

Approximate Solution of Delay Differential Equations Using the Collocation Method Based on Bernstein Polynomials

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Abstract:

In this paper a modified approach have been used to find the approximate solution of ordinary delay differential equations with constant delay using the collocation method based on Bernstein polynomials.

Key words: Bernstein polynomial, Delay differential equation.

Introduction:

The definition of the n -th order linear delay differential equations with one constant delay may be written as:

$$F(t, y(t), y(t - \tau_1), y(t - \tau_2), \dots, y(t - \tau_k), y'(t), y'(t - \tau_1), y'(t - \tau_2), \dots, y'(t - \tau_k), y''(t - \tau_1), y''(t - \tau_2), \dots, y''(t - \tau_k), y^{(n)}(t - \tau_1), y^{(n)}(t - \tau_2), \dots, y^{(n)}(t - \tau_k) = g(t) \dots(1)$$

where $g(t)$ is a given continuous function and the time lag τ is constant and a_0, a_1, b_0 and b_1 are constant coefficients.

Delay differential equations have a great importance in real life problems which found many applications in mechanics, physics, engineering, economics, biology and especially in the theory of automatic control, [1].

For this importance of delay differential equations many scientists and mathematicians worked on this field of mathematics applied by using several methods of solution like (the method of steps, Laplace transformation method, etc.), see [1], [2].

In this paper, we will solve the first order linear delay differential equations approximated by the collocation method with the Bernstein polynomials as a basis functions.

1- Bernstein Polynomials:

Now, the fundamental definition and some basic properties of Bernstein polynomials are given that will be used later in the definition and construction of the collocation method for solving delay differential equations.

1.1 Definition: [3]

The Bernstein polynomials of degree $n \in \mathbf{N}$ are defined by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad 0 \leq t < \infty$$

For $i = 0, 1, \dots, n$, where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

We usually set $B_{i,n} = 0$ if $i < 0$ or $i > n$.

1.2 Converting Bernstein Basis to Power Basis:[4]

Since the power basis $\{1, t, t^2, \dots, t^n\}$ form a basis for the space of polynomials of degree less than or equal to n , then any Bernstein polynomial of degree n may be rewritten in terms of the power basis, as follows:

$$B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

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$$\begin{aligned}
 &= \binom{n}{k} t^k \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} t^i \\
 &= \sum_{i=0}^{n-k} (-1)^i \binom{n}{k} \binom{n-k}{i} t^{i+k} \\
 &= \sum_{i=k}^n (-1)^{i-k} \binom{n}{k} \binom{n-k}{i-k} t^i
 \end{aligned}$$

or

$$B_{k,n}(t) = \sum_{i=k}^n (-1)^{i-k} \binom{n}{k} \binom{i}{i-k} t^i,$$

$0 \leq t < \infty, k=0,1,\dots,n.$

1.3 Differentiation of Bernstein Polynomials: [5]

Derivation of the n-th degree of Bernstein polynomials are polynomials of degree $n - 1$. By using the definition of the Bernstein polynomial, we can show that this derivative may be written also as a linear combination of Bernstein polynomials, as:

$$\frac{d}{dt} B_{k,n}(t) = n (B_{k-1,n-1}(t) - B_{k,n-1}(t)), 0 \leq k \leq n.$$

This can be shown by direct differentiation as following:

$$\begin{aligned}
 \frac{d}{dt} B_{k,n}(t) &= \frac{d}{dt} \binom{n}{k} t^k (1-t)^{n-k} \\
 &= \frac{k n!}{k!(n-k)!} t^{k-1} (1-t)^{n-k} + \frac{(n-k)n!}{k!(n-k)!} t^k (1-t)^{n-k-1} \\
 &= \frac{n(n-1)!}{(k-1)!(n-k)!} t^{k-1} (1-t)^{n-k} + \frac{n(n-1)!}{k!(n-k-1)!} t^k (1-t)^{n-k-1} \\
 \frac{d}{dt} B_{k,n}(t) &= n \left(\frac{(n-1)!}{(k-1)!(n-k)!} t^{k-1} (1-t)^{n-k} + \frac{(n-1)!}{k!(n-k-1)!} t^k (1-t)^{n-k-1} \right) \\
 &= n (B_{k-1,n-1}(t) - B_{k,n-1}(t)) \dots(2)
 \end{aligned}$$

2- The Collocation Method:

The collocation method is one of the most usually common methods used to approximate the solution of ordinary differential equations, integral equations, partial differential equations, etc., [Delves L.1985] and [Doyc D., 2001], see [6] and [7].

Here, we will use the collocation method to solve ordinary and linear delay differential equations with Bernstein polynomials as basis functions. For this objective consider the retarded delay differential equation of the form

$$y'(t) + y(t) + y(t - \tau) = g(t) \dots(3)$$

With the initial condition $y_0(t) = \phi(t)$ where $t_0 - \tau \leq t \leq t_0$ and the solution of this equation will be formed by using the collocation method

$$\text{Let } y(t) = \psi(t) + \sum_{i=0}^n a_i B_{i,n}(t) \dots(4)$$

where ψ is an arbitrary function satisfying the non homogenous initial conditions, then

$$y'(t) = \psi'(t) + \sum_{i=0}^n a_i B'_{i,n}(t)$$

from (2.2), we have

$$y'(t) = \psi'(t) + \sum_{i=0}^n n a_i (B_{i-1,n-1}(t) - B_{i,n-1}(t))$$

$$y'(t) = \psi'(t) + \sum_{i=1}^n n a_i B_{i-1,n-1}(t)$$

$$- \sum_{i=0}^{n-1} n a_i B_{i,n-1}(t) \dots(5)$$

$$\text{and } y(t - \tau) = \psi(t - \tau) \dots(6)$$

By substituting (4), (5) and (6) in (3), we have;

$$\begin{aligned}
 &\psi(t) + \psi'(t) + \psi(t - \tau) + \sum_{i=1}^n n a_i B_{i-1,n-1}(t) \\
 &- \sum_{i=0}^{n-1} n a_i B_{i,n-1}(t) + \sum_{i=0}^n a_i B_{i,n}(t) \\
 &= g(t)
 \end{aligned}$$

From (2.1), we have

$$\begin{aligned}
 &\psi(t) + \psi'(t) + \psi(t - \tau) + \sum_{i=1}^n n a_i \sum_{k=i-1}^{n-1} (-1)^{k-i+1} \binom{n-1}{k} \binom{k}{i-1} t^k - \\
 &\sum_{i=0}^{n-1} n a_i \sum_{k=i}^{n-1} (-1)^{k-i} \binom{n-1}{k} \binom{k}{i} t^k + \sum_{i=0}^n a_i \sum_{k=i}^n (-1)^{k-i} \binom{n}{k} \binom{k}{i} t^k = g(t) \dots(7)
 \end{aligned}$$

Simplifying equation (7) and substituting on t by $n + 1$ points that lies in the interval $[t_0 - \tau, t_0]$, we have $n + 1$ algebraic equations with $n + 1$ unknowns a_0, a_1, \dots, a_n and in matrix form as:

$$A X = B$$

where A is an $(n + 1) \times (n + 1)$ constant matrix, X is the column of the unknown elements a_0, a_1, \dots, a_n and B is a given vector, [8].

This system may be solved by using computer programs to get the values of a_0, a_1, \dots, a_n .

Then substituting this values back into the function $y(t)$ to get the approximate solution of the delay differential equation.

The next examples illustrate the above method of solution;

2.1 Example :

Consider the retarded delay differential equation

$$y'(t) + y(t) + y(t - 1) = t^2 \dots(8)$$

with the initial condition $y_0(t) = t,$

$$-1 \leq t \leq 0.$$

Then by letting

$$y(t) = t + \sum_{i=0}^n a_i B_{i,n}(t)$$

Hence, from the initial condition at $t = 0,$ we have

$$y(0) = a_0 \Rightarrow a_0 = 0.$$

$$y'(t) = 1 + \sum_{i=1}^n n a_i B_{i-1, n-1}(t) -$$

$$\sum_{i=1}^{n-1} n a_i B_{i, n-1}(t)$$

and

$$y(t - 1) = t - 1 + \sum_{i=0}^n a_i B_{i,n}(t - 1)$$

Substituting in the delay differential equation yields $t_0:$

$$2t + \sum_{i=1}^n n a_i B_{i-1, n-1}(t) - \sum_{i=1}^{n-1} n a_i B_{i, n}$$

$$-1(t) + \sum_{i=1}^n a_i B_{i,n}(t) = t^2$$

The solution will be found when $n = 3,$ from equation (7), we have;

$$2t + \sum_{i=1}^3 3a_i \sum_{k=i-1}^2 (-1)^{k-i+1} \binom{2}{k} \binom{k}{i-1} t^k$$

$$- \sum_{i=1}^2 3a_i \sum_{k=i}^2 (-1)^{k-i} \binom{2}{k} \binom{k}{i} t^k +$$

$$\sum_{i=1}^3 a_i \sum_{k=i}^3 (-1)^{k-i} \binom{3}{k} \binom{k}{i} t^k +$$

$$\sum_{i=1}^3 a_i \sum_{k=i}^3 (-1)^{k-i} \binom{3}{k} \binom{k}{i} (t-1)^k = t^2$$

$$2t + H_1 - H_2 + H_3 + H_4 = t^2 \dots(9)$$

where

$$H_1 = \sum_{i=1}^3 3a_i \sum_{k=i-1}^2 (-1)^{k-i+1} \binom{2}{k} \binom{k}{i-1} t^k$$

$$H_2 = \sum_{i=1}^2 3a_i \sum_{k=i}^2 (-1)^{k-i} \binom{2}{k} \binom{k}{i} t^k$$

$$H_3 = \sum_{i=1}^3 a_i \sum_{k=i}^3 (-1)^{k-i} \binom{3}{k} \binom{k}{i} t^k$$

$$H_4 = \sum_{i=1}^3 a_i \sum_{k=i}^3 (-1)^{k-i} \binom{3}{k} \binom{k}{i} (t-1)^k$$

and to simplify H_1, H_2, H_3 and H_4 as follows:

$$H_1 = \sum_{i=1}^3 3a_i \sum_{k=i-1}^2 (-1)^{k-i+1} \binom{2}{k} \binom{k}{i-1} t^k$$

$$H_1 = 3a_1 \sum_{k=0}^2 (-1)^k \binom{2}{k} \binom{k}{0} t^k +$$

$$3a_2 \sum_{k=1}^2 (-1)^{k-1} \binom{2}{k} \binom{k}{1} t^k +$$

$$3a_3 \sum_{k=2}^2 (-1)^{k-2} \binom{2}{k} \binom{k}{2} t^k$$

$$H_1 = 3a_1$$

$$\left[\binom{2}{0} \binom{0}{0} t^0 + (-1) \binom{2}{1} \binom{1}{0} t + (-1)^2 \binom{2}{2} \binom{2}{0} t^2 \right] +$$

$$3a_2 \left[\binom{2}{1} \binom{1}{1} t + (-1)^1 \binom{2}{2} \binom{2}{1} t^2 \right] +$$

$$3a_3 \binom{2}{2} \binom{2}{2} t^2$$

$$= 3a_1 [1 - 2t + t^2] + 3a_2 [2t - 2t^2] + 3a_3 t^2$$

$$= 3(a_1 - 2a_2 + 3a_3) t^2 + 6(-a_1 + a_2)t + 3a_1$$

$$\begin{aligned}
 H_2 &= \sum_{i=1}^2 3a_i \sum_{k=i}^2 (-1)^{k-i} \binom{2}{k} \binom{k}{i} t^k \\
 &= 3a_1 \sum_{k=1}^2 (-1)^{k-1} \binom{2}{k} \binom{k}{1} t^k + \\
 &3a_2 \sum_{k=2}^2 (-1)^{k-2} \binom{2}{k} \binom{k}{2} t^k \\
 &= 3a_1 \left[\binom{2}{1} \binom{1}{1} t - \binom{2}{2} \binom{2}{1} t^2 \right] + 3a_2 \binom{2}{2} \binom{2}{2} t^2 \\
 &= 3a_1 [2t - 2t^2] + 3a_2 t^2 \\
 &= 3(-2a_1 + 3a_2) t^2 + 6(-a_0 + a_1)t + 3a_0
 \end{aligned}$$

By the same way, we have

$$\begin{aligned}
 H_3 &= (3a_1 - 3a_2 + a_3) t^3 + (-6a_1 + 3a_2) t^2 \\
 &+ (3a_1) t + \text{and}
 \end{aligned}$$

$$\begin{aligned}
 H_4 &= (+3a_1 - 3a_2 + a_3) t^3 + (-15a_1 \\
 &+ 12a_2 - 3a_3) t^2 + (24a_1 - 15a_2 + 3a_3) t + \\
 &(-12a_1 + 6a_2 - a_3)
 \end{aligned}$$

By substituting H₁, H₂, H₃ and H₄ in equation (9), yields:

$$\begin{aligned}
 &2t + 3(a_1 - 2a_2 + 3a_3) t^2 + 6(-a_1 + a_2)t \\
 &+ 3a_1 - 3(-2a_1 + 3a_2) t^2 - 6a_1 t - 3a_0 + \\
 &(3a_1 - 3a_2 + a_3) t^3 + (-6a_1 + 3a_2) t^2 + \\
 &3a_1 t + (3a_1 - 3a_2 + a_3) t^3 + (-15a_1 \\
 &+ 12a_2 - 3a_3) t^2 + (24a_1 - 15a_2 + 3a_3) t + \\
 &(-12a_1 + 6a_2 - a_3) = t^2
 \end{aligned}$$

Thus;

$$(3a_1 - 3a_2 + a_3) t^3 + (3a_1 - 6a_2 + 3a_3 - 1)t^2 + (2 - 9a_1 + 6a_2)t + 3a_3 = 0$$

To find a₁, a₂, a₃, one may choose three points say: t₁ = 1/3, t₂ = 1/2, t₃ = 1

to get the following system:

$$\begin{aligned}
 0.444a_1 + 1.222a_2 + 0.37a_3 &= -0.55 \\
 -0.375a_1 + 1.125a_2 + 0.875a_3 &= -0.75 \\
 -3a_2 + 4a_3 &= -1
 \end{aligned}$$

and solving of equations produce

$$a_1 = -0.084, a_2 = -0.245 \text{ and } a_3 = -0.577$$

Hence, the solution is

$$\begin{aligned}
 y(t) &= t + \sum_{i=0}^3 a_i B_{i,3}(t) \\
 &= t + 0B_{0,3}(t) - 0.084 B_{1,3}(t) - 0.245 \\
 &B_{2,3}(t) - 0.577 B_{3,3}(t)
 \end{aligned}$$

$$y_1(t) = -0.094t^3 - 0.231 t^2 + 0.748 t = \psi_1(t), \text{ where } 0 \leq t \leq 1$$

Now, if 1 ≤ t ≤ 2 by the same way above from the initial condition at t = 1, we have:

$$y_1(1) = \psi_1(1) + \sum_{i=0}^3 a_i B_{i,3}(t)$$

that yields to a₃ = 0 then

$$y_2(t) = \psi_1(t) + \sum_{i=0}^2 a_i B_{i,2}(t)$$

$$y_2'(t) = \psi_1'(t) + \sum_{i=0}^2 a_i B'_{i,2}(t)$$

$$y_2(t-1) = y_1(t-1)$$

by substituting y₂(t), y'₂(t), y₂(t - 1) in the equation (8) we have the following equation

$$\begin{aligned}
 &(-0.137 - 2a_0 + 3a_1) + (1.214 + 3a_0 - \\
 &9a_1 + 6a_2)t + (-1.462 + 3a_1 - 6a_2)t^2 + \\
 &(-0.188 - a_0 + 3a_1 - 3a_2)t^3 = 0
 \end{aligned}$$

We can choose three points say t₁ = 1.25, t₂ = 1.5, t₃ = 2 to get the following system:

$$\begin{aligned}
 -0.203a_0 + 2.296a_1 - 7.743a_2 &= 1.271 \\
 -0.875a_0 + 6.375a_1 - 14.625a_2 &= 2.239 \\
 -4a_0 + 21a_1 - 36a_2 &= 5.061
 \end{aligned}$$

and solving of equations produce

$$a_0 = 0.108, a_1 = -0.054, a_2 = -0.183$$

Hence, the solution is

$$y_2(t) = 0.182 t^3 - 0.132 t^2 + 0.262 t + 0.108$$

by the same way for the other intervals [2,3], [3,4], ...

2.2 Example :

Consider the mixed linear delay differential equation

$$y'(t) + y'(t-1) + y(t) + y(t-1) = 2t \quad \dots(10)$$

with the initial condition y₀(t) = t, where -1 ≤ t ≤ 0.

Solution: Let y(t) = t + ∑_{i=0}ⁿ a_i B_{i,n}(t),

$$\text{then } y'(t) = 1 + \sum_{i=0}^n a_i B'_{i,n}(t)$$

$$= 1 + \sum_{i=1}^n n a_i B_{i-1, n-1}(t) - \sum_{i=0}^{n-1} n a_i B_{i, n-1}(t)$$

and $y(t-1) = t-1, y'(t-1) = 1$
 Substituting in the delay differential equation (8), give:

$$2t+1 - \sum_{i=0}^{n-1} n a_i B_{i, n-1}(t) + \sum_{i=0}^n a_i B_{i, n}(t) = 2t$$

When $n = 3$, then from equation (7), we have;

$$1 + \sum_{i=1}^3 3a_i \sum_{k=i-1}^2 (-1)^{k-i+1} \binom{2}{k} \binom{k}{i-1} t^k + \sum_{i=0}^2 3a_i \sum_{k=i}^2 (-1)^{k-i} \binom{2}{k} \binom{k}{i} t^k + \sum_{i=0}^3 a_i \sum_{k=i}^3 (-1)^{k-i} \binom{3}{k} \binom{k}{i} t^k = 0$$

From this equation and the initial condition that produce $a_0 = 0$, at $t = 0$, we get:

$$(1 + 3a_1) + (-9a_1 + 6a_2)t + (3a_1 - 6a_2 + 3a_3)t^2 + (3a_1 - 3a_2 + a_3)t^3 = 0 \dots(11)$$

To find a_1, a_2, a_3 on the interval $[0,1]$, we choose three points say: $t_1 = \frac{1}{3}, t_2 = \frac{1}{2}, t_3 = 1$ substituting in (11) we have:

$$\begin{aligned} 0.444a_1 + 1.222a_2 + 0.73a_3 &= -1 \\ -0.375a_1 + 1.125a_2 + 0.875a_3 &= -1 \\ -3a_2 + 4a_3 &= -1 \end{aligned}$$

by solving this system we have:
 $a_1 = -0.33, a_2 = -0.508$ and $a_3 = -0.63$

$$\text{Then } y_1(t) = -0.096 t^3 + 0.456 t^2 + 0.01 t$$

and for the interval $[1,2]$ from the new initial condition $y_1(t)$ for $t = 1$ we will get $a_3 = 0$ we choose the points $t_1 = 1.25, t_2 = 1.5, t_3 = 2$, by the same way above we get:

$$\begin{aligned} -0.203a_0 + 2.296a_1 - 7.734a_2 &= 0.113 \\ -0.875a_0 + 6.375a_1 - 14.625a_2 &= -0.244 \\ -4a_0 + 21a_1 - 36a_2 &= -1.066 \end{aligned}$$

and by solving this system we have:

$$a_0 = -3.626, a_1 = -0.743 \text{ and } a_2 = -0.14$$

$$\text{Then } y_2(t) = 1.721 t^3 - 6.384 t^2 + 8.659 t - 3.626$$

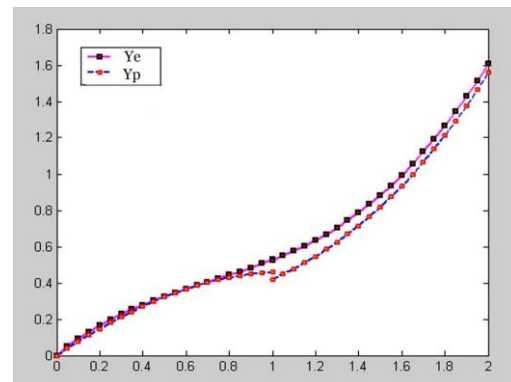


Fig. (1) Illustrate the approximate solution y_p and exact solution y_e of the retarded delay differential equation (8)

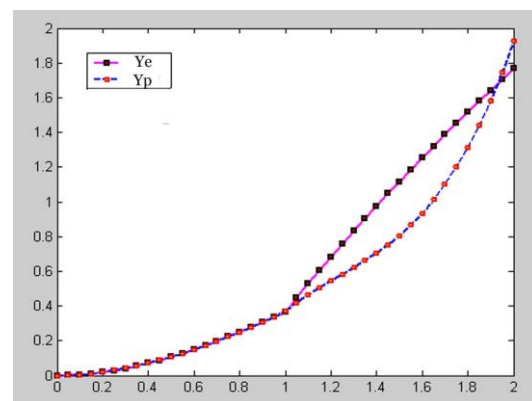


Fig. (2) illustrate the approximate solution y_p and exact solution y_e of the mixed delay differential equation (10)

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الحل التقريبي للمعادلات التفاضلية التباطؤية باستخدام طريقة الحشد المعتمدة على متعددات حدود بيرنشتاين

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الخلاصة:

في هذا البحث تم استخدام تقارب مُعدل لايجاد حل تقريبي للمعادلات التفاضلية الاعتيادية التباطؤية ذات تباطؤ ثابت باستخدام طريقة الحشد معتمدة على متعددات حدود بيرنشتاين كأساس للحل.