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NUMERICAL SOLUTION FOR ADVECTION-DIFFUSION EQUATION USING COLLOCATION BASED NEW RADIAL BASIS FUNCTION

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ABSTARCT

In the present paper, a composite radial basis function was developed by a combination between thin plate and logarithmic radial basis functions. This radial basis function was developed to overcome the singularity appeared when the source and field points coincide. In based new radial function, the thin plate function was used when evaluating diagonal elements of the matrix coefficients and the logarithmic function was used for evaluating the other elements. The advantage of using this new radial basis function represented by overcoming singularity from the diagonal elements when thin plate radialbasis function is used. The new function is a combination of both multiquadric and thin plate radial basis functions. One- and two-dimensional test examples were solved and the present results were compared with the analytical results and gave a good agreement in different applicant in applied mechanics and neural network .

Keywords: Collocation techniques, thin plate radial basis function, logarithmic radial basis function, advection-diffusion equation.

الحل العددي لمعادلة الطاقة باستخدام دالة الأساس الشعاعي المركب

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الخلاصة:

في هذا البحث تم تطوير دالة الاساس الشعاعي المركب بواسطة الربط بين صفيحة رقيقة ودوال الاساس اللوغاريتمي الشعاعي وهذه الدالة طورت لغرض التخلص من التفرد لنقاط محددة تظهر عند اتحاد المصدر والمجال . و على اساس الدالة الشعاعية الجديدة فقد استخدمت دالة الصفيحة الرقيقة لتقييم العناصر القطرية ضمن معاملات المصفوفة ، بينما استخدمت الدالة اللوغاريتمية لتقييم العناصر الاخرى .ان الدالة الجديدة هي اتحاد لكل من التقسيمات المتعددة والدالة الاساسية الشعاعية للصفيحة الرقيقة ، وان النتائج التي ظهرت من حلول امثلة احادية وثنائية الابعاد تم مقارنتها مع النتائج العددية واعطت مطابقة جيدة لمختلف التطبيقات لكل من كل التطبيقات الميكانيكية والشبكات العصبية

NOMENCLATURE

$u(\mathrm{x},t)$:	Temperature at position x and time t
x :	Vector position, $x=(x_1,x_2,x_3,.....,x_d)$
d :	Dimension of the problem
∇ :	Gradient differential operator
Ω :	Problem domain
$\partial\Omega$:	Boundary of the domain
k :	Diffusion coefficient
v :	Vector contains advection coefficients
$\varphi_1(r_{ij})$:	Thin plate radial basis function
$\varphi_2(r_{ij})$:	Logarithmic basis function
ε :	Control parameter
r_{ij} :	Distance between points i and j
$\mathrm{P}_e=(v/k)$:	Peclet number
$c_{1,2}$:	convection coefficient
λ :	Regularization parameter
$f(\mathrm{x,t})$	Function of position of x and t
$\mathrm{u}_\circ$ (t)	initial temperature of t
CRBF :	Composite Radial Basis Function

INTRODUCTION

A significant progress of the numerical methods has been achieved in the recent years. For long time the researchers recognized problems when using a mesh-based method such as finite element method **[**Chongjiang 2000**-**Lee JK1986**]**. One way for their efforts to overcome these problems was the automatic mesh generation. Automatic mesh generation is difficult to be fully generated in a wide range of engineering applications. The second way to overcome such problems was developing the meshless methods **[**Zhu T1999**]**. The initial idea of meshless methods dates back to the smooth particle hydrodynamics method for modeling astro-physical phenomena **[**Lucy LB1977**]**. Several domain type mesh free methods such as element free Galerkin method **[**Belytschko T 1994**]**, reproducing kernel particle method **[**Liu GR 1995**]**, the point interpolation method **[**Liu GR2001**]** and the meshless Petrov-Galerkin method **[**Allure SN1998**]** have been proposed and achieved remarkable progress in solving a wide range of static and dynamic problems for solid and structures. Advection-diffusion equation is one of the most important partial differential equations and observed in a wide range of engineering and industrial applications [Singh2000]. It has been used to describe heat transfer in a draining film [Isenberg J1972], water transfer in soil [Parlarge 1980], dispersion of tracers in porous media [Fattah 1985], contaminant dispersion in shallow lakes [Salmon1980], the spread of solute in a liquid flowing through a tube, long-range transport of pollutants in the atmosphere [Zlatev1984] and dispersion of dissolved salts in groundwater **[**Guvanasen1983**]**. Accurate numerical solution of the advection-diffusion equation is usually characterized by a dimensionless parameter, called Peclect number. These results become increasingly difficult as the Peclect number increases due to onset of spurious oscillations or excessive numerical damping if finite difference **[**Donea1984**]** or finite element formulations are used [Yu1986]. [S.G.Ahmed2006]a new radial basis functions were developed and based on a combination between multiquadric and logarithmic radial basis functions, while in [S.G.Ahmed2009**]** a new radial basis function with collocation and cartesian grid method was developed to solve partial differential equations. In the present paper, a composite radial basis function is developed by a combination between thin plate and logarithmic radial basis functions because this function appears to be more efficient ,is symmetric one. In this function the thin plate is used when evaluating diagonal elements of the matrix coefficients and the logarithmic function is used for evaluating the other elements. One- and two-dimensional test examples were solved and the present results were compared with the analytical results and gave a good agreement.

COLLOCATION BASED ON THE NEW COMPOSITE RADIAL BASIS FUNCTION

The New Radial Basis Function

In the collocation techniques, the approximate solution is applied at the scattered nodes over the domain of interest. After collocations, two different $(N \times N)$ matrices are obtained. When using logarithmic radial basis function, singularity appears in the diagonal elements of the matrices. The results were very good compared with other methods such as finite element method. A problem had been encountered with that function, was the choice of the shape parameter in the multiquadirc radial basis function. To overcome such difficulty, herein , the mutliquadric is replaced by the thin plate spline radial basis function and the new function is applied to the advection-diffusion problem in one-, and two-dimensions. The new radial basis function herein, named as composite radial basis function (**CRBF**) is assumed as follows [Boztosum2002]:

$$\Phi(r_{ij}) = (1-\varepsilon)\varphi_1(r_{ij}) + \varepsilon\varphi_2(r_{ij}) \quad (1)$$

$$\varphi_1(r_{ij}) = \text{TPS} = r_{ij}^{2m} \quad (2)$$

$$\varphi_2(r_{ij}) = \text{LBF} = r_{ij}^{2m}\log(r_{ij}) \quad (3)$$

2.2 Mathematical formulation

Consider the advection-diffusion equation [Boztosum2002]:

$$\frac{\partial u(\text{x},t)}{\partial \text{t}} = k\nabla^2(\text{x},t) + v.\nabla(\text{x},t) \quad (4)$$

with the following associated boundary and initial conditions:

$$c_1(x,t) + c_2(x,t) = f(x,t), \quad x \in \partial\Omega\ , t > 0 \quad (5)$$

$$u(x,t) = u_0(t) \quad (6)$$

The physical parameter which describes the relation between the diffusion coefficient k and the advection coefficient ν is the Peclet number defined as $\text{P}_e = (v/k)$. According to the values of P_e, equation (4) behaves as parabolic differential equation for small values of P_e and for large values of P_e it behaves as hyperbolic differential equation **[**Fornefett2001**].** The discretization of equations (4-6) starts with the approximation of time derivatives using forward difference and for the spatial derivatives, the Θ-weighted Crank- Nicholson scheme is used **[**Wright2006**].** Therefore, equations (4) and (5) will take the following form**[**Wright2006]:

$$\begin{aligned} & u(x,t+\Delta t)-k\Theta\Delta t\nabla^2 u(x,t+\Delta t)-\Delta t\Theta v\nabla u(x,t+\Delta t) \\ & = u(x,t)+\Delta t(1-\Theta)k\nabla^2 u(x,t)+\Delta t(1-\Theta)v.\nabla u(x,t) \end{aligned} \quad (7)$$

Re-arrange equation (7) in terms of time notation, leads to:

$$\begin{aligned} & \left(u(x,t)-k\Theta\Delta t\nabla^2 u(x,t)-\Delta t\Theta v\nabla u(x,t)\right)^{n+1} \\ & = \left(u(x,t)+\Delta t(1-\Theta)k\nabla^2 u(x,t)+\Delta t(1-\Theta)v.\nabla u(x,t)\right)^n \end{aligned} \quad (8)$$

The notation u^{n+1} stands for the potential at time $(t^n+\Delta t)$, Δt is the time step size. To simplify dealing with equation (8), let us introduce the following simplifications into consideration:

$$\begin{aligned} \alpha &= -k\Theta\Delta t \\ \beta &= -\Delta t\Theta v \\ \varsigma &= \Delta t(1-\Theta)k \\ \xi &= \Delta t(1-\Theta)v \end{aligned} \quad (9)$$

By making use of equation (9) into equation (8), the later takes the following simplified form:

$$[(1+\alpha\nabla^2+\beta\nabla)\,u(x,t)]^{n+1} = [(1+\zeta\nabla^2+\zeta\nabla)u(x,t)]^n \quad (10)$$

Equation (10) can be re-written as follows:

$$H_L\left\{u(x,t)^{n+1}\right\} = H_R\left\{u(x,t)^n\right\} \quad (11)$$

where

$$H_L = 1+\alpha\nabla^2+\beta\nabla \quad (12)$$

$$H_R = 1+\zeta\nabla^2+\zeta\nabla$$

The next step, is to approximate the solution by making use of equation (1) into equation (11), yeilds:

$$\sum_{j=1}^{N}\lambda_j^{n+1}H_L\Theta(r_{ij}) = \sum_{j=1}^{N}\lambda_j^{n}H_R\Theta(r_{ij}) \quad (13)$$

To simplify computation procedure, let us introduce the following:

$$H_L\Theta(r_{ij}) = \Psi_1(r_{ij})$$
$$H_R\Theta(r_{ij}) = \Psi_2(r_{ij}) \tag{14}$$

By making use of equation (14) into equation (13), the later takes the following simplified form:

$$\sum_{j=1}^{N}\lambda_j^{n+1}\Psi_1(r_{ij}) = \sum_{j=1}^{N}\lambda_j^{n}\Psi_2(r_{ij}) \tag{15}$$

SOLUTION PROCEDURE :

Let us start the solution procedure by manuplating the right hand side of equation (15) as follows:

$$[u(x_j)]^T = \sum_{j=1}^{N}\lambda_j^{n}\Psi_2(r_{ij}) \tag{16}$$

To solve the system given by equation (16), let us re-write it as follows:

$$\lambda^n\Psi_2 = U(\mathrm{x}) \tag{17}$$

The solution of equation (17) takes the following form:

$$\lambda^n = \Psi_2^{-1}U(\mathrm{x}) \tag{18}$$

By making use of the solution of equation (18) into equation (15), then the solution at the n+1 time step will be:

$$\lambda^{n+1} = \Psi_1^{-1}(r_{ij})\Psi_2(r_{ij})\lambda^n \tag{19}$$

TEST EXAMPLE :

To test the validaty of the CRBF function, on the solution of certain class of partial differential equations, two advection-diffusion problems are solved in the next two subsections.

1-D advection-diffusion

The test problem consists of the governing equation with the associated boundary and initial conditions as follows[Boztosum2002]:

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2} - v\frac{\partial u(x,t)}{\partial x},\ 0 \le x \le 1,\ t > 0$$

with

$$u(x=0,t) = 30$$
$$u(x=1,t) = 30$$
$$u(x,t=0) = 0$$

The analytical solution is given by **[**Boztosum I2002**]**:

$$u^{ex}(x,t) = a\exp(bt + cx) \quad \& \quad c = \frac{v \pm \sqrt{v^2 + 4kb}}{2k}$$

The test problem is solved at three different cases of convection coefficients and fixed value of diffusion coefficient. The numerical data corresponding to the three different cases are shown in **table (1).**

The results corresponding to the three cases are compared with the analytical results at time, $t = 0.5\sec$ to compare with another research ,[S.G.Ahmed 2009] as shown in **figure (1).** From the figure, one can conclude that at small value of the convection coefficient, the temperature profile behave linearly and by growing up its value, the profile behaves in an exponentially decaying and this agrees well with the expected physical behavior. Also, it is clear that, there is a good agreement with the analytical results.

2-D Advection-diffusion equation

To illustrate the application of the collocation method based on the proposed new CRBF for higher dimension, consider the following 2D advection-diffusion problem[Boztosum2002].

$$\frac{\partial u(x,y,t)}{\partial t} = k_x\frac{\partial^2 u(x,y,t)}{\partial x^2} + k_y\frac{\partial^2 u(x,y,t)}{\partial y^2} + v_x\frac{\partial u(x,y,t)}{\partial x} + v_y\frac{\partial u(x,y,t)}{\partial y}, 0 \le x,y \le 1$$

the initial condition with

$$u(x,y,0) = a\left(\exp(c_x x) + \exp(c_y y)\right)$$

and the boundary conditions

$$u(0,y,t) = a\left(\exp(bt) + \exp(bt + c_y y)\right),$$

$$u(1,y,t) = a\left(\exp(bt + c_x) + \exp(bt + c_y y)\right),$$

$$u(x,0,t) = a\left(\exp(bt + c_x x) + \exp(bt)\right),$$

$$u(x,1,t) = a\left(\exp(bt + c_x x) + \exp(bt + c_y)\right)$$

The analytic solution is given [Boztosum2002]

$$u(x,y,t) = a\left(\exp(bt + c_x x) + \exp(bt + c_y y)\right)$$

$$c_x = \frac{v_x \pm \sqrt{v_x^2 + 4k_x b}}{2k_x}$$

and

$$c_y = \frac{v_y \pm \sqrt{v_y^2 + 4k_y b}}{2k_y}$$

The problem is solved using the collocation method based on the CBRF, and a uniform distribution of nodes over a unit square. The computational domain is discretized with different arrangement of collocation points with a time step $\delta t = 0.05\sec$, the diffusion coefficient $k_x = 1.4, k_y = 1.7$, and the advection coefficients are $v_x = 0.1, v_y = 0.1$. we use this prameter to decreased the error , homogeneous results and compare with [Salmon,1980].

RESULTS AND DISCUSSION :

The results due to the present method are compared with the available analytical results[Guvanasen 1983]. Let us start analyzing the results by plotting the variation of temperature against space variables at two different times $t = 0.1\,\&\,0.5\,\text{sec}$ are shown in **figures (2)** and **(3),** respectively. From both figures, one observed a good agreement between the results due to the collocation based on the proposed CRBF and the analytical results. One can also observe that by growing up the time, the error between the two results is 13% slightly increases but still in the allowed tollerence and can be accepted to 17% [S.G Ahmed2009]. This error can be decreased by increaing the number of scattered node throughout the domain. Finally, the collocation based on the present CBRF promises well in this field of research. To test the results of the collocation based on the CBRF, the L_2- relative error norms(REN) indicator is used for comparison and it is defined as[S.G.Ahmed2009]:

$$L_2 - \text{REN} = \sqrt{\frac{\sum (u_{num} - u_{exact})^2}{\sum (u_{exact})^2}}$$

In the present computation, we check the comparison based on the increase of the scattered nodes**,** and shows the fixed value of diffusion coefficient compared with analytical result **table (1)** [Singh2000]. **Table (2)** shows the effect of increasing the number of the scattered nodes and the corresponding relative error norm**,** it is clear that by increasing the number of scattered nodes, the relative error norm decreases and this indicates that the error between the obtained results and the analytical one is acceptable. The collocation based on the CBRF gives acceptable results and so can solve problems with no solutions availble to compare.

CONCLUSIONS :

The present paper concered mainly with collocation based on a new composite radial basis function. In the present paper, a composite radial basis function is developed by a comination between thin plate and logarithmic radial basis functions. In this function the thin plate was used when evaluating diagonal elements of the matrix coefficients and the logarithmic function was used for evaluating the other elements. Finally, we can say that the collocation method is a very simple meshless method, also the results due to the the present CBRF was modified compared with the previous versions derived before.the temperature profile in this parameter more linearly and agrees with expected physical behavior.

Table 1: Numerical data for coefficients

	Case (1)	Case (2)	Case (3)
v	1	6	12
k	1	1	1

Table 2: Relative errors norm

Number of nodes	L_2-REN
81	368×10^{-5}
169	351×10^{-5}
256	319×10^{-5}
400	64×10^{-5}

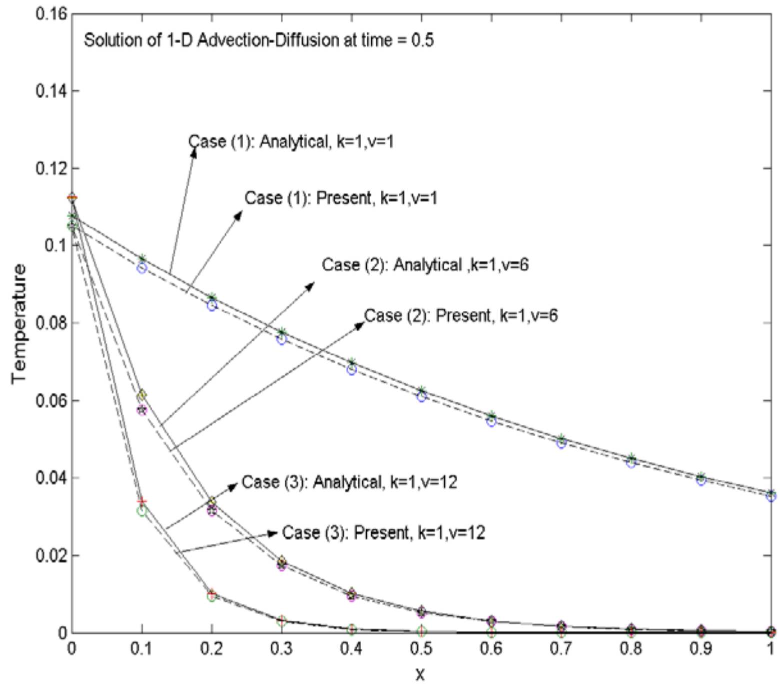

Figure 1: Results of test problem 1 at three different cases of k and v

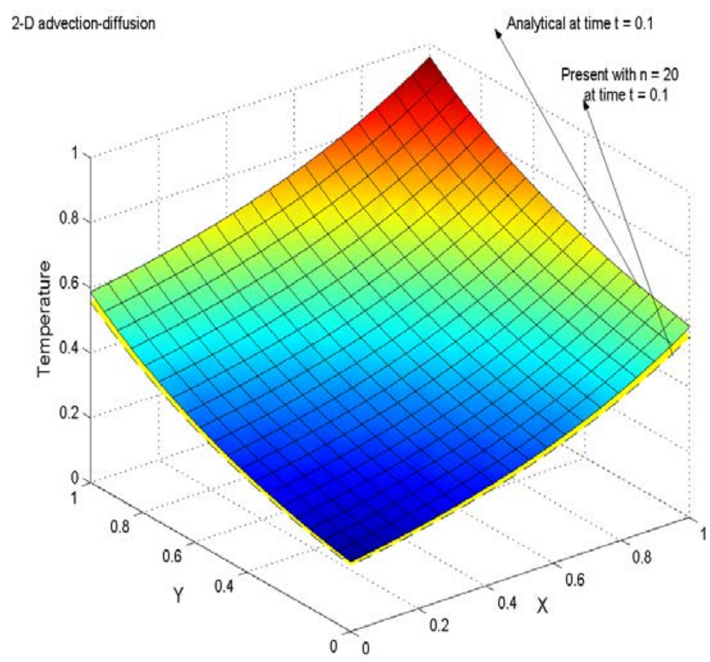

Figure 2: Temperature distribution at time t = 0.1

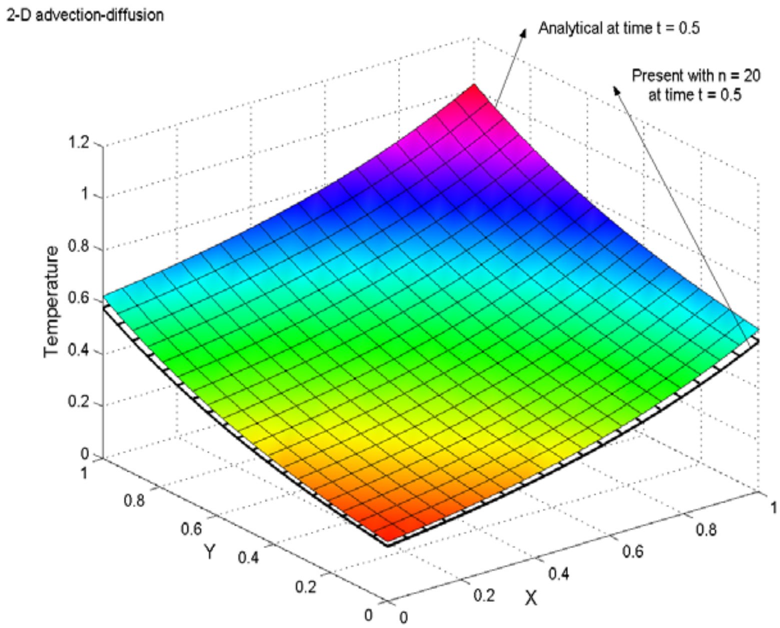

Figure 3: Temperature distribution at time t = 0.5

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