة الرائية لمؤثر متعدد متغيرات برنستاين الكسرية. أولا، ثم قدمت مبرهنات التقريب النقطي والمنتظم ودراستها في فضاء كل الدوال الحقيقية المستمرة على $\mathcal{R}$. ثم تمت دراسة المبرهنتين ادناه في فضاء لبشتز في v. أيضا، تم إعطاء بعض الأمثلة العددية لبرهنة تقريب تلك الشبكات العصبية الى دالتين اختياريتين. أظهرت النتائج العددية ان الشبكة العصبية تعطي تقريبا أفضل كلما زادت قيمة r . أخيرا، تم وصف تقريب تلك الشبكات العصبية بواسطة الرسوم البيانية ومعدل الخطأ الحاصل بين التقريب ودالة الاختبار.