Tikrit Journal of Pure Science Vol. 13 No.(1) 2008, 185 - 188

Generalized SF-Rings

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Abstract:

In the present paper we define a new generalization of flatness it is called generalized flat, also we define a new generalization of SF-rings due to Rege M. B. in [8], it is called generalized SF-rings. Furthermore, we find several properties for them. Finally, we find some main results for generalized SF-rings and we find the relation between such modules with certain rings.

Introduction:

Throughout this paper, R denotes an associative ring with identity and all modules are unital right R-module unless otherwise stated. For any non zero element a of a ring R, r(a) and ℓ(a) are denote the right and left annihilators of a, respectively. J(R) will be stand for the Jacobson radical of R. Recall that:

(1) A ring R is called reduced if it contains no nonzero nilpotent elements
(2) A ring R is called semi-prime if it contains no nozero nilpotent ideals.
(3) A ring R is called π-regular (strongly π-regular) [2] if for every $a \in R$, there exists a positive integer n, depending on a and an element $b \in R$ such that $a^n = a^n b a^n$ ($a^n = a^{n+1} b$). Or equivalently R is strongly π-regular if and only if $a^n R = a^{2n} R$ [1]. It is easy to see that R is strongly π-regular if and only if $Ra^n = Ra^{2n}$.
(4) A ring R is called fully right(left) idempotent if every right(left) ideal of R is idempotent.
(5) An ideal I of a ring R is called left semi π-regular [1] if for every $a \in I$, there exists $b \in I$ and a positive integer n such that $a^n = b a^n$ and $\ell(a^n) = \ell(b)$.
(6) A ring R is called right(left) SF-ring[8] if every simple right(left) R-module is flat.

Now, the following lemma which is due to Rege in [8] is the basic result for our new definition.

Lemma:
Let I be a right(left) ideal of R. Then, R/I is a flat right(left) R-module if and only if for each $a \in I$, there exists $b \in I$ such that a = ba (a = ab).

Generalized SF-Rings:

In this section we define a new generalization of flatness, it is called generalized flat right(left) R-module and depend on it we define a new generalization of SF-rings due to Rege in [8], it is called generalized SF-ring. Furthermore, we find several properties for them.
We start this section with the following definitions.

Definition 2.1:
Let I be a right(left) ideal of R. Then, R/I is a generalized flat right(left) R-module if and only if for each $a \in I$, there exists $b \in I$ such that $a^n = ba^n$ ($a^n = a^n b$), for some positive integer n.

Definition 2.2:
A ring R is called generalized SF-ring if every simple right R-module is generalized flat.
We define a condition (*) as follows:

Definition 2.3:
A ring R is satisfy condition (*) if $ba^n = 0$ implies $a^n b = 0$, for each $a, b \in R$ and a positive integer n.
Clearly every right SF-ring is a right generalized SF-ring, however the converse is not true in general as the following example shows:

Example (1):
Let $\mathbf{Z}_{12}$ be the ring of integers modulo 12, M = {0, 2, 4, 6, 8, 10} and N = {0, 6} are both maximal ideals of $\mathbf{Z}_{12}$, then both $\mathbf{Z}_{12}$/M and $\mathbf{Z}_{12}$/N are generalized simple flat, but they are not simple flat. Thus, $\mathbf{Z}_{12}$ is generalized SF-ring, but it is not SF-ring.

Proposition 2.4:
If R is a right generalized flat and I is a two-sided ideal of R, then the factor ring R/I is a right generalized flat.

Proof:
Let R be a right generalized SF-ring and I a two-sided ideal of R. Therefore, for each $a \in I$, there exists $b \in I$ such that $a^n = ba^n$, for some positive integer n. We show that R/I is a right generalized SF-ring. Now, (b + I) $(a^n + I) = ba^n + I = a^n + I$. Therefore, the factor ring R/I is a right generalized SF-ring.
We now consider a necessary and sufficient condition for generalized SF-ring to be SF-ring.

Theorem 2.5:
Let R be a ring such that for every $a \in I$ and every positive integer n, $\ell(a^n) \subseteq \ell$ (a), then every generalized SF-ring is SF-ring.

Proof:
Let M be a maximal right ideal of R and for every $a \in M$. Since R is generalized SF-ring, there exists $b \in M$ such that $a^n = ba^n$, for some positive integer n. This implies that $(1\text{-}b) \in \ell(a^n) \subseteq \ell$ (a) and hence a = ba. Whence R is a right SF-ring.

Theorem 2.6:
Let R be a right generalized SF-ring and M a maximal right ideal of R. If $\ell(a^n) = a^n R$, for every $a \in M$ and a positive integer n, then $R = a^n R + bR$, for some $b \in M$.

Proof:
Let M be a maximal right ideal of R containing a. Since R is generalized SF-ring, there exists $b \in M$ such that $a^n = ba^n$, for some positive integer n. Therefore, $(1\text{-}b) \in \ell(a^n) = a^n R$. Thus, $1\text{-}b = a^n R$. Whence $R = a^n R + bR$.

Theorem 2.7:

Let R be generalized SF-ring. If $\ell(a^n) = 0$, then a^n is a right invertible in R, for some positive integer n.

Proof:

Let $a \in R$ with $\ell(a^n) = 0$. If $a^n R \neq R$, there exists a maximal right ideal M of R containing $a^n R$, for every positive integer n. Since $a \in M$ and R/M is generalized flat, there exists $b \in M$ such that $a^n = ba^n$. Whence $1-b \in \ell(a^n) = 0$, yielding $1 \in M$ which contradicts $M \neq R$. Therefore,$a^n R=R$ and hence a^n is a right invertible in R.

Theorem 2.8:
If R is a commutative ring such that R / I and R / J are generalized flat. Then, R / $(I \cap J)$ is generalized flat.
Proof:
Let $x \in I \cap J$, then $x \in I$ and $x \in J$. Since R / I and R / J are generalized flat. Then, there exists $y \in I$ and a positive integer n such that $x^n = yx^n$ and there exists $z \in J$ and a positive integer m such that $x^m = zx^m$. Hence, $x^{n+m} = yz\, x^{n+m}$. Since $yz \in I.J \subseteq I \cap J$, therefore, R / $(I \cap J)$ is generalized flat.

Theorem 2.9:

Let R be a ring such that $\ell(a^n)$ is also a right ideal. If R is generalized SF-ring, then R is π-regular.

Proof:

Let R be generalized SF-ring and $a \in R$. If $K = \ell(a^n)$, for some positive integer n, then K is also a right ideal by hypothesis. If M is a maximal right ideal of R containing $a^n R + K$. Since R/M is generalized flat, then $a^n = x\, a^n$, for some $x \in M$ and a positive integer n. Hence $(1-x) \in \ell(a^n) = K \subseteq M$. This implies that $1 \in M$ a contradiction. Therefore, $a^n R + K = R$. So, $a^n r + k = 1$, for some $r \in R$ and $k \in K$. Since $ka^n = 0$, this gives $a^n = a^n r\, a^n$. Thus, R is a π-regular ring.
Recall that R is said to be right (left) sπ-weakly regular [7], if for each $a \in R$, there exists a positive integer n = n(a), depending on a such that $a^n \in a^n R\, a^{2n} R$ ($a^n \in R\, a^{2n} R\, a^n$). R is called s$\pi$-weakly regular if it is both right and left sπ-weakly regular.
The following result is a connection between generalized SF-ring and sπ-weakly regular ring by adding that R satisfy condition (*).

Theorem 2.10:
If R is a generalized SF-ring and satisfy condition (*), then R is sπ-weakly regular ring.

Proof:

Let $b \in R$. We claim $Rb^{2n} R + r(b^n)$, for some positive integer n. If not, there exists a maximal right ideal M of R containing $Rb^{2n} R + r(b^n)$. Since R is generalized SF-ring. Therefore, R/M is a right generalized flat. Since $b \in M$, there exists $c \in M$ such that $b^n = c\, b^n$, for some positive integer n. Then, $1-c \in \ell(b^n) \subseteq r(b^n) \subseteq M$ and hence $1 \in M$, a contradiction. Whence, R is sπ-weakly regular.

Proposition 2.11:
Let R be generalized SF-ring. Then,

(1) Any reduced $a^n R$ of R is idempotent.

(2) $R = Rc^n R$, for any nonzero divisor c of R.

Proof:

(1) Let $a^n R$ be a reduced principal right ideal of R, for some positive integer n. For any $b \in a^n R$, $\ell(b^n) \subseteq r(b^n)$ and if $R\, b^n R + r(b^n) \neq R$. Let M be a maximal right ideal containing $R\, b^n R + r(b^n)$. If R/M is generalized flat, then $b^n = c\, b^n$, for some $c \in R$ and a positive integer n. Then, $1-c \in \ell(b^n) \subseteq r(b^n) \subseteq M$ yields $1 \in M$, a contradiction. Therefore, $R\, b^n R + r(b^n) = R$, for any $b \in R$ and a positive integer n, which proves $a^n R = (a^n R)^2$.

(2) If $Rc^n R \neq R$, let M be a maximal right ideal containing $Rc^n R$. Since $\ell(c^n) = r(c^n) = 0$, the proof of (1) shows that R/M is generalized flat leads to a contradiction. This proves that $Rc^n R = R$.

Corollary 2.12:
If R is a right generalized SF-ring, then any reduced principal right ideal $a^n R$ of R is a fully right idempotent.

Proof:

Let T be a reduced principal right ideal $a^n R$ of R, I a right ideal of T. Then, by Proposition 2.11 (1), for any $b^n \in I$, $b^n R = (b^n R)^2 = (b^n R)^4$ and since $(b^n R)^2 \subseteq b^n T \subseteq I$, then $b^n \in I^2$, which proves that T is a fully right idempotent.

Theorem 2.13:
If R is a right generalized SF-ring without zero divisors, then R is a division ring.
Proof:
Let a be a nonzero element of R and $a^n R \neq R$. Then, there exists a maximal right ideal M of R containing $a^n R$. Since R/M is a right generalized flat, there exists $b \in M$ such that $a^n = ba^n$, for some positive integer n. This implies that $(1-b)\, a^n = 0$ and hence $1-b \in \ell(b^n)$. Since a is a nonzero divisor so is a^n. Therefore, $\ell(a^n) = 0$ implies $1 \in M$ and hence $M = R$, a contradiction. Therefore, $a^n R = R$. Hence $a^n c = 1$. Likewise we can show that $ca^n = 1$. Therefore, a is invertible. Whence it follows that R is a division ring.

Main Results:
In this section we find some main results for generalized SF-ring and we connect it with condition (*), completely semi-prime ideal, strongly π-regular ring and other types of rings and ideals.

Recall that a right ideal I of a ring R is said to be completely semi-prime if for every positive integer n and $a \in R$ such that $a^n \in I$, implies $a \in I$.

Theorem 3.1:
Let R be a generalized SF-ring. If M is a maximal right ideal and completely semi-prime ideal of R. Then, $a \in M$ if and only if $M + \ell(a^n) = R$, for some positive integer n.

Proof:
Let M be a maximal right ideal of R and $a \in M$. Since R is generalized SF-ring, there exists $b \in M$ such that $a^n = ba^n$, for some positive integer n. This implies that $1-b \in \ell(a^n)$. Now, consider $1 = b + (1-b)$. Therefore, $R = M + \ell(a^n)$.

Conversely, assume that $R = M+\ell(a^n)$, for every $a \in R$ and a positive integer n. Then, $1 = b+c$ with $b \in M$ and $c \in \ell(a^n)$. Multiplying from the right by a^n we obtain $a^n = b\,a^n + c\,a^n$, so $a^n = b\,a^n$, this implies $a^n \in M$. Since M is completely semi-prime ideal, then $a \in M$.

Theorem 3.2:
Let M be a maximal right ideal of R. Then, R is generalized SF-ring if and only if for every $a \in M$, $M + \ell(a^n) = R$, for some positive integer n.

Proof:
Let M be a maximal right ideal of R and $a \in M$. Since R is generalized SF-ring, then by Theorem 3.1, $M+\ell(a^n) = R$, for some positive integer n.

Conversely, assume that $M+\ell(a^n) = R$, for every $a \in M$ and a positive integer n. Then, $t+r = 1$, for some $t \in M$ and $r \in \ell(a^n)$, so $ta^n + r\,a^n = a^n$, and this implies $a^n = ta^n$. Whence R is generalized SF-ring.

Theorem 3.3:
Let R be a ring satisfy condition (*) and generalized SF-ring. Then R is a strongly π-regular.

Proof:
Let R be a right generalized SF-ring and a be any element in R. We shall prove that $a^n R+R(a^n) = R$. Suppose $a^n R+r(a^n) \neq R$. Let M be a maximal right ideal of R containing a. Since R is generalized SF-ring, there exists $b \in M$ such that $a^n = ba^n$, for some positive integer n. This implies that $(1-b) \in \ell(a^n)$. Since R satify condition (*), then $\ell(a^n) \subseteq r(a^n)$, for every $a \in R$ and a positive integer n. Therefore, $(1-b) \in r(a^n) \subseteq M$, thus $1 \in M$ a contradiction. Therefore, $a^n R+R(a^n) = R$. Inparticular, $a^n t + u = 1$, for some $t \in R$ and $u \in r(a^n)$. So, $a^n = a^{2n} t$. Thus, R is strongly π-regular.

We recall the following result of [5].

Lemma 3.4:
Let R be a reduced ring. Then, for every $0 \neq a \in R$ and every positive integer n,

(1) $r(a^n) = \ell(a^n)$.

(2) $a^n R \cap \ell(c^n) = (0)$.

The following result is the extension of Proposition 2.2 in [6].

Proposition 3.5:
If R is a right generalized SF-ring, then any reduced principal right ideal $I = a^n R$ of R is a direct summand for any $a \in R$ and a positive integer n.

Proof:
Let $I = a^n R$ be a principal right ideal of R, for any $a \in R$ and a positive integer n and let $a^n R+r(a^n) \neq R$. By a similar method of proof is usedn in Theorem 3.3, R is strongly π-regular, whence $a^n = a^{2n} r$. If we set $d = a^n r^{2n}$, then $a^n = a^{2n} r$. Clearly $(a^n - a^n d\, a^n)^2 = 0$ implies $a^n = a^n d\, a^n$ and hence $I = eR$, where $e = a^n d$ is idempotent element. Thus, I is a direct summed.

Theorem 3.6:
Let R be a reduced ring. Then, $R/a^n R$ is generalized SF-ring if and only if Ra^n is a left semi-π-regular ideal.

Proof:
Let a be a non zero element in R. Then, $R/a^n R$ is a right generalized SF-ring, therefore, for every $c \in a^n R$, there exists $b \in a^n R$ such that $c^n = b\,c^n$, for some positive integer n. Now, to prove that $\ell(c^n) = \ell(b)$. Let $y \in \ell(b)$, then $yb = 0$ and hence $ybc^n = 0$, but $c^n = bc^n$, so $yc^n = 0$, thus $y \in \ell(c^n)$ so $\ell(b) \subseteq \ell(c^n)$. On the other hand, let $z \in \ell(c^n)$, then $z\,c^n = 0$, but $c^n = bc^n$. This implies that $z\,bc^n = 0$ and hence $zb \in \ell(c^n)$. But, $b \in c^n R$, $z \in R$ implies $zb \in c^n R$, so $zb \in c^n R \cap \ell(c^n)$. Since R is reduced, then by Lemma 3.4(2), $zb \in c^n R \cap \ell(c^n) = (0)$, this gives $zb = 0$, hence $z \in \ell(b)$, therefore, $\ell(c^n) \subseteq \ell(b)$. Thus, $\ell(c^n) = \ell(b)$ and hence $c^n = bc^n$, therefore, Ra^n is a left semi-π-regular ideal. The converse part directly it is true.

Lemma 3.7:
Let R be a right generalized SF-ring. If M is a maximal right ideal of R and $M \subseteq J(R)$, then M is a nilideal.

Proof:
Let M be a maximal right ideal of R and $a \in M$. Since R is a right generalized SF-ring, there exists $b \in M$ and a positive integer n such that $a^n = ba^n$. This implies that $(1-b)a^n = 0$. Since $b \in M \subseteq J(R)$, then $b \in J(R)$ and hence (1-b) is invertible, so there exists $u \in R$ such that $u(1-b) = 1$, and this implies that $u(1-b)a^n = a^n$. Therefore, $a^n = 0$. Whence M is a nilideal.

Theorem 3.8:
If R is a local generalized SF-ring such that $\ell(a^n) = 0$, for every $a \in R$ and a positive integer n, then the maximal ideal of R contains a unit element.

Proof:
Assume that R is a local ring. Since R is generalized SF-ring, then for the unique maximal ideal M of R, R/M is generalized flat. Therefore, for every $a \in M$, there exists $b \in M$ such that $a^n = ba^n$, for some positive integer n. Then, $(1-b) \in \ell(a^n) = 0$, hence $b = 1$. Thus, b is unit.

Theorem 3.9:
Let R be a reduced ring. If R is generalized SF-ring, then $r(a^n)$ is a direct summand for every $a \in R$ and a positive integer n.

Proof:
To prove $r(a^n)$ is a direct summand, for every $a \in R$ and a positive integer n. We claim that $a^n R + r(a^n) = R$. If this is not true, let M be a maximal right ideal containing $a^n R + r(a^n)$. By a similar method of proof is used in Theorem 3.3 we obtain $a^n R + r(a^n) = R$. Since R is reduced, then by Lemma 3.4(2), $a^n R \cap r(a^n) = (0)$. Whence $r(a^n)$ is a direct summand for every $a \in R$ and a positive integer n.

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الملخص:

في هذا البحث نحن عرفنا المسطح المعمم وكذلك قمنا بتعميم الحلقة -SF المستخدمة من قبل الباحث Rege M. B. في المصدر [8] وسميت الحلقة SF المعممة نحن وجدنا بعض الخواص لهذه الحلقة. وأخيرا وجدنا بعض النتائج للحلقة المعممة وأيجاد العلاقة بين هذه الحلقة مع حلقات أخرى.