

Some Results on The Complete Arcs in Three Dimensional Projective Space Over Galois Field

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Abstract

The aim of this paper is to introduce the definition of projective 3-space over Galois field $GF(q)$, $q = p^m$, for some prime number p and some integer m .

Also the definitions of (k,n) -arcs, complete arcs, n -secants, the index of the point and the projectively equivalent arcs are given.

Moreover some theorems about these notations are proved.

Keywords: arcs, index, plane.

Introduction: [1]

A projective 3 – space $PG(3,K)$ over a field K is a 3 – dimensional projective space which consists of points, lines and planes with the incidence relation between them.

The projective 3 – space satisfies the following axioms:

- A.** Any two distinct points are contained in a unique line.
- B.** Any three distinct non-collinear points, also any line and point not on the line are contained in a unique plane.
- C.** Any two distinct coplanar lines intersect in a unique point.
- D.** Any line not on a given plane intersects the plane in a unique point.
- E.** Any two distinct planes intersect in a unique line.

A projective space $PG(3,q)$ over Galois field $GF(q)$, $q = p^m$, for some prime number p and some integer m , is a 3 – dimensional projective space.

Now, some theorems on $PG(3,q)$ proved in [1] and [2] are given in the following

Theorem 1:

Every line in $PG(3,q)$ contains exactly $q + 1$ points.

Theorem 2:

Every point in $PG(3,q)$ is on exactly $q + 1$ lines.

Theorem 3:

Every plane in $PG(3,q)$ contains exactly $q^2 + q + 1$ points.

Theorem 4:

Every plane in $PG(3,q)$ contains exactly $q^2 + q + 1$ lines.

Theorem 5:

Every point in $PG(3,q)$ is on exactly $q^2 + q + 1$ planes.

Theorem 6:

There exist $q^3 + q^2 + q + 1$ points in $PG(3,q)$.

Theorem 7:

There exist $q^3 + q^2 + q + 1$ planes in $PG(3,q)$.

Theorem 8:

Any line in $PG(3,q)$ is on exactly $q + 1$ planes.

Definition 1: [1]

A (k,n) – arc A in $PG(3,q)$ is a set of k points such that at most n points of which lie in any plane, $n \geq 3$. n is called the degree of the (k,n) – arc.

Definition 2:

In $PG(3,q)$, if A is any (k,n) – arc, then an (m -secant) of A is a plane ℓ such that $|\ell \cap A| = m$.

Definition 3: [1,2]

A point N not on a (k,n) -arc A has index i if there exists exactly i (n – secants) of A through N , one can denote the number of points N of index i by C_i .

Definition 4:

(k,n) -arc A is complete if it is not contained in any $(k + 1,n)$ -arc.

From definitions 3 and 4, it is concluded that the (k,n) -arc is complete iff $C_0 = 0$. Thus the (k,n) -arc is complete iff every point of $PG(3,q)$ lies on some n -secant of the (k,n) -arc.

Definition 5: [1,3]

Let T_i be the total number of the i – secants of a (k,n) – arc A , then the type of A denoted by $(T_n, T_{n-1}, \dots, T_0)$.

Definition 6: [1]

Let (k_1,n) – arc A is of type $(T_n, T_{n-1}, \dots, T_0)$ and (k_2,n) – arc B is of type $(S_n, S_{n-1}, \dots, S_0)$, then A and B have the same type iff $T_i = S_i$, for all i , in this case they are projectively equivalent.

Theorem 9:

Let $t(P)$ represents the number of 1-secants (planes) through a point P of a (k,n) – arc A and let T_i represent the numbers of i – secants (planes) for the arc A in $PG(3,q)$, then:

1. $t = t(P) = q^2 + q + 2 - k - \frac{(k-1)(k-2)}{2} - \dots - \frac{(k-1)(k-2)\dots(k-(n-1))}{(n-1)!}$
2. $T_1 = k t$
3. $T_2 = \frac{k(k-1)}{2}$
4. $T_3 = \frac{k(k-1)(k-2)}{3!}$

$$5. T_n = \frac{k(k-1)\cdots(k-n+1)}{n!}$$

$$6. T_0 = q^3 + q^2 + q + 1 - k t - \frac{k(k-1)}{2} - \frac{k(k-1)(k-2)}{3!} - \dots - \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

Proof :

1. there exist $(k-1)$ 2-secants to A through P and there exist $\binom{k-1}{2}$ (3-secants) to A

through P, and so there exist $\binom{k-1}{n-1}$ n -secants to A through P, and since there exist exactly $q^2 + q + 1$ planes through P, then the number of the 1-secants through P:

$$\begin{aligned} t(P) &= q^2 + q + 1 - (k-1) - \binom{k-1}{2} - \dots - \binom{k-1}{n-1} \\ &= q^2 + q + 2 - k - \frac{(k-1)(k-2)}{2} - \dots - \frac{(k-1)(k-2)\cdots(k-n+1)}{(n-1)!} = t. \end{aligned}$$

2. T_1 = the number of 1-secants to A, since each point of A has t (1-secants) and the number of the points is k , then $T_1 = k t$.

3. T_2 = the number of 2-secants to A, which is the number of planes passing through any two points of A. Hence $T_2 = \binom{k}{2} = \frac{k(k-1)}{2}$.

4. T_3 = the number of 3-secants of A, which is the number of planes passing through any three points of A. Hence $T_3 = \binom{k}{3} = \frac{k(k-1)(k-2)}{3!}$.

5. T_n = the number of n -secants planes to A, $T_n = \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$.

6. $q^3 + q^2 + q + 1$ represents the number of all planes, then in a (k,n) -arc of PG(3,q),
 $q^3 + q^2 + q + 1 = T_0 + T_1 + T_2 + T_3 + \dots + T_n$
 $T_0 = q^3 + q^2 + q + 1 - T_1 - T_2 - T_3 - \dots - T_n$
 So

$$T_0 = q^3 + q^2 + q + 1 - k t - \frac{k(k-1)}{2} - \frac{k(k-1)(k-2)}{3!} - \dots - \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}.$$

Theorem 10:

Let T_i represents the total number of the i -secants for a (k,n) -arc A in PG(3,q), then the following equations are satisfied:

$$1. \sum_{i=0}^n T_i = q^3 + q^2 + q + 1$$

$$2. \sum_{i=1}^n i! T_i = k t + k(k-1) + k(k-1)(k-2) + \dots + k(k-1)\cdots(k-n)$$

$$3. \sum_{i=2}^n i(i-1) T_i = k(k-1) + k(k-1)(k-2) + \frac{1}{2} k(k-1)(k-2)(k-3) + \dots + \frac{1}{(n-2)!} k(k-1) \dots (k-n).$$

Proof :

1. $\sum_{i=0}^n T_i$ represents the sum of numbers of all i – secants to A, which is the number of all planes in the space. Hence $\sum_{i=0}^n T_i = q^3 + q^2 + q + 1$.

$$2. T_1 = k t, t = q^2 + q + 2 - k - \frac{(k-1)(k-2)}{2} - \dots - \frac{(k-1) \dots (k-n+1)}{(n-1)!},$$

$$T_2 = \frac{k(k-1)}{2}, \quad T_3 = \frac{k(k-1)(k-2)}{3!}, \quad T_4 = \frac{k(k-1)(k-2)(k-3)}{4!}, \quad \dots,$$

$$T_n = \frac{k(k-1) \dots (k-n+1)}{n!}$$

$$\sum_{i=1}^n i! T_i = T_1 + 2! T_2 + 3! T_3 + \dots + n! T_n$$

$$= k t + k(k-1) + k(k-1)(k-2) + \dots + k(k-1) \dots (k-n+1)$$

$$3. \sum_{i=2}^n i(i-1) T_i = 2 T_2 + 6 T_3 + 12 T_4 + \dots + n(n-1) T_n$$

$$= k(k-1) + k(k-1)(k-2) + \frac{1}{2} k(k-1)(k-2)(k-3) + \dots + \frac{1}{(n-2)!} k(k-1) \dots (k-n+1)$$

Theorem 11:

Let $R_i = R_i(P)$ represents the number of the i – secants (planes) through a point P of a (k,n) – arc A, in PG(3,q) then the following equations are satisfied:

$$1. \sum_{i=1}^n R_i = q^2 + q + 1$$

$$2. \sum_{i=2}^n (i-1)! R_i = (k-1) + (k-1)(k-2) + \dots + (k-1)(k-2) \dots (k-n-1)$$

$$= \sum_{i=1}^{n-1} (k-1) \dots (k-i)$$

Proof :

1. $\sum_{i=1}^n R_i = R_1 + R_2 + \dots + R_n$, $\sum_{i=1}^n R_i$ represents the sum of numbers of all the i – secants through a point P of the arc A, which is the number of the planes through P. Thus,

$$\sum_{i=1}^n R_i = q^2 + q + 1.$$

$$2. \sum_{i=2}^n (i-1)! R_i = R_2 + 2! R_3 + 3! R_4 + \dots + (n-1)! R_n$$

From proof (1) of theorem 9, there exist $(k-1)$ 2-secants to A through P, and there exist $\binom{k-1}{2}$ 3-secants to A through P, and so there exist $\binom{k-1}{n-1}$ n-secants to A through P.

$$\text{Thus } R_2 = k-1, R_3 = \binom{k-1}{2}, R_4 = \binom{k-1}{3}, \dots, R_n = \binom{k-1}{n-1}$$

$$R_3 = \frac{(k-1)!}{2!(k-3)!}, R_4 = \frac{(k-1)!}{3!(k-4)!}, \dots, R_n = \frac{(k-1)!}{(n-1)!(k-n)!}$$

$$R_3 = \frac{(k-1)(k-2)}{2}, R_4 = \frac{(k-1)(k-2)(k-3)}{3!}, \dots, R_n = \frac{(k-1)\dots(k-(n-1))}{(n-1)!}$$

$$\begin{aligned} \sum_{i=2}^n (i-1)! R_i &= k-1 + \frac{2!(k-1)(k-2)}{2!} + \frac{3!(k-1)(k-2)(k-3)}{3!} + \dots + \\ &\quad \frac{(n-1)!(k-1)(k-2)\dots(k-(n-1))}{(n-1)!} \\ &= (k-1) + (k-1)(k-2) + (k-1)(k-2)(k-3) + \dots + (k-1)(k-2)\dots(k-(n-1)) \\ &= \sum_{i=1}^{n-1} (k-1)\dots(k-i) \end{aligned}$$

Theorem 12:

Let $S_i = S_i(Q)$ represent the numbers of the i -secants (planes) of a (k,n) -arc A through a point Q not in A, then the following equations are satisfied:

$$1. \sum_{i=0}^n S_i = q^2 + q + 1$$

$$2. \sum_{i=1}^n i S_i = k$$

Proof :

$$1. \sum_{i=0}^n S_i \text{ represents the sum of the total numbers of all } i\text{-secants to A through a point Q}$$

not in A, which is equal to the number of all planes through Q. Thus $\sum_{i=0}^n S_i = q^2 + q + 1$.

$$2. \sum_{i=1}^n i S_i = S_1 + 2 S_2 + 3 S_3 + \dots + n S_n$$

$S_1, S_2, \dots, S_n$ represent the numbers of the i -secants of the arc A through the point Q not in A.

S_1 is the number of the 1-secants to A, each one passes through one point of A.

S_2 is the number of the 2-secants to A, each one passes through two points of A.

S_3 is the number of the 3-secants to A, each one passes through three points of A.

Also, S_n is the number of the n -secants to A, each one passes through n points of A.

Since the number of points of the (k,n) -arc A is k , then $\sum_{i=1}^n i S_i = k$.

Theorem 13:

Let C_i be the number of points of index i in $S = PG(3,q)$ which are not on a complete (k,n) -arc A , then the constants C_i of A satisfy the following equations:

$$(i) \sum_{\alpha}^{\beta} C_i = q^3 + q^2 + q + 1 - k$$

$$(ii) \sum_{\alpha}^{\beta} i C_i = \frac{k(k-1)\cdots(k-n+1)}{n!} (q^2 + q + 1 - n)$$

where α is the smallest i for which $C_i \neq 0$, β be the largest i for which $C_i \neq 0$.

Proof :

The equations express in different ways the cardinality of the following sets

$$(i) \{Q \mid Q \in S \setminus A\}$$

$$(ii) \{(Q, \pi) \mid Q \in \pi \setminus A, \pi \text{ an } n\text{-secant of } A\}$$

for in (i), $\sum_{\alpha}^{\beta} C_i$ represents all points in the space which are not in A , then

$\sum_{\alpha}^{\beta} C_i = q^3 + q^2 + q + 1 - k$, and in (ii) $\sum_{\alpha}^{\beta} i C_i$ represents all points in the space not in A , which are on n -secants of A , that is, each n -secant contains $q^2 + q + 1 - n$ points, and the number of the n -secants is $\binom{k}{n}$, then

$$\sum_{\alpha}^{\beta} i C_i = \binom{k}{n} (q^2 + q + 1 - n) = \frac{k(k-1)\cdots(k-n+1)}{n!} (q^2 + q + 1 - n).$$

Theorem 14:

If P is a point of a (k,n) -arc A in $PG(3,q)$, which lies on an m -secant (plane) of A , then the planes through P contain at most $(n-1)q(q+1) + m$ points of A .

Proof :

If P in A lies on an m -secant (plane), then every other plane through P contains at most $n-1$ points of A distinct from P . Hence the $q^2 + q + 1$ planes through P contain at most $(n-1)(q^2 + q) + m$ points of A .

References

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بعض النتائج حول الاقواس الكاملة في فضاء اسقاطي ثلاثي الابعاد حول حقل كالوا

آمال شهاب المختار

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الخلاصة

الهدف من هذا البحث هو تقديم تعريف الفضاء الثلاثي الاسقاطي حول حقل كالوا $GF(q)$ ، $q = p^m$ ، لبعض قيم p و m اذ ان p عدد أولي و m عدد صحيح.

كذلك أعطيت تعاريف الاقواس (k, n) ، الاقواس الكاملة، القاطع $n -$ ، دليل النقطة، والاقواس المتكافئة اسقاطيا.

فضلا على ذلك برهنت بعض المبرهنات حول هذه المفاهيم.

الكلمات المفتاحية: الاقواس، الدليل، المستوي.