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MCD-Domain of type A+x B[x] and A+xB[[x]]

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Abstract:

In this paper we study MCD-Domain of type A+xB[x] and A+xB[[x]].

Let R be a commutative Ring with unity and S be a subset of R. The set of all nonzero common divisors of the elements in S by $CD_R(S)$. An element m in $CD_R(S)$ is called a maximal common divisor (for short an MCD) of S if m is associated with any element in $CD_R(S)$ which is divisible by m. A domain R is called MCD-Domain if every finite set of nonzero elements in R has an MCD.

In corollary (III) and corollary (IIII) we shall given a new proof of [2,Theorem 1.1] , and see also [5, corollary 1.5] and for its power series analogue.

Introduction:

In this paper we study maximal common divisor (for short an MCD) domain of type A+XB[x] and A+XB[[x]] , and we shall given a new proof of [2 ,Theorem 1.1] , and see also [5 ,corollary 1.5] , and its power series an analogue .

First we recall some basic definitions and facts. We let R be a(commutative) domain with quotient field k .An element of R is called reducible if it is a product of two nonzero non units in R ,it is called irreducible (or an atom) if it is a nonzero non unit and is not reducible .The set of units is denoted by U(R). two nonzero elements a and b of R are called associate if a=ub for some unit u ,we denote the this by a ~ b .

A domain R is called pre-schreier if whenever an nonzero $x\in R$ divides a_1a_2 with $a_1, a_2 \in R$,x can be written as $x=x_1x_2$ s.t. x_i divides a_i , i= 1,2 .

Let S be a subset of R . We denote the set of all nonzero common divisor of the elements in S by $CD_R(S)$. An element m in $CD_R(S)$ is called a maximal common divisor (for short an MCD) of S , if m is associate with any element in $CD_R(S)$ which is divisible by m. thus an element m of R is an MCD of S if and only if m divides each element in S and the set (1/m) S={s/m\ s $\in$ S }has GCD 1 (GCD=greatest common divisor). More particularly , 1 is an MCD of S if an only if is a GCD of S. The set maximal common divisor of S is denoted by $MCD_R(S)$ .it is easy to show that a nonzero element m of R is an MCD of two nonzero elements a and b if and only if the element ab/m belongs to R and is minimal common multiple of a and b .

A domain R is called an MCD domain if every finite set of nonzero elements in R has an MCD . Recall that R is a weak GCD domain if every two nonzero elements in R have an MCD.[4]

A nonzero polynomial in R[x] is called indecomposable if it is not a product of two non constant polynomials in R[x].

An element $\alpha \in R\backslash\{0\}$ is LCM –prime to S if $\alpha R \cap tR = \alpha tR$ (equivalent to $tR : \alpha = tR$) for each $t \in S$.

A set S is called splitting multiplicative system in R if for each nonzero element of R can be written as product between an element of S and an element LCM-prime to S.

O MCD –domains:

We recall from [3,Theorem 1.4] that the following conditions are equivalent :

1- R is an atomic MCD –domain , another world (for each $n \geq 2$ and $a_1,\ldots\ldots,a_n \in R\backslash\{0\}$, there are $c_1,\ldots\ldots,c_n \in R$ with no common factors and irreducible $b_1,\ldots\ldots,b_n \in R$ such that $a_i=b_1\ldots\ldots b_m c_1$ for each $1\leq i \leq n$).

2- any polynomial extension of R is atomic .

3-R[x ,y] is atomic

Proof :

We have already observed that (1) $\Rightarrow$ (2) ,and (2) $\Rightarrow$ (3) is clear . for (3) $\Rightarrow$ (1) , we first observed that for any field F and any b, $a_0,\ldots\ldots,a_n \in F\backslash\{0\}$, $a_nx^n+\ldots\ldots\ldots\ldots a_1x+a_0 +bY \in F[X,Y]$ is irreducible . hence , given $a_1,\ldots\ldots,a_n \in R\backslash\{0\}$, $a_nx^{n-2}+\ldots\ldots\ldots+ a_2+a_1Y= b_1\ldots\ldots b_m (c_nx^{n-2}+\ldots\ldots\ldots+ c_2+c_1Y)$, where $b_1,\ldots\ldots,b_m \in R$ are irreducible and $c_1,\ldots\ldots,c_n \in R$ have no common factors , since R]X,Y] is atomic . Thus (1) holds.

Similarly , we have the following :

Proposition (1) :

Let R be a domain . The following two conditions are equivalent :

1-R is atomic , and the set of coefficients of any indecomposable polynomial in R has an MCD .

2-R[x] is atomic .

Proof :

(1) $\Rightarrow$ (2)

We prove by induction on the degree , that any nonzero noninvertible polynomial f in R[x] is a product of atoms . the assertion is clear if deg f =0 .

Let deg. f $\neq$ 0 .since f is a product of indecomposable polynomials , we may assume that f is indecomposable . Let m be an MCD in R of the coefficients of f . write f = mg with $g \in R[x]$. If m is not a unit then it is a product of atoms in R and so also in R[x] .

We now show that g is an atom. If not ,let $g=g_1g_2$ be a nontrivial decomposition of g .since g is indecomposable .we may that $g_1 \in R$ .Hence ,mg_1 is common divisor of all the coefficient of f ,so g_1 is invertible in R ,a contradiction . We conclude that R[x] is atomic.

(2) $\Rightarrow$ (1)

Let $f= a_nx^n+\ldots\ldots\ldots\ldots+a_0$ be an indecomposable polynomial in R[x] of positive degree .Let $f=f_1f_2\ldots\ldots\ldots\ldots f_k$ be an irreducible decomposition of f . since f is indecomposable ,we may assume that $f_1\ldots\ldots\ldots\ldots f_{k-1}$ are in R . Let $m =f_1\ldots\ldots\ldots\ldots\ldots f_{k-1}$, so m is a common divisor of the coefficients of f .Let c be a common divisor of the coefficients of f .Such that $m \mid c$ in R. Let c=md with $d \in R$. Thus in R[x], the element c=md divides $f=mf_k$, so $d \mid f_k$. since f_k is irreducible of positive degree ,we obtain that d is invertible so m is an MCD of the coefficient of f .

In particular ,if R[x] is atomic ,then R is weak GCD [3,Theorem 1.3(a)] . And for its power series R[[x]] analogue .

Proposition (2) :

The Following conditions are equivalent :

1- R is an MCD-domain.

2- R[x] is an MCD –domain .

3- Any polynomial extension of R is an MCD –domain.

4- R[x] is a weak GCD domain .

5- any polynomial extension of R is a weak GCD .domain .

Proof:

It is enough to prove the implication $(4) \Rightarrow (1) \Rightarrow (3)$.

$(4) \Rightarrow (1)$.

Let $r_1,\ldots\ldots,r_n$ be elements in R\{0} (n≥2) and let $f(x) = r_1+r_2x+\ldots\ldots+r_{n-1}x^{n-2}$ .We have $CD_R(\{r_1\ldots\ldots r_n\})=MCD_{R[x]}(\{f,r_n\})$,and also $MCD_R(\{r_1,\ldots\ldots,r_n\}) =MCD_{R[x]}(\{f,r_n\})$ .It follows that R is an MCD domain.

$(1) \Rightarrow (3)$

Let x be a set of independent in determinates over R .And F be a finite nonempty subset of R[x]\{0}. There exists a polynomial g in $CD_{R[x]}(F)$ of highest total degree .

Let c be an MCD in R of all coefficient of the polynomial in $(1/g)F=\{f/g \backslash f \in F\}$.Thus $c \in MCD_{R[x]}((1/g)F)$,and so $cg \in MCD_{R[x]}(F)$ .We conclude that R[x] is an MCD domain .

We have already observed that $(4) \Rightarrow (5)$, and $(5) \Rightarrow (4)$ is clear by [3, Theorem 1.3] .

We conjecture that if R[x] is atomic ,then R[x ,y] is atomic ,that is, we have the following .

Conjecture I :

The following conditions are equivalent :

1- R[x] is atomic.

2- R[x ,y] is atomic.

3- R is an atomic MCD domain.

By propositions (2) , conjecture I is equivalent to the assertion that if the domain R[x] is atomic ,then R is an MCD domain .In view of proposition (1) ,the previous conjecture follows from the following conjecture restricted to atomic domains .

Conjecture II :

For any domain R and nonzero finitely generated ideal I of R there exists an indecomposable polynomial f in R[x] with content I.

Now the following some remarks and some proposition conjecture to prove [2,Theorem 1.1] .

Remarks:

Let R=A+XB[X] and A be a domain and $S \subseteq A$ a multiplicative system:

i- if $a \in A$ is LCM –prime to S and a divides a product bs with $b \in A$ and $s \in S$,then a divides b ,because $b \in A_a : S= A_a$, bs $=\{b,s \in A_a\}$

ii- if $a,b \in A$,a is LCM – prime to S and a divides b in A_s, then a divides b in A ,because $b \in A_a: S= A_a$ for some $s \in S$.

iii- By (ii) if f ,g are in $A +xA_s[x]$,f(0) ≠0 is LCM –prime to S and f divides g in A_s [x] .

iiii- By (ii) if (f,g) are in A +x A_s [[x]],f(0) ≠0 is LCM-prime to S and f divides g A_s [[x]] ,then f divides g in A +x A_s [[x]].(A_s [[x]] =power series).

Proposition (3) :

Let A be a domain .then A is a GCD-domain if and only if A is pre-Schreier MCD-domain.

Proof:

Assume that A is a pre-Schreier MCD domain. Let $a_1,b_1 \in A\backslash\{0\}$.Then factoring out an MCD of a_1 and $b_1$it suffices to show that $a_1A \cap b_1 A = a_1 b_1 A$ provided a_1 and b_1 are relatively prime.

Assume that m is a common multiple of a_1 and b_1 say $m=a_1a_2=b_1b_2$ with $a_2,b_2 \in A$, there exist $c_{ij} \in A$ s.t.

$c_{1j}\, c_{2j} = a_j$,$c_{i1}\, c_{i2} =b_i$,1≤i , j≤2.

By Theorem 2.2[1] (A ring R is a Schreier Ring if and only if it is an integrally closed integral domain s.t. For any two Factorizations of an element a $(\neq 0)$of R ,a = $p_1p_2\ldots\ldots p_m = q_1q_2\ldots\ldots q_n$ $\exists$ elements r_{ij} (i=1................m ,j=1.........n) s.t.

$P_i = \prod_j r_{ij}$, $q_j = \prod_i r_{ij}$.

In other words :any two factorization of a have a common refinement).

Then c_{11} is invertible , hence a_1b_1 divides m , because $a_1 b_1 c^{-1}_{11} c_{22} = a_1 c_{12} c_{22} = a_1 a_2 = m$. Converse is obvious.

Proposition (4):

Let A be a domain and $S \subseteq A$ a splitting multiplicative system .If A is an MCD –Domain ,then A_s is MCD –Domain .

Proof:

Since $S \subseteq \bigcup(A_s)$, A is an MCD –Domain and S is splitting ,it suffices to show that ,if x_1 ,........,x_n is a set of nonzero LCM-prime to S elements of A s.t.

GCD_A (x_1,..........,x_n)=1 .then any LCM-prime to splitting multiplicative system of A which divides each x_i in A_s is a unit of A . But this is a consequence of remark (ii) before .

Proposition (5):

Let A be a domain and $S \subseteq A$ a splitting multiplicative system .If A is an MCD –Domain then so is A +x A_s [x].

Proof:

Since A is an MCD –Domain and S is splitting multiplicative system ,A_s[x] is MCD –Domain (propositions (2),(4) and [4,prposition 1.2]).

Let R =A+ x A_s[x].

Let us show that any finite set $f_1,f_2,\ldots\ldots,f_n$ of nonzero elements of R has MCD in R

Let $f \in R$ be an MCD of $f_1,\ldots,f_n$ in A_s[x] .We may assume that f divides $f_1,\ldots,f_n$ in R. Indeed if f(0)=0 , $\exists$ $s \in S$ s.t. f/s divides $f_1,\ldots\ldots,f_n$ in R . In this case we replace f by f/s . If f(0)≠0 , write f(0) = as with $s \in S$,$a \in A$ and a LCM-prime to S.

Factoring out S ,we may assume that f(0) is LCM-prime to S .

By Remark (iii) ,f divides $f_1,f_2,\ldots,f_n$ in R .

Factoring out of f ,we may assume that each common factor of f_1 ,...,f_n in R lies in S .Then $f_1(0),\ldots,f_n(0)$ are not all zero other wise some x/s with $s \in S$ is a common factor of $f_1,\ldots,f_n$ in R .Let a be an MCD in A of

$f_1(0),...,f_n(0)$ and write a=bs with $s \in S$,$b \in A$ and b LCM –prime to S.

Factoring out S from $f_1,...,f_n$ we may assume that S=1 .Then $f_1(0),...,f_n(0)$ have no non unit common divisor in S . Indeed if $t \in S$ divides $f_1(0),...,f_n(0)$ in A ,then ta divides $f_1(0),..,f_n(0)$ in A, because a is LCM-prime to S ,hence t is a unit in A, because a is an MCD of $f_1(0),...,f_n(0)$. let f be a common divisor of $f_1,...,f_n$ in R .then f(0) divides $f_1(0),..,f_n(0)$ in A and B a preceding reduction f(0) S . hence f(0) is a unit of A , thus f is a unit of R . There for $f_1,f_2,\ldots,f_n$ because relatively prim in R.

Proposition (6):

Let A be domain and $S \subseteq A$ a splitting multiplicative system. If A and $A_s[[x]]$ an are MCD-Domain then so is $A+A_s[[x]]$.

Proof:

Since is an MCD-Domain and S is splitting multiplicative system , $A_s[x]$ is MCD-Domain (proposition (2),(4) and [4, proposition 1.2]).

Let R $A+xA_s[[x]]$

Let us show that any finites set $f_1,\ldots,f_n$ of nonzero elements of R has MCD in R.

Let $f \in R$ be an MCD of $f_1,\ldots,f_n$ in $A_s[[x]]$.We may assume that f divides $f_1,\ldots,f_n$ in R. Indeed if f(0) =0 $\exists$ $s \in S$ s.t. f/s divides $f_1,\ldots,f_n$ in R. In this case we replace f by f/s .If f(0) $\neq$0 ,write f(0) =as with $s \in S$, $a \in A$ and a LCM-prime to S .

Factoring out S , we may assume that f(0) is LCM –prime to S.

By remark (iiii) if divides $f_1,\ldots,f_n$ in R.

Factoring out of f ,we may assume that each common factor of $f_1,...,f_n$ in R has the constant term in S .Then $f_1(0),...,f_n(0)$ are not all zero other wise some x/s with $s \in S$ is a common factor of $f_1,...,f_n$ in R .Let a be an MCD in A of $f_1(0),...,f_n(0)$ and write a=bs with $s \in S$,$b \in A$ and b LCM –prime to S.

Factoring out S from $f_1,...,f_n$ we may assume that S=1 .Then $f_1(0),...,f_n(0)$ have no non unit common divisor in S . Indeed if t S divides $f_1(0),..,f_n(0)$ in A ,then ta divides $f_1(0),..,f_n(0)$ in A, because a is LCM-prime to S ,hence t is a unit in A, because a is an MCD of $f_1(0),...,f_n(0)$. let f be a common divisor of $f_1,...,f_n$ in R .then f(0) divides $f_1(0),..,f_n(0)$ in A and B a preceding reduction $f(0) \in S$. hence f(0) is a unit of A , thus f is a unit of R . There for $f_1,f_2,\ldots,f_n$ because relatively prim in R.

We conjecture that A+XB[X] and A+XB[[X]] are GCD-Domain if and only if S is a splitting multiplicative system ,B= A_s and A is GCD-Domain.

In corollary III and IIII that proof of [2,Theorem 1.1] and [5,Corollary 1.5]

Corollary III:

Let $A \subseteq B$ be an extension domain and $S=U(B) \bigcap A$. then A+XB[X] is a GCD-Domain if and only if S is a splitting multiplicative system ,B= A_s and A is GCD-Domain.

Corollary IIII:

Let $A \subseteq B$ be an extension domain and $S=U(B) \bigcap A$ .Then A+XB[[X]] is a GCD-Domain if and only if S is a splitting multiplicative system ,B= A_s , A and $A_s[[x]]$ are GCD-Domain.

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مجال الـ MCD من النوع A+x B[x] و A+xB[[x]]

ماهرة ربيع قاسم

قسم الرياضيات، كلية التربية، جامعة تكريت، تكريت، جمهورية العراق

الملخص:

في هذا البحث دراسة مجال الـ MCD من النوع A+x B[x] و A+xB[[x]].

لتكن R حلقة أبدالية ذات عنصر محايد و S مجموعة جزئية من R.

$CD_R(S)$هي مجموعة كل القواسم المشتركة غير الصفرية في S .

العنصر m من هذه المجموعة $CD_R(S)$ يسمى بالقاسم المشترك الأعظم (اختصارا MCD لـ S إذا كان m مكافئ لأي عنصر من عناصر $CD_R(S)$ وهو الذي يقسم m. يسمى المجال R بمجال القاسم المشترك الأعظم (MCD) إذا كان لأي مجموعة منتهية غير صفرية من عناصر R قاسم مشترك أعظم(MCD) .

النتيجة III و IIII إعطاء برهان جديد للنظري[١،١، ٢] والنتيجة[5,١,٥] وبطريقة مشابهة بالنسبة لمتسلسلة القوى .