π -Generalized b- Closed Sets in Topological Spaces

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Abstract

In this paper we introduce a new class of sets called - π generalized b- closed (briefly π gb closed) sets. We study some of its basic properties. This class of sets is strictly placed between the class of π gp- closed sets and the class of π gsp- closed sets. Further the notion of π b- $T_{\frac{1}{2}}$ space is introduced and studied.

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1. Introduction and Prelimimnaries.

Park[1] introduced the class of π -generalized pre-closed(briefly π gp closed) sets and the class of π -generalized semipreopen closed (briefly π gsp closed) sets was introduced by Sarsak [2] as a generalization of closed sets. In this paper we define and study a new class of π -generalized closed sets, we denote by π -generalized b- closed (briefly π gb- closed) sets, which is strictly placed between the class of π gp- closed set and π gsp- closed sets. Moreover, we define π b- $T_{\frac{1}{2}}$ space as the space in which every π gb- closed set is b- closed.

Throughout this paper (X,τ) and (Y,σ) represent nonempty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), *cl*(A), *int*(A) and *P*(X) denote the closure , the interior and power set of A respectively. (X,τ) will be replaced by X if there is no confusion.

Let us recall the following definitions which are useful in the sequel.

Definition 1.1. A subset A of a space X is called:

(1) *semi- open* if $A \subset cl(\text{int}(A))$ and *semi- closed* if $\text{int}(cl(A)) \subset A$.[3]

(2) α - *open* if $A \subset \text{int}(cl(\text{int}(A)))$ and α - closed if $cl(\text{int}(cl(A))) \subset A$.[4]

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(3) *preopen* if $A \subset \text{int}(cl(A))$ and *preclosed* set if $cl(\text{int}(A)) \subset A$.[5]

(4) *semi- preopen* if $A \subset cl(\text{int}(cl(A)))$ and a *semi- preclosed* if $\text{int}(cl(\text{int}(A))) \subset A$.[6]

(5) *regular open* if $A = \text{int}(cl(A))$ and a *regular closed* set if $A = cl(\text{int}(A))$.[7]

(6) *b- open* if $A \subset \text{int}(cl(A)) \bigcup cl(\text{int}(A))$ and *b- closed* if $\text{int}(cl(A)) \bigcap cl(\text{int}(A)) \subset A$.[8]

(7) π - *open* if *A* is the union of regular open sets, and π -closed if A is the intersection of regular closed sets. [9]

The b- interior (briefly *bint*) of a subset A of X is the union of all b- open sets contained in A. The b- closure (resp. pre-closure, semipre- closure) of A is the intersection of all b-closed (resp. preclosed, semipre- closed) sets containing A, and is denoted by bcl(A) (resp.pcl(A), spcl(A)). The collection of all b- open (resp. b- closed) sets is denoted by BO(X) (resp. BC(X)).[8]

It is well known that:

(1) α - open set $\Rightarrow$ preopen set $\Rightarrow$ b- open set $\Rightarrow$ semi- preopen.[8]

(2) The intersection of a b- open set with α - open set is b- open.[8]

Definition 1.2. A subset A of a space X is called:

(1) generalized closed (briefly g- closed) if $cl(A) \subset U$ whenever $A \subset U$ and U is open in X.[10]

(2) π - generalized closed (briefly π g- closed) if $cl(A) \subset U$ whenever $A \subset U$ and U is π -open.[11]

(3) π - generalized pre cosed (briefly π gp- closed) if $pcl(A) \subset U$ whenever $A \subset U$ and U is π - open.[1]

(4) π -generalized semipre- closed (briefly π gsp- closed) if $spcl(A) \subset U$ whenever $A \subset U$ and U is π - open.[2]

Lemma 1.3. [12]

Let $A \subset X$ then,

(1) $A \subset B \Rightarrow bcl(A) \subset bcl(B)$.

(2) A is b- closed $\Leftrightarrow bcl(A) = A$.

(3) Let $x \in X$, then $x \in bcl(A)$ if and only if every $U \in BO(X)$ such that $x \in U$, $U \bigcap A \neq \phi$.

2. π - Generalized b- Closed Sets.

Definition 2.1.

A subset A of a space X is called π - generalized b- closed (breifly π gb closed) if $bcl(A) \subset U$ whenever $A \subset U$ and U is π - open. The complement of π gb- closed set is called π gb- open.

The family of all π gb- closed (resp. π gb- open) subsets of the space X is denoted by π GBC(X) (resp. π GBO(X)).

Definition 2.2.

The π - kernel (π - ker (A)) of A is the intersection of all π - open sets containing A.

Remark 2.3.
A subset A of a space X is π gb- closed if and only if bcl(A) $\subset \pi$ - ker(A).

Remark 2.4.
Every b- closed set is π gb- closed.

Proposition 2.5.
Every π gp- closed set is π gb- closed.
Proof.
Let A be π gp- closed subset of X and U be π - open such that $A \subset U$. Then pcl (A) $\subset$U. Since every preclosed set is b-closed. Therefore bcl(A) $\subset pcl$ (A). Hence A is π gb- closed.

Proposition 2.6.
Every π gb- closed set is π gsp- closed.
Proof.
Let A be π gb- cosed and U be π - open such that $A \subset U$, then bcl(A) $\subset$U. Since every b-closed set is π gsp-closed. Therefore $spcl$(A) $\subset bcl$(A). Hence, A is π gsp- closed.
The following diagram summarizes the implications among the introduced concept and other related concepts.

$$
\begin{array}{ccccc}
\pi\text{g- closed} & \rightarrow & \pi\text{gp- closed} & & \\
 & & \downarrow & & \\
\text{b- closed} & \rightarrow & \pi\text{gb- closed} & \rightarrow & \pi\text{gsp- closed}
\end{array}
$$

Diagram (1)
The following three examples show that the converses of Remarks 2.4 and Proposition 2.5 are not true in general.

Example 2.7.
Let X= {a, b, c}, τ = {X,ϕ, {a}}and A = {a, b}. Then X is the only regular open (π - open) set containing A. Hence A is π gb- closed, but A is not b- closed, since bcl (A) = X.

Example 2.8.
Let X= {a, b, c}, τ = {X,ϕ, {a}, {b}, {a, b}}. Let A= {a}. Then A is b- closed. Hence A is π gb- closed, but A is not π gp- closed, since A is regular open (π - open) and pcl(A)= {a,c} $\not\subset$ A.

3. Some Properties of π gb- Closed Sets.

Proposition 3.1.
If A is π - open and π gb- closed, then A is b- closed and hence gb- closed.
Proof.
Since A is π - open and π gb- closed. So bcl(A) $\subset$A. But A $\subset bcl$(A). So A= bcl(A). Hence A is b- closed. Hence gb- closed.

Proposition 3.2.
Let A be a π gb- closed in X. Then bcl(A)\ A does not contain any nonempty π - closed set.

Proof.

Let F be a π- closed set such that F $\subset$ *bcl*(A)\ A, so F $\subset$ X\A. Hence A $\subset$X\F. Since A is πgb- closed and X\ F is π- open. So *bcl*(A) $\subset$X\ F. That is F$\subset$X\ *bcl*(A). Therefore F$\subset$*bcl*(A)$\bigcap$(X\ *bcl*(A)) = ϕ. Thus F =ϕ.

Corollary 3.3.

Let A be πgb- closed set in X. Then A is b- closed if and only if *bcl*(A)- **A** is π- closed.

Proof.

Let A be πgb- closed. By hypothesis *bcl*(A)= A and so *bcl*(A)\A=ϕ, which is π- closed. Conversely, suppose that *bcl*(A)\A is π- closed. Then by Theorem 3.2, *bcl*(A)\A= ϕ, that is *bcl*(A)= A. Hence A is b- closed.

Proposition 3.4.

If A is πgb- closed and A$\subset$B$\subset$*bcl*(A). Then B is πgb- closed.

Proof.

Let B$\subset$U, where U is π- open. Then A$\subset$B implies A $\subset$U. Since A is πgb- closed, so *bcl*(A) $\subset$U and since B$\subset$*bcl*(A), then *bcl*(B) $\subset$*bcl*(*bcl*(A))= *bcl*(A). Therefore *bcl*(B) $\subset$U. Hence B is πgb- closed.

Definition 3.5.[13]

Let (X, τ) be a topological space, A $\subset$X and x$\in$X. Then x is said to be a b- limit point of A and only if every b- open set containing x contains a point of A different from x, and the set of all b- limit points of A is said to be the b- derived set of A and is denoted by D_b(A).

Usual derived set of A is denoted by D(A).

The proof of the following result is analogous to the well known ones.

Lemma 3.6.

Let (X, τ) be a topological space and A $\subset$X. Then bcl(A) = A$\bigcup$ D_b(A).

Remark 3.7.

The union of two πgb- closed sets is not necessarily a πgb- closed set as the following example shows.

Example 3.8.

Consider the space (X, τ) in Example 2.8, the sets A= {a} and B= {b} are πgb- closed. But A$\bigcup$B= {a, b} is not πgb- closed.

Proposition 3.9.

Let A and B be πgb- closed sets in (X,τ) such that *cl*(A)= *bcl*(A) and *cl*(B)= *bcl*(B). Then A$\bigcup$B is πgb-closed.

Proof.

Let (A$\bigcup$B) $\subset$U and U is π- open in (X,τ). Then *bcl* (A) $\subset$U and *bcl*(B) $\subset$U. Now, *cl* (A$\bigcup$B) = cl (A) $\bigcup$*cl* (B) = *bcl* (A) $\bigcup$*bcl*(B) $\subset$U. But *bcl* (A$\bigcup$B) $\subset$ *cl* (A$\bigcup$B). So, *bcl* (A$\bigcup$B) $\subset$U and hence A$\bigcup$B is πgb- closed.

From the fact that D_b(A) $\subset$ D(A) and Lemma 3.6 we have the following,

Remark 3.10.
For any subset A of X such that $D(A) \subset D_b(A)$. Then cl(A)= *bcl*(A).
We get the following,

Corollary 3.11.
Let A and B be πgb- closed sets in (X, τ) such that $D(A) \subset D_b(A)$ and $D(B) \subset D_b(B)$. Then A$\bigcup$B is πgb- closed.

Proposition 3.12.
For every x$\in$X its complement X\{x} is πgb- closed or π-open in (X, τ).
Proof.
Suppose X\{x} is not π- open. Then X is the only π- open set containing X\{x}. This implies *bcl*(X\{x}) $\subset$X. Hence X\{x} is πgb- closed.

4. πgb- Open Sets.
The following result is analogous to well known corresponding ones.

Lemma 4.1.
bcl(X\ A)= X\ *bint*(A).
By Lemma 4.1 and definition 2.1 we get the following which is similar to Corollary 4.1 of [2].

Corollary 4.2.
A subset A of X is πgb- open if and only if F$\subset$*bint*(A) whenever F is π-closed in X and F$\subset$A.

Proposition 4.3.
If *bint*(A) $\subset$B$\subset$A and A is πgb- open, then B is πgb- open.
Proof.
Since bint(A) $\subset$B$\subset$A. Hence X\ A $\subset$X\ B$\subset$*bcl*(X\ A), by Lemma 4.1. Since X\ A is πgb- closed, so by Theorem 3.4, X\ B is πgb- closed. Thus B is πgb- open.

Proposition 4.4.
Let A be πgb- open in X and let B be α- open. Then A$\bigcap$B is πgb- open in X.
Proof.
Let F be any π- closed subset of X such that F$\subset$ A$\bigcap$B. Hence F$\subset$A and by Theorem 4.2, F$\subset$bint(A)= $\bigcup${U: U is b- open and U$\subset$A}. Then F$\subset\bigcup$(U$\bigcap$B), where U is a b- open set contained in A. Since U$\bigcap$B is a b- open set contained in A$\bigcap$B for each b- open set U contained in A, F$\subset$bint(A$\bigcap$B), and by Theorem 4.2, A$\bigcap$B is πgb- open in X.

Lemma 4.5.
For any A$\subset$X, *bint*(*bcl*(A)\ A)= ϕ.
Proof.
If *bint*(*bcl*(A)\ A)$\neq\phi$. Then there is an element x$\in$*bint*(*bcl*(A)-A), so there is U$\in$BO(X) such that x$\in$U$\subset$*bcl*(A)-A. Therefore U$\subset$*bcl*(A) and U$\not\subset$A. Thus U $\subset$ *bcl*(A) and U$\subset$X-A. Hence there is U$\in$BO(X), U$\bigcap$A=ϕ, a contradiction, since x$\in$*bcl*(A).

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Proposition 4.6.

Let A$\subset$B$\subset$X and let *bcl*(A)\A be πgb- closed set. Then *bcl*(A)\B is also πgb- open.

Proof.

Suppose *bcl*(A)\A is πgb- open and let F be a π- closed subset of X with F$\subset$*bcl*(A)\B. Then F$\subset$*bcl*(A)\A. By Theorem 2.4 and Lemma 4.5, F $\subset$*bint*(*bcl*(A)\A)= ϕ. So, F=ϕ.Consequently, F$\subset$*bint*(bcl(A)\B).

Proposition 4.7.

Let A$\subset$X be a πgb-closed. Then bcl(A)\ A is πgb- open.

Proof.

Let F be a π- closed such that F$\subset$*bcl*(A)- A. Then by Theorem 3.2, F=ϕ. So F$\subset$ *bint* (*bcl*(A)\A). Therefore *bcl*(A)-A is πgb- open, by Theorem 4.2.

5. πB- $T_{\frac{1}{2}}$ Spaces

In this section we define a new class of spaces, named πb- $T_{\frac{1}{2}}$ space which is a generalization of $T_{\frac{1}{2}}$ [14].

Definition 5.1.

A space (X,τ) is called a πb- $T_{\frac{1}{2}}$ space if every πgb- closed set is b- closed.

Example 5.2.

If X$\neq\phi$ be any set. Then (X, $\tau_{ind.}$) is πb- $T_{\frac{1}{2}}$ space.

Recall that X is $T_{\frac{1}{2}}$ space if every g- closed set is closed or equivalently if every singleton is open or closed.

The notions of πb-$T_{\frac{1}{2}}$ and $T_{\frac{1}{2}}$ are independent as it can be seen through the following examples.

Example 5.3.

Let X= {a, b, c}, τ={X,ϕ, {c}, {a, b}}. Then RO(X) =τ, BO(X) = P(X) = BC(X) = πGBC(X). Then X is πb-$T_{\frac{1}{2}}$ but not $T_{\frac{1}{2}}$.

Example 5.4.

Consider (N,τ) where N is the set of natural numbers and τ = {U$\subset$N: 1 $\in$U}$\bigcup${ϕ}, then τ is a topology on N, and (N,τ) is $T_{\frac{1}{2}}$ but not πb-$T_{\frac{1}{2}}$.

Next, we recall the following,

Definition 5.5.

A space X is πgsp-$T_{\frac{1}{2}}$ (or πgsp in [2]) if every πgsp- closed subset of X is semi- preclosed

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Remark 5.6.
It seems that the notions of π gsp- $T_{\frac{1}{2}}$ and π b- $T_{\frac{1}{2}}$ are independent of each other, but we could not disprove it.
The following result is analogous to Proposition 3.7 in [2].

Proposition 5.7.
A space X is π b- $T_{\frac{1}{2}}$ if and only if every singleton of X is either π - closed or b- open.

Proof.
Necessity: Let $x \in X$ and assume that $\{x\}$ is not π - closed, then $X \setminus \{x\}$ is not π - open, so the only π - open set containing $X \setminus \{x\}$ is X, hence $X \setminus \{x\}$ is π gb- closed. By assumption $X \setminus \{x\}$ is b- closed. Thus $\{x\}$ is b- open.
Sufficiency: Let A be a π gb- closed subset of X and $x \in bcl(A)$. By assumption, we have the following two cases:
(i) $\{x\}$ is b- open. Since $x \in bcl(A)$, So $\{x\} \cap A \neq \phi$. Thus $x \in A$.
(ii) $\{x\}$ is π - closed. Then by Theorem 3.2, $x \notin (bcl(A)- A)$. But $x \in bcl(A)$, so $x \in A$. Therefore in both cases $x \in A$. This shows that $bcl(A) \subset A$ or equivalently A is b-closed.

Proposition 5.8.
(i) $BO(X) \subset \pi GBO(X)$.
(ii) A space X is π b- $T_{\frac{1}{2}}$ if and only if $BO(X) = \pi GBO(X)$.

Proof.
(i) Let A be a b- open. Then X- A is b- closed and so π gb- closed. Thus A is π gb- open. Therefore $BO(X) \subset \pi GBO(X)$.
(ii) *Necessity:* Let X be π b-$T_{\frac{1}{2}}$. Let $A \in GBO(X)$. Then X-A is π gb-closed. By hypothesis, X-A is b-closed. Thus $A \in BO(X)$. Hence $\pi GBO(X) = BO(X)$.
Suficiency: Let $BO(X) = \pi GBO(X)$ and A be π gb- closed. Then X-A is π gb- open. Hence $X\text{-}A \in BO(X)$. Thus A is b- closed. Therefore X is π b-$T_{\frac{1}{2}}$.

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المجموعات المغلقة من النمط $gb\pi$

عذية خليفة العبيدي
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المقدمة

في هذا البحث قدمنا صنفا جديدا من المجموعات اسميناها المجموعات المغلقة من النمط ( ) $gb\pi$ ودرسنا بعض الخواص الاساسية لها.إذ ان هذا النوع يقع بين صنفين من المجموعات هما المجموعات المغلقة من النمط ( ) $gp\pi$ والمجموعات المغلقة من النمط ( ) $gsp\pi$. كما عرفنا ودرسنا نوعا من الفضاءات اسميناه الفضاء $T_{\frac{1}{2}}$ -b π .

الكلمات المفتاحية: المجموعة المفتوحة من النمط b، المجموعة المفتوحة المنتظمة، المجموعة المغلقة من النمط $gb\pi$.