Engineering and Technology Journal 40 (10) (2022) 1318-1324

Engineering and Technology Journal
Journal homepage: https://etj.uotechnology.edu.iq

The Alternative Representation of the Bipolar Sugeno Integral

Jabbar Abbas [a*], **Hossein Jafari** [b]

[a] Department of Applied Sciences, University of Technology-Iraq, Alsina'a Street, 10066 Baghdad, Iraq.
[b] Department of Mathematical Sciences, University of South Africa, UNISA0003, South Africa

*Corresponding author Email: jabbar.a.ghafil@uotechnology.edu.iq

HIGHLIGHTS

- An alternative representation of the bipolar Sugeno integral was proposed to be suitable for bipolar scales.
- Study the main properties of this integral.
- This representation is consistent as a generalization of the expression concerning the classical Sugeno integral.

ARTICLE INFO

Handling editor: Qusay F. Alsalhy

Keywords:
Bi-capacities; Decision support systems; Fuzzy integrals; Bipolar Sugeno integral.

ABSTRACT

In the context of decision support systems, bi-capacities were introduced as an extension of classical capacities. Many bipolar fuzzy integrals related to the bi-capacities have been presented in recent years. One of these integrals is the Sugeno integral concerning aggregation on bipolar scales. The paper aims to build an equivalent representation of the bipolar Sugeno integral. Therefore, we first employ in this paper the framework based on a ternary-criterion set for proposing an alternative formula of the bipolar Sugeno integral to be suitable for bipolar scales. Then, we discuss some basic properties and give an illustrative example of this representation. This representation is consistent as an extension of the representation concerning the classical capacities and aggregation on the Sugeno integral unipolar scales.

Introduction

With the increase of applications of fuzzy integrals in different fields of science and engineering, such as computational intelligence, image processing, biology, face recognition, pattern recognition, economics, multi-criteria decision-making problems, data fusion, etc. (see, e.g. [1-10]), many fuzzy integrals related to capacities (or ``non-additive measures") have been provided by several research works (see, e.g. [11-15]). Among other fuzzy integrals, the Sugeno integral was introduced by Sugeno in 1974 [6] to become one of the important analytical tools of integration theory for measuring uncertain information.

Many generalizations of the fuzzy integrals concerning aggregation on bipolar scales were presented in recent years [16-19]. Furthermore, bipolarity and its potential applications have been studied in many recent works of literature [20-23]. In this respect, alternative formulas for the case of the Choquet integral on bipolar scales have been proposed in [24- 26], so these formulas allow for a generalization of several results around the bipolar Choquet integral.

The paper aims to build an equivalent representation of bipolar Sugeno integral different from the Sugeno integral framework introduced in [18]. Therefore, we first employ the framework based on a ternary-criterion set for proposing an alternative formula of the bipolar Sugeno integral in this paper. Then, we study the main properties of this representation. This representation is consistent as an extension of the representation concerning the classical capacities and aggregation on the Sugeno integral unipolar scales.

The following section recalls some basic concepts that we need in this contribution. In Section 3, we introduce the bi-capacities defined in the new approach. Then, section 4 proposes an alternative representation of bipolar Sugeno integral with illustrated example. In Section 5, we study the main properties of this representation. Finally, in section 6, some conclusions are described.

http://doi.org/10.30684/etj.2022.135012.1256
Received 03 August 2022; Accepted 27 August 2022; Available online 14 September 2022
2412-0758/University of Technology-Iraq, Baghdad, Iraq

Basic Concepts

2.1 Capacities and Sugeno Integral

In this paper, we denote by $(X, Å)$ for a measurable space, where Å is a σ- algebra of subsets of the universal set X. A capacity [11] is an extension of a classical measure and is defined as follows:

Definition 1: [10] Let $(X, Å)$ be a measurable space. A capacity is a function $\mu: Å \rightarrow [0,1]$ that satisfies the requirements:
i) $\mu(X) = 1, \ \mu(\emptyset) = 0,$
ii) if $A \subseteq B \implies \mu(A) \leq \mu(B)$, for all $A, B \in Å$.

Let us denote by Ж to a class of nonnegative real-valued input on measurable space $(X, Å)$. For any $\boldsymbol{x} = (x_1, \dots, x_i, \dots, x_m) \in$ Ж, the Sugeno integral of $\boldsymbol{x}$ related to μ is defined as follows.

Definition 2: [10] To any real-valued input $\boldsymbol{x} \in$ Ж. The Sugeno integral of $\boldsymbol{x}$ with respect to capacity μ is given by

$$(Su) \int (\boldsymbol{x}, \mu) d\mu = \vee_{i \in \boldsymbol{x}} \{ \wedge \{ x_i, \mu(\{ j \in \boldsymbol{x} | x_j \geq x_i \}) \} \} \quad (1)$$

2.2 Bi-capacities

Although capacities can capture a wide range of decision-making applications, they are incompetent in some circumstances, especially when the capacities are defined on bipolar scales. Therefore, in several workable cases, it is normal to employ a scale that goes from bad ``negative'' to good ``positive'' values, which includes the middle neutral amount. This scale is called a bipolar scale because of encodes the bipolarity of the impact, and exemplary examples are R (unbounded cardinal), [−1, 1] (bounded cardinal), or {excellent, good, medium, bad, very bad} (ordinal). For simplicity, we use the [−1, 1] scale in this paper, with a neutral value of zero.

Considering that the independence between the negative and positive partitions does not hold, for this reason, we have to treat the triple alternatives $(1_A, -1_B, 0_{(A \cup B)^c})$, and give each of them a value in [−1, 1]. This value is denoted as $\upsilon_b(A, B)$, that is, a function with two arguments, the first argument being the set of criteria that are completely satisfied, the second being the set of criteria that are not completely satisfactory, and the remaining criteria being on the neutral level. Grabisch and Labreuche [16] gave the following definition and called it bi-capacity.

Suppose $Q(X) := \{(A_1, A_2) \in P(X) \times P(X) | \ A_1 \cap A_2 = \emptyset\}$. Then, for any two disjoint pairs of sets $(A_1, A_2), (B_1, B_2) \in Q(X)$ the binary relation $\sqsubseteq$ is defined by:

$$(A_1, A_2) \sqsubseteq (B_1, B_2) \ iff \ A_1 \subseteq B_1 \ and \ A_2 \supseteq B_2 \quad (2)$$

The Supremum (*Sup*) and Infimum *(Inf)* on the structure $(Q(X), \sqsubseteq)$ are denoted by $\sqcup, \ \sqcap$, respectively. The *Sup* is given by

$$(A_1, A_2) \sqcup (B_1, B_2) = (A_1 \cup B_1, A_2 \cap B_2) \quad (3)$$

Inf is given by

$$(A_1, A_2) \sqcap (B_1, B_2) = (A_1 \cap B_1, A_2 \cup B_2) \quad (4)$$

Hence, this ordered set becomes a lattice with the top being $(X, \emptyset)$ and the bottom being $(\emptyset, X)$, (for more details, see [16]). The structure $Q(X)$ has another order relation introduced by Bilbao et al. [27]. This order relation is defined as follows:

$$(A_1, A_2) \sqsubseteq (B_1, B_2) \ iff \ A_1 \subseteq B_1 and \ A_2 \subseteq B_2 \quad (5)$$

Inf is given by

$$(A_1, A_2) \sqcap (B_1, B_2) = (A_1 \cap B_1, A_2 \cap B_2), \quad (6)$$

and the *Sup* does not exist. Thus, this order relation on the structure $Q(X)$ is an Inf-semilattice with the bottom being $(\emptyset, \emptyset)$. (see, [16], [27] for more details).

Definition 3: [16] A set function $\upsilon: Q(X) \rightarrow [-1,1]$ is called a bi-capacity on X if satisfies the following requirements:

i. $\upsilon(\emptyset, \emptyset) = 0, \ \upsilon(N, \emptyset) = 1 \ and \ \upsilon(\emptyset, N) = -1$

ii. $\forall \ (A_1, A_2), (B_1, B_2) \in Q(X), \ (A_1, A_2) \sqsubseteq (B_1, B_2) \implies \upsilon(A_1, A_2) \leq \upsilon(B_1, B_2)$.

2.3 Bipolar Sugeno Integral Based on Bi-capacities With Order ⊑

The bipolar maximum of $\boldsymbol{x} = (\boldsymbol{x_1}, \dots, \boldsymbol{x_i}, \dots, \boldsymbol{x_m}) \in [-\mathbf{1}, \mathbf{1}]^{\boldsymbol{n}}$, denoted by

$$\bigvee_{i\in x}^{b} \boldsymbol{x} = (\vee_{i=1}^{m} \boldsymbol{x_i}) \curlyvee (\wedge_{i=1}^{m} \boldsymbol{x_i}) \quad (7)$$

Note that the operator $\curlyvee : [-\mathbf{1}, \mathbf{1}]^{\mathbf{2}} \to [-\mathbf{1}, \mathbf{1}]$ is the symmetric maximum introduced in [28]. Thus, the bipolar Sugeno integral based on bi-capacities with order ⊑ is defined as follows.

Definition 4: [18] Given a vector $\boldsymbol{x} = (\boldsymbol{x_1}, \dots, \boldsymbol{x_i}, \dots, \boldsymbol{x_m}) \in [-\mathbf{1}, \mathbf{1}]^{\boldsymbol{n}}$, the bipolar Sugeno integral of $\boldsymbol{x}$ related to the bi-capacity $\boldsymbol{v}$ on $\boldsymbol{x}$ is defined by

$$BSu_b \int (x, v)\, dv = \vee_{i\in x}^{b} \{sign(v(\{ j \in x : x_j \geq |x_i|\}, \{ j \in x : x_j \leq -|x_i|\})) . \wedge \{ v(\{ j \in x : x_j \geq |x_i|\}, \{ j \in x : x_j \leq -|x_i|\})|, |x_i|\}\}. \quad (8)$$

$$BSu_b \int (x, v)\, dv = \bigvee_{i\in x}^{b} \{BSu_{b^+} \int (x, v)\, dv,\ BSu_{b^-} \int (x, v)\, dv\}$$

Where, $BSu_{b^+} \int (x, v)\, dv$ is the right bipolar Sugeno integral is defined by

$$BSu_{b^+} \int (x, v)\, dv = \vee_{i\in x}^{b+} \{sign(v(\{ j \in \boldsymbol{x} : x_j \geq |x_i|\}, \{ j \in \boldsymbol{x} : x_j \leq -|x_i|). \wedge \{ v(\{ j \in x : x_j \geq |x_i|\}, \{ j \in x : x_j \leq -|x_i|\})|, |x_i|\}\}. \quad (9)$$

and $BSu_{b^-} \int (x, v)\, dv$ is the left bipolar Sugeno integral is defined by

$$BSu_{b^-} \int (x, v)\, dv = \vee_{i\in x}^{b-} \{sign(v(\{ j \in x : x_j \geq |x_i|, \{ j \in x : x_j \leq -|x_i|\})) . \wedge \{ v(\{ j \in x : x_j \geq |x_i|\}, \{ j \in x : x_j \leq -|x_i|\})|, |x_i|\}\}. \quad (10)$$

Bi-capacities Defined on The Approach of Ternary-Criterion Sets

In this section, we begin by recalling the basic concepts of the ternary-criterion set and the equivalent definition of bi-capacities (for more details, see [24- 26]).

We consider every criterion $i \in X$ that has either a positive impact, a negative, or no impact. So that we symbolize the criterion $\boldsymbol{i}$ as $\boldsymbol{i}^{+}$whenever $\boldsymbol{i}$ is positively significant, as i^{-} whenever i is negatively significant and as $i^{\emptyset}$ whenever $\boldsymbol{i}$ is neutral, and we call this criterion a ternary-criterion. For all i, $i \in \{1,2 \cdots, n\}$, the ternary-criterion set is the set that contains only out of $\{i^{\emptyset}, i^{-}, and\ \ i^{+}\}$.

Hence, in this approach, we denote by $T(X) = \{\{\tau_1, \dots, \tau_2\} \mid \forall \tau_i \in \{i^{+}, i^{-}, i^{\emptyset}\},\ i = 1, \dots, n\}$ for the set of all possible combinations of ternary criterion set of $\boldsymbol{n}$ elements.

We have $T\ (X)$ can be identified with $\{1, 0, -1\}^n$, hence $|T\ (X)| = 3^n$. Also, simply remarked that for any ternary criterion set $A \in T\ (X)$, A is an alternative to a ternary vector $(\tau_1, \dots, \tau_n)$ with $\tau_i = 1\ if\ i^{+} \in A$, $\tau_i = 0\, if\ i^{\emptyset} \in A, and\ \tau_i = -1\ \ if\ i^{-} \in A$, $\forall\ \ i = 1,2 \cdots, n$.

Definition 5: For any two sets A, $B \in T\ (X)$. The order relation ⊑ between ternary-criterion sets A, $B \in T\ (X)$, Then $A \sqsubseteq B \Leftrightarrow \forall\, i \in X$,

$$``if\ i^{\emptyset} \in A\ implies\ i^{+}\ or\ i^{\emptyset} \in B",\ and\ ``if\ i^{+}\ \ A\ implies\ i^{+}\ \ B" \quad (11)$$

The next definition is an alternative definition of bi-capacities defined on the approach of ternary-criterion sets.

Definition 6: A mapping $v : T\ (X) \to [-1, 1]$ is called bi-capacity based on the ter-criterion sets if it satisfies the following requirements:

1) $v(X^{-}) = v(\{1^{-}, 2^{-} \cdots, n^{-}\}) = -1$, $v(X^{+}) = v(\{1^{+}, 2^{+} \cdots, n^{+}\}) = 1$, $and\ \ v(X^{\emptyset}) = v(\{1^{\emptyset}, 2^{\emptyset} \cdots, n^{\emptyset}\}) = 0$.

2) $A \sqsubseteq B\ \ implies\ \ v(A) \leq v(B)$, $\forall\, A$, $B \in T(X)$.

Bi-capacities are functions defined on the structure $T\ (X)$. Hence, we can introduce another order relation on the structure $T\ (X)$, we denote by ⊆, which is an alternative to the order relation in a bi-cooperative game [27].

Definition 7: For any two sets $A,\ B \in T\ (X)$. The order relation ⊑ between ternary-criterion sets $A,\ B \in T\ (X)$, Then, $A \subseteq B \ \Leftrightarrow \ \forall\, i \in X$,

$$``if\ i^- \in A\ implies\ i^- \in B'' \text{ and } ``if\ i^+ \in A\ implies\ i^+ \in B'' \tag{12}$$

The Alternative Representation of Bipolar Sugeno Integral

Let $\boldsymbol{X}$ be a non empty finite set, the binary operators ∧, ∨ on [0,1] is defined as follows: For any $s, t \in [0,1]$, $s \wedge t := \min\{s, t\}$ and $s \vee t := \max\{s, t\}$. According to [28, 18], the symmetric minimum ⋏ and the symmetric maximum ⋎ are operations have been introduced as follows:

For any $s, t \in [-1,1]$, $\boldsymbol{s} \curlywedge \boldsymbol{t} = sign(s + t) \cdot (|s| \wedge |t|)$ and $\boldsymbol{s} \curlyvee \boldsymbol{t} = sign(s \cdot t) \cdot (|s| \vee |t|)$,

where $sign(r) = -1$ if $r < 0$, $= 0$ if $r = 0$, and $= 1$ if $r > 0$.

Moreover, for any subset I of the interval $[-1,1]$.

$$\curlyvee_{s_i \in I}\ s_i = \bigvee_{s_i \geq 0}\ s_i \ \curlyvee \ \bigwedge_{s_i \leq 0}\ s_i \tag{13}$$

Here, we propose an alternative representation of bipolar Sugeno integral related to bi-capacity defined on the ternary-criterion set.

For an input vector $\boldsymbol{x} = (x_{\tau_1}, \dots, x_{\tau_i}, \dots, x_{\tau_n});\ x_{\tau_i} \in R,\ i \in \{1,2 \cdots, n\}$. We denote a ternary-criterion set $X^* = \{\tau_1, \cdots, \tau_n\}$ with $\tau_i = i^+\ if\ x_i > 0;\ \tau_i = i^-\ if\ x_i < 0;\ and\ \tau_i = i^{\emptyset}\ if\ x_i = 0,\ \forall\, i = 1, \dots, n$.

Therefore, we can define the alternative representation of bipolar Sugeno integral of real input $\boldsymbol{x}$ related to bi-capacity as follows.

Definition 8: Let $T\ (X)$ be the set of all ternary-criterion sets and $v:\ T\ (X) \to [-1, 1]$ be a bi-capacity defined on a ternary-criterion set. Then, the alternative representation of bipolar Sugeno integral of $\boldsymbol{x}$ related to $\boldsymbol{v}$ is defined by

$$(BSu) \int (x, v)\, dv = \curlyvee_{i=1}^{n} \left[|x_{\sigma(\tau_i)}| \ \curlywedge \ v\left(A_{\sigma(\tau_i)}\right) \right], \quad \tau_i \ \in \left\{ i^-, i^+, i^{\emptyset} \right\} \tag{14}$$

where σ is a permutation on X^* so that $|x_{\sigma(\tau_i)}| \geq \cdots \geq |x_{\sigma(\tau_n)}|$, and $A_{\sigma(\tau_i)} = \left\{\sigma(\tau_1), \dots, \sigma(\tau_i), \sigma\left((i+1)^{\emptyset}\right), \sigma\left((i+2)^{\emptyset}\right), \dots \right\}$ is ternary-criterion set $\subseteq X^*$.

For the sake of clarity, we give an illustrative example of the alternative representation of the bipolar Sugeno integral.

Example 1: Let us consider $S = \{1,2,3\}$, and we define the bi-capacity values $v : T\ (X) \to [-1, 1]$ as shown in Table 1. The function $\boldsymbol{x}$ on $\boldsymbol{S}$ defined by $\boldsymbol{x} = (0.1, -0.7, 0.4)$. That is, $x_{\tau_1} = 0.1,\ x_{\tau_2} = -0.7, x_{\tau_3} = 0.4$.

Then, the ternary criterion set corresponding to $\boldsymbol{x}$ is $X^* = \{1^+, 2^-, 3^+\}$.

Using definition 4, $|x_{\sigma(\tau_1)}| = 0.7,\ |x_{\sigma(\tau_2)}| = 0.4,\ |x_{\sigma(\tau_3)}| = 0.1$,

$$(BSu) \int (\boldsymbol{x}, v)\, dv = \curlyvee_{i=1}^{n} \left[|x_{\sigma(\tau_i)}| \ \curlywedge \ v\left(A_{\sigma(\tau_i)}\right) \right]$$

We obtain,

$$(BSu) \int (\boldsymbol{x}, v)\, dv = [0.7 \curlywedge v(\{1^{\emptyset}, 2^-, 3^{\emptyset}\})] \curlyvee [0.4 \curlywedge v(\{1^{\emptyset}, 2^-, 3^+\})] \curlyvee [0.1 \curlywedge v(\{1^+, 2^-, 3^+\})]$$
$$(BSu) \int (\boldsymbol{x}, v)\, dv = [0.7 \curlywedge -0.6] \curlyvee [0.4 \curlywedge 0.5] \curlyvee [0.1 \curlywedge 0.8]$$
$$(BSu) \int (\boldsymbol{x}, v)\, dv = [-0.6] \curlyvee [0.4\,] \curlyvee [0.1]$$
$$(BSu) \int (\boldsymbol{x}, v)\, dv = -0.6.$$

Table 1: Bi-capacity values

Ternary-criterion sets	$\{\mathbf{1}^{\emptyset},\mathbf{2}^{\emptyset},\mathbf{3}^{\emptyset}\}$	$\{\mathbf{1}^{+},\mathbf{2}^{\emptyset},\mathbf{3}^{\emptyset}\}$	$\{\mathbf{1}^{\emptyset},\mathbf{2}^{+},\mathbf{3}^{\emptyset}\}$
Bicapacity values	0	0.6	0.4
Ternary-criterion sets	$\{\mathbf{1}^{\emptyset},\mathbf{2}^{-},\mathbf{3}^{-}\}$	$\{\mathbf{1}^{-},\mathbf{2}^{-},\mathbf{3}^{\emptyset}\}$	$\{\mathbf{1}^{-},\mathbf{2}^{\emptyset},\mathbf{3}^{-}\}$
Bicapacity values	-0.7	-0.8	-0.8
Ternary-criterion sets	$\{\mathbf{1}^{\emptyset},\mathbf{2}^{\emptyset},\mathbf{3}^{+}\}$	$\{\mathbf{1}^{+},\mathbf{2}^{+},\mathbf{3}^{\emptyset}\}$	$\{\mathbf{1}^{+},\mathbf{2}^{\emptyset},\mathbf{3}^{+}\}$
Bicapacity values	0.6	0.7	0.8
Ternary-criterion sets	$\{\mathbf{1}^{\emptyset},\mathbf{2}^{+},\mathbf{3}^{+}\}$	$\{\mathbf{1}^{-},\mathbf{2}^{\emptyset},\mathbf{3}^{\emptyset}\}$	$\{\mathbf{1}^{\emptyset},\mathbf{2}^{-},\mathbf{3}^{\emptyset}\}$
Bicapacity values	0.7	-0.6	-0.6
Ternary-criterion sets	$\{\mathbf{1}^{\emptyset},\mathbf{2}^{\emptyset},\mathbf{3}^{-}\}$	$\{\mathbf{1}^{+},\mathbf{2}^{-},\mathbf{3}^{\emptyset}\}$	$\{\mathbf{1}^{-},\mathbf{2}^{+},\mathbf{3}^{\emptyset}\}$
Bicapacity values	-0.4	0.3	0.2
Ternary-criterion sets	$\{\mathbf{1}^{\emptyset},\mathbf{2}^{-},\mathbf{3}^{+}\}$	$\{\mathbf{1}^{\emptyset},\mathbf{2}^{+},\mathbf{3}^{-}\}$	$\{\mathbf{1}^{+},\mathbf{2}^{\emptyset},\mathbf{3}^{-}\}$
Bicapacity values	0.5	-0.2	-0.3
Ternary-criterion sets	$\{\mathbf{1}^{-},\mathbf{2}^{\emptyset},\mathbf{3}^{+}\}$	$\{\mathbf{1}^{+},\mathbf{2}^{+},\mathbf{3}^{-}\}$	$\{\mathbf{1}^{+},\mathbf{2}^{-},\mathbf{3}^{+}\}$
Bicapacity values	0.2	0.8	0.5
Ternary-criterion sets	$\{\mathbf{1}^{-},\mathbf{2}^{+},\mathbf{3}^{-}\}$	$\{\mathbf{1}^{-},\mathbf{2}^{+},\mathbf{3}^{+}\}$	$\{\mathbf{1}^{-},\mathbf{2}^{-},\mathbf{3}^{+}\}$
Bicapacity values	-0.4	0.4	-0.5
Ternary-criterion sets	$\{\mathbf{1}^{+},\mathbf{2}^{-},\mathbf{3}^{-}\}$	$\{\mathbf{1}^{-},\mathbf{2}^{-},\mathbf{3}^{-}\}$	$\{\mathbf{1}^{+},\mathbf{2}^{+},\mathbf{3}^{+}\}$
Bicapacity values	-0.4	-1	1

Basic Properties of The Alternative Representation of Bipolar Sugeno Integral

Here, we introduce the basic properties satisfied by the alternative representation of bipolar Sugeno integral concerning the bi-capacity.

Proposition 1: Consider a bi-capacity (v) defined on $T\ (X)$, then

$$(BSu)\int\big((1_A\,,-1_A,0_A),v\big)\ dv\ =\ v(A),\qquad \forall\ A\ \in\ T\ (X).$$

Proof: For any ternary input $(1_A\,,-1_A,0_A)$, $|x_{\sigma(\tau_i)}|\ =1\ or\ |x_{\sigma(\tau_i)}|\ =0,\ \forall\ \tau_i\in\{\,i^-,i^+,i^{\emptyset}\}$ and $v\big(\{A_{\sigma(\tau_i)}\}\big)\ =\ v(1_A\,,-1_A,0_A)\ =\ v(A)$. Thus, from the definition of bipolar Sugeno integral concerning the bi-capacity (Eq. (14)), we have

$$(BSu)\int\big((1_A\,,-1_A,0_A),v\big)\ dv\ =\bigvee_{i=1}^{n}\big[|x_{\sigma(\tau_i)}|\ \wedge\ v\big(A_{\sigma(\tau_i)}\big)\big]$$

$$(BSu)\int\big((1_A\,,-1_A,0_A),v\big)\ dv\ =\ v(A),\ \ \forall\ A\ \in\ T\ (X)$$

The following proposition presents that the bipolar Sugeno integral concerning the bi-capacity satisfies the monotonic characteristic.

Proposition 2: Consider a bi-capacity (v) defined on $T\ (X)$, and $\forall\,\boldsymbol{x},\ \boldsymbol{x}'\in\ R,\,if\ x_{\tau_i}\ \geq\ x'_{\tau_i}\ \ \forall\,\tau_i\in\big\{\,i^-,i^+,i^{\emptyset}\big\}$, then

$$(BSu)\int(\boldsymbol{x},v)dv\ \geq\ (BSu)\int(\boldsymbol{x}',v)dv.$$

Proof: First, we suppose that for any $\tau_i\in\big\{i^+,i^-,i^{\emptyset}\big\}$, $x_{\tau_i}\ >\ x'_{\tau_i}$ and $\forall\,k\ \in\{\,1,\dots,i-1,i+1\dots,n\}$, $x_{\tau_k}\ =\ x'_{\tau_k}$.

And, we suppose that for all elements of X, the order of each element is the same, i.e., $|x_{\sigma(\tau_i)}|\ \geq\cdots\geq|x_{\sigma(\tau_n)}|$ and $|x'_{\sigma(\tau_i)}|\ \geq\cdots\geq|x'_{\sigma(\tau_n)}|$.

In the beginning, we demonstrate the monotonicity of this case.

According to the alternative representation of bipolar Sugeno integral (Eq. (14)), we have

$$(BSu)\int(\boldsymbol{x},v)dv\ =\bigvee_{i=1}^{n}\big[|x_{\sigma(\tau_i)}|\ \wedge\ v\big(A_{\sigma(\tau_i)}\big)\big]\tag{15}$$

also,

$$(BSu)\int(\boldsymbol{x}',v)dv\ =\bigvee_{i=1}^{n}\big[|x'_{\sigma(\tau_i)}|\ \wedge\ v\big(A_{\sigma(\tau_i)}\big)\big]\tag{16}$$

$A_{\sigma(\tau_i)}\ and\ A_{\sigma(\tau_{i-1})}$ are the ternary-criterion sets with $A_{\sigma(\tau_{i-1})}\ \subseteq\ A_{\sigma(\tau_i)}$.

Hence, $v(A_{\sigma(\tau_i)})-v(A_{\sigma(\tau_{i-1})})\geq\ 0.$

Now, since $x_{\tau_i}\ \geq\ x'_{\tau_i}\ \forall\,\tau_i\in\big\{i^+,i^-,i^{\emptyset}\big\}$, it is clear that

$(BSu)\int(\boldsymbol{x},v)\,dv\ \geq\ (BSu)\int(\boldsymbol{x}',v)dv.$

So that, if $x_{\tau_i}\ >\ x'_{\tau_i}$ then $(BSu)\int(\boldsymbol{x},v)dv\ \geq\ (BSu)\int(\boldsymbol{x}',v)\,dv$ is proved inside the scope that the order of each element of $\boldsymbol{x}$ and $\boldsymbol{x}'$ does not change.

Therefore, by repeating the above process 2 times at the point of the replacement of the order,

if $x_{\tau_i}\ >\ x'_{\tau_i}$ then, the following can be proved even in the range with the order replacement.

$$(BSu)\int(\boldsymbol{x},v)dv \geq (BSu)\int(\boldsymbol{x}',v)dv$$

Thus, by applying this procedure successively for each i, the monotonic property can be proved.

Conclusion

In this contribution, we have proposed an alternative representation of bipolar Sugeno integrals related to bi-capacities. Then, we studied the basic properties of this representation. According to the definitions of bi-capacities and their integrals through the concept of ternary-criterion sets, we can see these definitions are significant because associating polarity with each criterion is easier than associating polarity with a set of criteria. Therefore, we look forward to future work on these definitions to allow an easy way to introduce new integrals related to the bi-capacities.

Author contribution

The contribution in the current work was equally made by all authors.

Funding

No grant is provided for the present work and given by any funding organization.

Data availability statement

Upon any demand made by the corresponding author, the data which supports the conclusions of the current work can be made available.

Conflicts of interest

There is no conflict of interest in the current work.

References

[1] D. Candeloro, R. Mesiar,A.R. Sambucini, A special class of fuzzy measures: Choquet integral and applications, Fuzzy Sets Syst., 355(2019) 83–99. https://doi.org/10.1016/j.fss.2018.04.008

[2] G. Lucca, J. A. Sanz, G. P. Dimuro, E. N. Borges, H. Santos ,H. Bustince, Analyzing the performance of different fuzzy measures with generalizations of the Choquet integral in classification problems, 2019 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), New Orleans, LA, USA, 2019, 1-6. https://doi.org/10.1109/FUZZ-IEEE.2019.8858815

[3] M. Hesham , J. Abbas, Multi-criteria decision making on the optimal drug for rheumatoid arthritis disease, Iraqi J. Sci., 62 (2021) 1659–1665.. https://doi.org/10.24996/ijs.2021.62.5.28

[4] P. Karczmarek, A. Kiersztyn , W. Pedrycz, Generalized Choquet integral for face recognition, Int. J. Fuzzy Syst., 20 (2018)1047-1055. https://doi.org/10.1007/s40815-017-0355-5

[5] A. Naeem, J. Abbas, A Computational Model for Multi-Criteria Decision Making in Traffic Jam Problem, J. Autom. Mob. Robot. Intell. Syst., 15 (2022) 39-43. https://doi.org/10.14313/JAMRIS/2-2021/12

[6] Sugeno, M. Theory of fuzzy integrals and its applications, Ph.D. Thesis, Tokyo Institute of Technology, 1974.

[7] J.C. S. Bueno ,C. A. Dias , G.P. Dimuro, H. Santos , E. N. Borges ,G. Lucca ,H. Bustince, Aggregation functions based on the Choquet integral applied to image resizing, 11th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT) ,1, 2019,460–466. https://doi.org/10.2991/eusflat-19.2019.64

[8] S. Dey, R. Bhattacharya, S. Malakar, S. Mirjalili, R.Sarkar, Choquet fuzzy integral-based classifier ensemble technique for COVID-19 detection, Comput. Biol. Med., 135 (2021) 104585. https://doi.org/10.1016/j.compbiomed.2021.104585

[9] A. Banerjee, P. K. Singh, R. Sarkar, Fuzzy integral based CNN classifier fusion for 3D skeleton action recognition, in IEEE Trans. Circuits Syst. Video Technol., 31 (2021) 2206-2216. https://doi.org/10.1109/TCSVT.2020.3019293

[10] M. Grabisch, T. Murofushi,, Sugeno, M., Fuzzy Measures and Integrals. Theory and Applications, Physica Verlag, Berlin Heidelberg, 2000.

[11] G. Choquet, Theory of capacities, Ann. Inst. Fourier, 5(1953) 131-295. https://doi.org/10.5802/aif.53

[12] J. Abbas, Shilkret Integral Based on Binary-Element Sets and its Application in The Area of Synthetic Evaluation, Eng. Technol. J., 33 (B) (2015) 571-577.

[13] I. Chitescu, the Sugeno integral. A point of view, Inf. Sci., 582 (2022) 648-664. https://doi.org/10.1016/j.ins.2021.10.034

[14] J. A. Ghafil, Logical twofold integral, Eng. Technol. J., 28 (2010) 477-483.

[15] B.Mihailović , E. Pap , M. Štrboja , A. Simićević , A unified approach to the monotone integral-based premium principles under the CPT theory, Fuzzy Sets Syst., 398 (2020) 78-97. https://doi.org/10.1016/j.fss.2020.02.006

[16] M. Grabisch, C. Labreuche, Bi-capacities—I: definition, Möbius transform and interaction, Fuzzy Sets Syst., 151 (2005) 211-236. https://doi.org/10.1016/j.fss.2004.08.012

[17] M. Grabisch, C. Labreuche, Bi-capacities II: The Choquet integral, Fuzzy Sets Syst., 151(2005) 237-259. https://doi.org/10.1016/j.fss.2004.08.013

[18] S. Greco, F. Rindone, Bipolar fuzzy integrals, Fuzzy Sets Syst., 220 (2013) 21-33. http://dx.doi.org/10.1016/j.fss.2012.11.021

[19] F. Kareem, J. Abbas ,A Generalization of the Concave Integral in Terms of Decomposition of the Integrated Function for Bipolar Scales, J. Appl. Sci. Nanotechnol., 1 (2021) 81–90. http://dx.doi.org/10.53293/jasn.2021.3985.1065

[20] S M. Grabisch, S. Greco, M.Pirlot, Bipolar and bivariate models in multi-criteria decision analysis: descriptive and constructive approaches, Int. J. Intell. Syst., 23 (2008) 930-969. https://doi.org/10.1002/int.20301

[21] S. Greco, F. Rindone, The bipolar Choquet integral representation, Springer, 77 (2014) 1-29. https://doi.org/10.1007/s11238-013-9390-3

[22] J. Abbas, The Banzhaf interaction index for bi-cooperative games, Int. J. Gen. Syst., 50 (2021) 486-500. https://doi.org/10.1080/03081079.2021.1924166

[23] M. Štrboja, E. Pap, B. Mihailović, Discrete bipolar pseudo-integrals, Inf. Sci., 468 (2018) 72–88. https://doi.org/10.1016/j.ins.2018.07.075

[24] J. Abbas, the Bipolar Choquet Integrals Based On Ternary-Element Sets. jaiscr., 6(2016)13-21. https://doi.org/10.1515/jaiscr-2016-0002

[25] J. Abbas, A New Framework of bi-capacities and its Integrals, J. Al-Nahrain Univ. Sci., 18 (2015) 132-137. https://doi.org/10.22401/JNUS.18.4.18

[26] J. Abbas, the Balancing Bipolar Choquet Integrals, Int. J. Innov. Comput. Inf. Control, 17 (2021) 949–957. https://doi.org/10.24507/ijicic.17.03.949

[27] Bilbao, J.M., Fernández, J.R., Jiménez, N., López, J.J. 2008. A survey of bicooperative games, In: Chinchuluun, A., Pardalos, P.M., Migdalas, A., Pitsoulis, L. (eds) Pareto Optimality, Game Theory And Equilibria. Springer Optimization and Its Applications, vol 17, pp 187–216. Springer, New York, NY. https://doi.org/10.1007/978-0-387-77247-9_8

[28] M. Grabisch, The symmetric Sugeno integral, Fuzzy Sets Syst., 139 (2003) 473–490. https://doi.org/10.1016/S0165-0114(02)00499-2