

Essentially Quasi-Invertible Submodules and Essentially Quasi-Dedekind Modules

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Abstract

Let R be a commutative ring with identity. In this paper we study the concepts of essentially quasi-invertible submodules and essentially quasi-Dedekind modules as a generalization of quasi-invertible submodules and quasi-Dedekind modules. Among the results that we obtain is the following: M is an essentially quasi-Dedekind module if and only if M is a K -nonsingular module, where a module M is K -nonsingular if, for each $f \in \text{End}_R(M)$, $\text{Ker} f \leq_e M$ implies $f = 0$.

Key words : Essentially quasi-invertible submodules , Essentially quasi-Dedekind Modules .

Introduction

The concepts of a quasi-invertible submodule of an R -module and quasi-Dedekind module were introduced in [5]. Where a submodule N of an R -module M is called quasi-invertible if $\text{Hom}(M/N, M) = 0$, and an R -module M is called quasi-Dedekind if each nonzero submodule of M is quasi-invertible. As a generalizations to these concepts we introduce the following concepts: We call a submodule N of M is essentially quasi-invertible if, $N \leq_e M$ and N is quasi-invertible. And an R -module M is called essentially quasi-Dedekind if every essential submodule N of M is quasi-invertible; (i.e. $\text{Hom}(M/N, M) = 0$). This paper consists of two sections, §₁ is devoted to study essentially quasi-invertible submodules, in §₂ we study and give the basic properties of essentially quasi-Dedekind modules.

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1 . Essentially Quasi-Invertible Submodules

In this section we introduce the concept of essentially quasi-invertible submodules. We develop basic properties of essentially quasi-invertible submodule .

We start with the following definition :

Definition (1.1)

Let M be an R -module and $N \leq_e M$, then N is called an essentially quasi-invertible submodule of M if , $\text{Hom}(M/N, M) = 0$; that is N is essentially quasi-invertible if , $N \leq_e M$ and N is quasi-invertible . An ideal J in a ring R is called an essentially quasi-invertible ideal of R if , J is an essentially quasi-invertible R -submodule of R .

Remarks and Examples (1.2)

1) It is clear that every essentially quasi-invertible submodule is quasi-invertible submodule .

Recall that an R -module M is called a semisimple if every submodule of M is a direct summand of M , [3, p.189] .

2) If M is a semisimple R -module , then M is the only essentially quasi-invertible submodule of M .

3) Consider Z_4 as a Z -module , $N = (\bar{2}) \leq_e Z_4$, but $\text{Hom}(Z_4/(\bar{2}), Z_4) \cong Z_2 \neq 0$, so $N = (\bar{2})$ is not essentially quasi-invertible submodule of Z_4 , similarly in the Z -module Z_{20} , $N = (\bar{2}) \leq_e Z_{20}$, but it is not quasi-invertible .

4) If N is an essentially quasi-invertible R -submodule of an R -module M , then $\text{ann}_R M = \text{ann}_R N$.

Proof : It is clear . $\square$

The converse of (Rem.and.Ex. 1.2(4)) is not true in general , for example : Let $M = Z \oplus Z$, considered as a Z -module and let $N = Z \oplus (0) \leq M$, then it is clear that $\text{ann}_R M = \text{ann}_R N = (0)$, but N is not essentially quasi-invertible submodule of M , since $N \not\leq_e M$ and also N is not quasi-invertible .

5) Let J be an ideal of a ring R . Then J is an essentially quasi-invertible if and only if $\text{ann}_R(J) = 0$.

Proof : It is easy .

6) Let J be an ideal of a ring R . The following statements are equivalent :

- J is an essentially quasi-invertible ideal of R .
- J is a quasi-invertible ideal of R .
- $\text{ann}_R(J) = 0$.

Proof :

(a) $\Leftrightarrow$ (c) : It follows by (Rem.and.Ex. 1.2(5)) .

(b) $\Leftrightarrow$ (c) : It follows by [5 , prop. 2.2] . $\square$

7) Let R be a ring . The following statements are equivalent :

- a) R is an integral domain .
 b) R is quasi-Dedekind .

Proof : It follows by (Rem.and.Ex. 1.2(6)) . $\square$

- 8) If $M = M_1 \oplus M_2$ is an R -module , and K be an essentially quasi- invertible submodule in M_i for some $i=1,2$, then it is not necessarily that K is an essentially quasi-invertible submodule of M , for example :

Let $M = Z \oplus Z_2$ as Z -module , then $K = Z_2$ is an essentially quasi- invertible submodule of Z_2 as Z -module , but $Z_2 \cong (0) \oplus Z_2$ which is not essentially quasi-invertible of $M = Z \oplus Z_2$, since $(0) \oplus Z_2 \not\leq_e Z \oplus Z_2$.

Proposition (1.3)

Let M be an R -module , and let N_1, N_2 be an essentially quasi- invertible R -submodules of M , then $N_1 \cap N_2$ is an essentially quasi-invertible R -submodule of M .

Proof :

Since $N_1 \leq_e M$, $N_2 \leq_e M$ then $\text{Hom}(M/N_1, M) = 0$ and $\text{Hom}(M/N_2, M) = 0$. Also $N_1 \leq_e M$, $N_2 \leq_e M$ imply $N_1 \cap N_2 \leq_e M$. But $\text{Hom}(M/N_1 \cap N_2, M) \subseteq \text{Hom}(M/N_1, M) + \text{Hom}(M/N_2, M)$. Hence $\text{Hom}(M/N_1 \cap N_2, M) = 0$ and so that $N_1 \cap N_2$ is an essentially quasi- invertible R -submodule of M . $\square$

The following lemma is needed for the next proposition .

Lemma (1.4)

Let M be an R -module such that for each nonzero submodule K of M , $0_P \neq K_P \leq M_P$ for each maximal ideal P of R . If $N_P \leq_e M_P$ implies $N \leq_e M$.

Proof :

Suppose that there exists $0 \neq U \leq M$ such that $U \cap N = 0$. Hence $(U \cap N)_P = 0_P$ which implies that $U_P \cap N_P = 0_P$, but $0_P \neq U_P \leq M_P$ by hypothesis , so that $N_P \not\leq_e M_P$ which is a contradiction . $\square$

Proposition (1.5)

Let M be an R -module , $N \leq M$. If N_P is an essentially quasi-invertible R_P -submodule of R_P -module M_P (for each maximal ideal P of R) , then N is an essentially quasi-invertible submodule of an R -module M .

Proof :

Since N_P is an essentially quasi-invertible R_P -submodule of M_P , $\text{Hom}(M_P/N_P, M_P) = 0$. But by [4 , Ex.3 , p.75] , $(\text{Hom}(M/N, M))_P \subseteq \text{Hom}(M_P/N_P, M_P) = 0$, thus $(\text{Hom}(M/N, M))_P = 0$ and by [4, Prop.3.13 , p.70] , $\text{Hom}(M/N, M) = 0$; that is N is a quasi-invertible

submodule of M . Beside this, by (Lemma (1.4)), $N \leq_e M$. Thus N is an essentially quasi-invertible submodule of M . $\square$

Recall that an R -submodule N of an R -module M is called a SQI-submodule if, for each $f \in \text{Hom}(M/N, M)$, $f(M/N)$ is a small submodule in M , [6, p.44]. And an R -submodule N of an R -module M is called a small submodule of M ($N \ll M$, for short) if, for all $K \leq M$ with $N+K = M$ implies $K = M$, [3, P.106].

Remark (1.6)

It is clear that every quasi-invertible submodule is an SQI-submodule and hence every essentially quasi-invertible submodule is an SQI-submodule.

The converse of (Remark 1.6) is not true in general, consider the following example.

Example (1.7)

Consider the Z -module Z_4 , $N = (\bar{2})$, then N is an SQI-submodule of Z_4 , since for all $f \in \text{Hom}(Z_4/(\bar{2}), Z_4)$, then $f(Z_4/(\bar{2})) \not\leq Z_4$, and every proper submodule of Z_4 is a small in Z_4 , so $f(Z_4/(\bar{2})) \ll Z_4$, but it is known that $N = (\bar{2})$ is not essentially quasi-invertible in Z_4 , (see Rem.and.Ex. 1.2(3)).

2. Essentially Quasi-Dedekind Modules

In this section we give the definition of essentially quasi-Dedekind module with some examples. We prove that essentially quasi-Dedekind module and K -nonsingular module which is introduced by [8] are equivalent. We give conditions under which submodule (resp. quotient module) of essentially quasi-Dedekind is essentially quasi-Dedekind.

Definition (2.1)

An R -module M is called essentially quasi-Dedekind if, $\text{Hom}(M/N, M) = 0$ for all $N \leq_e M$. A ring R is essentially quasi-Dedekind if R is an essentially quasi-Dedekind R -module.

Remarks and Examples (2.2)

- 1) It is clear that every quasi-Dedekind module is an essentially quasi-Dedekind module, but the converse is not true in general, for example:
Each of Z_{10} , Z_{15} are essentially quasi-Dedekind as a Z -module, but it is not quasi-Dedekind.
- 2) Every integral domain R is an essentially quasi-Dedekind R -module, by [5, Ex 1.4, p.24] and (Rem.and.Ex 2.2(1)).
- 3) Z_4 as a Z -module is not essentially quasi-Dedekind, since $(\bar{2}) \leq_e Z_4$,
but $\text{Hom}(Z_4/(\bar{2}), Z_4) \cong Z_2 \neq 0$.
- 4) Let $M = Z_p^\infty$ as a Z -module. Then M is not essentially quasi-Dedekind, but $\text{End}_Z(M)$ (is the ring of P -adic integers) is a commutative domain [see Ex 4.1.2, 8], so $\text{End}_Z(M)$ is essentially quasi-Dedekind, by (Rem.and.Ex 2.2(2)).
- 5) Let M be a uniform R -module. Then M is a quasi-Dedekind R -module if and only if M is an essentially quasi-Dedekind R -module.

Proof : It is clear . $\square$

Roman C.S in [8] , introduce the following : " An R-module M is called K-nonsingular if , for each $f \in \text{End}_R(M)$, $\text{Ker} f \leq_e M$ implies $f = 0$ ". However we prove the following :

Theorem (2.3)

Let M be an R-module . Then M is an essentially quasi-Dedekind R-module if and only if M is a K-nonsingular R-module .

Proof : $\Rightarrow$ Let $f \in \text{End}_R(M)$, $f \neq 0$. Suppose that $\text{Ker} f \leq_e M$, defined

$g : M/\text{Ker} f \longrightarrow M$ by $g(m + \text{Ker} f) = f(m)$ for all $m \in M$. It is easy to see that g is well-defined and g is a nonzero homomorphism . Thus $\text{Hom}(M/\text{Ker} f, M) \neq 0$ which is a contradiction , since M is an essentially quasi-Dedekind R-module .

$\Leftarrow$ $N \leq_e M$. Suppose that there exists $f : M/N \longrightarrow M$ and $f \neq 0$. we have $M \xrightarrow{\pi} M/N \xrightarrow{f} M$, where π is the canonical projection . Let $\psi = f \circ \pi \in \text{End}_R(M)$. $N \subseteq \text{Ker} \psi$ and $N \leq_e M$ implies $\text{Ker} \psi \leq_e M$, $\psi(M) = f \circ \pi(M) = f(M/N) \neq 0$ which is a contradiction with M is a K-nonsingular R-module . $\square$

Although the concepts of essentially quasi-Dedekind module and K-nonsingular module are equivalent ,but we see that it is convenient to use the notion essentially quasi-Dedekind in this paper .

Proposition (2.4)

Every semisimple R-module is an essentially quasi-Dedekind R-module.

Proof : It is easy . $\square$

The converse of (Prop 2.4) is not true in general, consider the following example .

Example (2.5)

It is known that $\mathbb{Z}$ as a $\mathbb{Z}$ -module is essentially quasi-Dedekind , but it is not semisimple .

Recall that an ideal I of a ring R is semiprime if , for all $r \in R$ with $r^2 \in I$ implies $r \in I$ [or , for all ideal A of R with $A^2 \subseteq I$ implies $A \subseteq I$] . And a ring R is called semiprime if (0) is a semiprime ideal of R ; i.e R does not contain nonzero nilpotent ideals , [2] .

Proposition (2.6)

Let R be a ring . The following statements are equivalent :

- 1) R is an essentially quasi-Dedekind ring .
- 2) R is a semiprime ring .
- 3) $Z(R) = 0$ (R is a nonsingular ring) .

Proof :

(2) $\Leftrightarrow$ (3) : It follows by [2 , Prop 1.27, p.35]

(2) $\Rightarrow$ (1) : Let $f \in \text{End}_R(R)$ such that $\text{Ker} f \leq_e R$. To prove $f = 0$.

Suppose that $f \neq 0$, there exists $0 \neq r \in R$ such that $f(a) = ra$ for all $a \in R$. Since $\text{Ker} f \leq_e R$ and $0 \neq r \in R$, then there exists $0 \neq t \in R$ such that $0 \neq rt \in \text{Ker} f$, hence $0 =$

$f(rt) = rf(t) = r^2t$. This implies $(rt)^2 = 0$ and since R is semiprime, $rt = 0$ which is a contradiction. Thus $f = 0$ and R is essentially quasi-Dedekind.

(1) $\Rightarrow$ (3) : Suppose that $Z(R) \neq 0$. Then there exists $0 \neq a \in Z(R)$ and hence $\text{ann}_R(a) \leq_e R$, this implies $\text{ann}_R(a)$ is a quasi-invertible ideal and so that by (5, Prop 2.2), $\text{ann}_R(\text{ann}_R(a)) = 0$, but $(a) \subseteq \text{ann}_R(\text{ann}_R(a))$, hence $a = 0$ which is a contradiction. $\square$

Proposition (2.7)

Let R be a ring. Then R is essentially quasi-Dedekind if and only if $R[x]$ is essentially quasi-Dedekind, where $R[x]$ is the ring of polynomials with one indeterminate x .

Proof :

$\Rightarrow$) Suppose that R is essentially quasi-Dedekind, so by (Prop 2.6) R is a nonsingular ring, and hence by [2, Ex. 13, p.37], $R[x]$ is a nonsingular ring. Thus $R[x]$ is essentially quasi-Dedekind, by (Prop 2.6).

$\Leftarrow$) Suppose that R is not essentially quasi-Dedekind, so by (Prop 2.6), R is not a semiprime ring; that is there exists $a \in L(R)$ and $a \neq 0$, where $L(R) = \{x \in R : x^n = 0, \text{ for some } n \in \mathbb{N}\}$, then $a^n = 0$, for some $n \in \mathbb{N}$. Define $f(x) = a \neq 0$, so $f(x) \in R[x]$, and $R[x]$ is a semiprime ring, by (Prop 2.6). On the other hand $[f(x)]^n = a^n = 0$, implies $f(x) \in L(R[x]) = 0$. It follows that $f = 0$ which is a contradiction. Thus R is essentially quasi-Dedekind. $\square$

Proposition (2.8)

Let M be a faithful R -module. Then R is essentially quasi-Dedekind if and only if $N \oplus \frac{M}{N}$ is a faithful R -module, for all $N \leq M$.

Proof :

$\Rightarrow$) Suppose that R is essentially quasi-Dedekind, so by ((Prop 2.6), R is semiprime. Let $r \in \text{ann}_R(N \oplus \frac{M}{N})$, then $r \in \text{ann}_R(N) \cap \text{ann}_R(\frac{M}{N})$; that is $rN = 0$ and $rM \subseteq N$, so $r^2M \subseteq rN = 0$ implies $r^2 \in \text{ann}_R(M) = 0$ then $r^2 = 0$, thus $r = 0$, since R is a semiprime ring. Therefore $N \oplus \frac{M}{N}$ is a faithful R -module for all $N \leq M$.

$\Leftarrow$) Suppose that $N \oplus \frac{M}{N}$ is a faithful R -module, for all $N \leq M$. To prove that R is essentially quasi-Dedekind. We shall prove that R is a semiprime ring. Let $r \in R$ with $r^2 = 0$, suppose that $r \neq 0$, so $r \notin \text{ann}_R(M)$, since M is a faithful R -module, then $rM \neq 0$. Let $N = rM \leq M$, hence $rN = r^2M = 0$, so $r \in \text{ann}_R(N)$, but $r \in \text{ann}_R(\frac{M}{N})$ (since $rM \subseteq rM = N$), so

$r \in \text{ann}_R(N) \cap \text{ann}_R(\frac{M}{N}) = \text{ann}_R(N \oplus \frac{M}{N}) = 0$, thus $r = 0$ which is a contradiction.

Hence R is essentially quasi-Dedekind. $\square$

Proposition (2.9)

Let M be an R -module and let $\bar{R} = R/J$, where J is an ideal of R such that $J \subseteq \text{ann}_R(M)$. Then M is an essentially quasi-Dedekind R -module if and only if M is an essentially quasi-Dedekind $\bar{R}$ -module.

Proof :

By [3, p.51], we have $\text{Hom}_R(M/N, M) = \text{Hom}_{\bar{R}}(M/N, M)$ for all $N \leq M$. Suppose that M is an essentially quasi-Dedekind R -module, then $\text{Hom}_{\bar{R}}(M/N, M) = \text{Hom}_R(M/N, M) = 0$ for all $N \leq_e M$, implies M is an essentially quasi-Dedekind $\bar{R}$ -module.

The converse follows similarly. $\square$

Let R be an integral domain, and let M be an R -module. An element $x \in M$ is called a torsion element of M if, $\text{ann}_R(x) \neq 0$. The set of all torsion elements of M denoted by $T(M)$ and it is a submodule of M . If $T(M) = 0$ the R -module M is said to be torsion-free, [1, p.45].

The following result shows that essentially quasi-Dedekind preserves under isomorphism.

Proposition (2.10)

Let M_1, M_2 be R -modules such that $M_1 \cong M_2$. Then M_1 is an essentially quasi-Dedekind R -module if and only if M_2 is an essentially quasi-Dedekind R -module.

Proof :

$\Rightarrow$) Suppose that M_1 is an essentially quasi-Dedekind R -module. Let $\phi: M_1 \longrightarrow M_2$, ϕ is an isomorphism. To prove that M_2 is an essentially quasi-Dedekind R -module. Let $f \in \text{End}_R(M_2)$, $f \neq 0$. We have $M_1 \xrightarrow{\phi} M_2 \xrightarrow{f} M_2 \xrightarrow{\phi^{-1}} M_1$, let $h = \phi^{-1} \circ f \circ \phi \in \text{End}_R(M_1)$, and hence $h \neq 0$, then $\text{Ker} h \not\leq_e M_1$. To prove $\text{Ker} f \not\leq_e M_2$, we claim that $\text{Ker} f = \{y \in M_2 : \phi^{-1}(y) \in \text{Ker} h\}$, to prove our assertion. Let $y \in \text{Ker} f$, $f(y) = 0$, $h(\phi^{-1}(y)) = (\phi^{-1} \circ f \circ \phi)(\phi^{-1}(y)) = (\phi^{-1} \circ f)(y) = \phi^{-1}(f(y)) = \phi^{-1}(0) = 0$. Then for all $y \in \text{Ker} f$, $\phi^{-1}(y) \in \text{Ker} h$, so $\phi^{-1}(\text{Ker} f) \subseteq \text{Ker} h \not\leq_e M_1$ which implies $\phi^{-1}(\text{Ker} f) \not\leq_e M_1$, so $\text{Ker} f \not\leq_e M_2$. Thus M_2 is an essentially quasi-Dedekind R -module.

$\Leftarrow$) The proof is similarly. $\square$

Remark (2.11)

Let M be an R -module and let $N \leq M$. If M/N is an essentially quasi-Dedekind R -module. Then M is not necessarily an essentially quasi-Dedekind R -module, as we can see by the following example.

Example (2.12)

Let $M = Z_4$ as a Z -module, and $N = (\bar{2}) \leq Z_4$, then $Z_4/(\bar{2}) \cong Z_2$ is an essentially quasi-Dedekind Z -module, but $M = Z_4$ is not an essentially quasi-Dedekind Z -module.

Now, we turn our attention to a submodule of essentially quasi-Dedekind. First consider the following remark:

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Remark (2.13)

Let M be an essentially quasi-Dedekind R -module, $N \leq M$. Then it is not necessarily that N be an essentially quasi-Dedekind R -module. To show this, consider the following example which appeared in [7].

Let $M = Q \oplus Z_2$ as a Z -module is essentially quasi-Dedekind.

Take $N = Z \oplus Z_2 \leq Q \oplus Z_2$ as a Z -module, then N is not essentially quasi-Dedekind as a Z -module, since if $f: N \longrightarrow N$ define by $f(x, \bar{y}) = (0, \bar{x})$, $x \in Z$, $\bar{y} \in Z_2$, then $f \neq 0$ and

$\text{Ker} f = \{(x, \bar{y}) \in N : f(x, \bar{y}) = (0, \bar{0})\} = \{(x, \bar{y}) \in N : \bar{x} = \bar{0}\} = 2Z \oplus Z_2$. Hence $\text{Ker} f \leq_e N$. Thus $N = Z \oplus Z_2$ is not an essentially quasi-Dedekind as a Z -module.

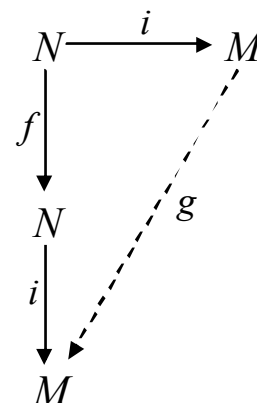
Now, in the next proposition we give a condition which makes R -submodule of an essentially quasi-Dedekind R -module is essentially quasi-Dedekind.

Proposition (2.14)

Let M be an essentially quasi-Dedekind R -module, and M is quasi-injective. If $N \leq_e M$ then N is an essentially quasi-Dedekind R -module.

Proof :

Let $f \in \text{End}_R(N)$, $f \neq 0$, to prove that $\text{Ker} f \leq_e N$. Assume that $\text{Ker} f \leq_e N$. Since M is quasi-injective, then there exists $g \in \text{End}_R(M)$ such that $goi = iof$, (where i is the inclusion mapping).



It follows that $g \neq 0$, and this implies $\text{Ker} g \leq_e M$, since M is essentially quasi-Dedekind. But $\text{Ker} f \subseteq \text{Ker} g$, so $\text{Ker} f \leq_e M$. On the other hand $N \leq_e M$ and by assumption $\text{Ker} f \leq_e N$ imply $\text{Ker} f \leq_e M$. To show this, since $N \leq_e M$ then for all $U \leq M$, $U \neq 0$ then $N \cap U \neq 0$ and $N \cap U \leq N$. But $\text{Ker} f \leq_e N$, hence $\text{Ker} f \cap (N \cap U) \neq 0$; that is $(\text{Ker} f \cap U) \cap N \neq 0$ which implies that $\text{Ker} f \cap U \neq 0$ which is a contradiction. Thus $\text{Ker} f \leq_e N$ and hence N is an essentially quasi-Dedekind R -module. $\square$

Corollary (2.15)

Let M be an R -module . If $\overline{M}$ is an essentially quasi-Dedekind R -module then M is an essentially quasi-Dedekind R -module .

Proof : Suppose that $\overline{M}$ is an essentially quasi-Dedekind R -module , and since $\overline{M}$ is a quasi-injective R -module and $M \leq_e \overline{M}$, so by (Prop 2.14) , M is an essentially quasi-Dedekind R -module . $\square$

Corollary (2.16)

Let M be an R -module . If $E(M)$ is an essentially quasi-Dedekind R -module then M is an essentially quasi-Dedekind R -module .

Proof : It is clear . $\square$

The converse of (Coro2.16) is not true in general, consider the following example .

Example (2.17)

Let $M = Z_2$ as a Z -module . M is an essentially quasi-Dedekind Z -module. But $E(Z_2) = Z_2^\infty$ is not an essentially quasi-Dedekind Z -module , (see Rem.and.Ex 2.2(4)) .

Now we prove the following proposition :

Proposition (2.18)

Let M be an R -module such that ,for each $f \in \text{Hom}(M, E(M))$, $f \neq 0$ implies $\text{Ker} f \not\leq_e M$. Then M is essentially quasi-Dedekind .

Proof : Let $g \in \text{End}_R(M)$, $g \neq 0$. Then $io g \in \text{Hom}(M, E(M))$, and $io g \neq 0$, where i is the inclusion mapping. Hence $\text{Ker}(io g) \not\leq_e M$. But $\text{Ker} g = \text{Ker}(io g)$. Thus $\text{Ker} g \not\leq_e M$ and M is essentially quasi-Dedekind . $\square$

Next we study the behavior of the quotient module of essentially quasi-Dedekind module . First we have the following .

Remark (2.19)

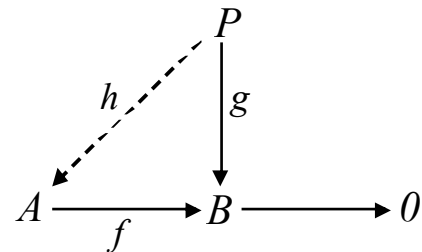
Let M be an R -module , $N \leq M$. If M is an essentially quasi- Dedekind R -module , then M/N is not necessarily essentially quasi- Dedekind R -module , consider the following example .

Example(2.20)

It is well-known that Z as a Z -module is essentially quasi- Dedekind .

Let $N = (4) \leq Z$, $Z/N = Z/(4) \cong Z_4$ is not essentially quasi-Dedekind as a Z -module , (see Rem.and.Ex 2.2(3)) .

We need to recall that an R -module P is projective if and only if , for any R -modules A, B and for any epimorphism $f : A \longrightarrow B$ and for any homomorphism $g : P \longrightarrow B$, there exists a homomorphism $h : P \longrightarrow A$ such that $foh = g$ (i.e the following diagram is a commutative) , [3 , p.117] .



Now , in the next proposition we give a condition under which the (Remark 2.19) is true .

Proposition (2.21)

Let M be an R -module such that M/K is a projective R -module for all $K \leq_e M$. If M is an essentially quasi-Dedekind R -module , then

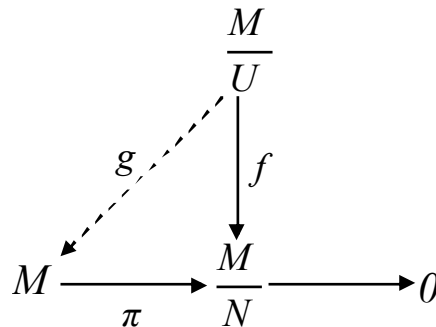
M/N is an essentially quasi-Dedekind R -module for all $N \leq M$.

proof :

Let $U/N \leq_e M/N$. Then $U \leq_e M$ and hence by hypothesis M/U is a projective R -module . Suppose that there exists $f \in \text{Hom}(\frac{M/N}{U/N}, \frac{M}{N})$, $f \neq 0$. But

$\text{Hom}(\frac{M/N}{U/N}, \frac{M}{N}) \cong \text{Hom}(\frac{M}{U}, \frac{M}{N})$ and since M/U is projective , so there exists

$g: \frac{M}{U} \longrightarrow M$ such that $\pi \circ g = f$, where π is the canonical projection mapping.



Since $f \neq 0$ then $g \neq 0$, thus $\text{Hom}(\frac{M}{U}, M) \neq 0$, $U \leq_e M$; that is M is not an essentially quasi-Dedekind R -module ,which is a contradiction. Thus M/N is an essentially quasi-Dedekind R -module for all $N \leq M$. $\square$

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المقاسات الجزئية شبه-معكوسة الواسعة و المقاسات شبه - ديديكندية الواسعة

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الخلاصة

لتكن R حلقة أبدالية ذا عنصر محايد . في هذا البحث درسنا مفهومي المقاسات الجزئية شبه-معكوسة الواسعة والمقاسات شبه - ديديكندية الواسعة أعمام إلى المقاسات الجزئية شبه-معكوسة و المقاسات شبه - ديديكندية. ومن بين النتائج التي حصلنا عليها النتيجة الآتية " M مقياس شبه-ديديكندي واسع اذا كان M مقياس غير منفرد من النمط - K " ، اذ المقاس M هو مقياس غير منفرد من النمط - K اذا كان لكل تشاكل f من M إلى M على الحلقة R بحيث $\text{Ker} f \leq_e M$ يؤدي إلى أن $f = 0$.

