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Studying behavior of the asymptotic solutions to P-Laplacian type diffusion-convection model .

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ABSTRACT

The rescaling method is presented to allow us to establish nonnegative local solutions to the evolution of the Cauchy problem (CP) of the nonlinear degenerate parabolic p-Laplacian process with conservation laws that are posed in one-dimensional space. This equation has specific restrictions in the range of parameters, the non-negative advection coefficient, and a self-similarity representing the main feature. In this study, there are several regions to discuss the qualitative analysis for the local weak solutions and the asymptotic interfaces in irregular domains. The solutions of the CP for degenerate parabolic p-Laplacian type diffusion-advection equations are asymptotically equal to the solutions of p-Laplacian type diffusion or advection equations under some restrictions. Moreover, the blow-up technique, comparison method, and characteristic method are used to estimate the asymptotic local solutions to the CP and the interface functions. The results of this paper can be used to solve problems in the oil and gas industries, such as estimating and controlling the size of oil and gas resources as they evolve through time.

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Introduction with Statement of Problem

The mathematical equation that describes the diffusion difficulties has attracted the attention of numerous academics over the years. Cherniha and Serov [1] were giving the non-linear diffusion equations a fresh analysis and accurate solution. New modification equations were derived by Kuske and Mileniski [2] for the hexagon-style in reaction-diffusion systems. These systems exhibit more non-linearities than Smith-Hohenberg models or Rayleigh-Bernard convection. Matano et al. [3] investigated the interaction and diffusion equations using the spatially heterogeneous interaction term. If this reaction's term coefficient is far higher than the dispersion coefficient, the strong interface between two separate phases will be visible. They demonstrated that the motion equation for this interface includes a drift term even though drift was absent from the original diffusion equations.

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The researchers in [4] investigated the uniqueness and existence of the solution to the self-similarity of diffusion equation. A study was performed to examine fast gas flow models by heating various materials using a microwave and by porous media. A source function and nonlinear diffusion-advection equation is being investigated in Alvarez et al. 1988 [13]; Aal-Rkhais et al. 2018 [6], and Abdulla et al. 2019 [7].

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(\frac{\partial u^{m}}{\partial x}\right)+c\frac{\partial u^{\lambda}}{\partial x}+bu^{\beta}=0; \qquad (1)$$

According to [17, 19, 24], the general theory of the CP for equation (1) and qualitative analysis are established in irregular domains with compactly supported initial data. This paper represents the continuity of our previous study that we started in our earlier study, see [5]. We provide rescaling and blowing up techniques that are effective to estimate the self-similar solutions. Also, comparison principle and characteristic methods are significant to handle our results. Let us introduce the nonlinear degenerate parabolic p-Laplacian type diffusion PDEs equation as follows,

$$\mathrm{L}u \equiv \frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)+\mathrm{c}\frac{\partial u^{\lambda}}{\partial x}=0; \qquad (2)$$

$$u(x,0)=u_0(x),\ t{=}0,\ x\in\mathrm{R} \qquad (3)$$

where the parameters $\lambda>0, \mathrm{p}>2,\ 0<\mathrm{T}<\infty, c>0$ and $u_0(x)\geq 0$ is continuous. Because the non-positive sign of the advection coefficient c. In [14, 19, 23], There are several uses for fluid mechanics, plasma and quantum physics, and many other fields. One among the key characteristics of process (2) is the nonlinearity quality, and due to the gap created by nonlinear variables and irregular domains, the CP is occasionally discussed with peculiar growth condition. The equation (2) with $c=0$, is interpreted as a particular case of the p-Laplacian type diffusion equation. Recently it has attracted a lot of attention additionally it has established itself similar to a key concept during the study of parabolic PDEs. We provide the readers with more details in several studies [16, 20, 26]. Also, the interface functions of the solutions to the CP(2)-(3) are separated regions and more studies to the behavior of interfaces, [5, 6, 7]. The local case for the initial data is

$$u_0(x)\approx \mathrm{A}(-x)_+^{\alpha} \quad as\ \mathrm{A}>0; x\to 0^-, \qquad (4)$$

where α is a positive and $\approx$ is an inequality in two sides. Direction and behavior of the interface's movement are determined by a conflict between these two forces, p-Laplacian and advection. As demonstrated in [17], because u_0 is a bounded initial function that satisfies certain parameter restrictions as $x\to 0$, it is suitable for satisfying the general theory. Additionally, the study of a porous medium equation (PME) and a growth rate conditions to nonlinear parabolic PDEs clearly introduces the global initial data in

$$u_0(x)=\mathrm{A}(-x)_+^{\alpha},\ \ x\in\mathrm{R} \qquad (5)$$

This study is regarded as a classification of the evolution of the interfaces in our scenario, where advection dominates over p-Laplacian type diffusion force. Our main focus in the case when the advection force dominates over the diffusion force of the p-Laplacian type. To estimate the interface and the solution near a shrinking or waiting time interface under restrictions, in Table1. Also, studying the initial growth of interface $\eta(t)=\{x: u(x,t)>0\}$ and $\eta(0)=0$, will be considered. To classify developing interfaces and the local weak solutions of CP (2)–(5) close to the interfaces, we have to use the plane (α,λ) in Figure 1 where $\mathrm{c}>0$.

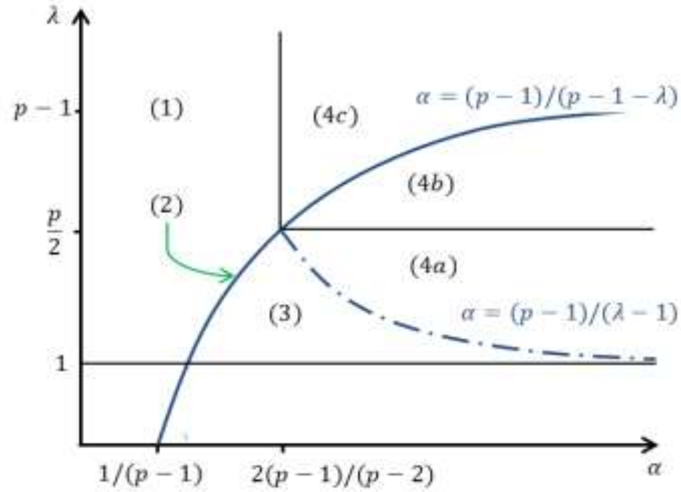

Figure 1. To classify qualitative analysis of the solution and interface to the CP(2)-(3).

To make the calculation process easy, we impose $q=p-1$, the study's discussion of interfaces and solutions are limited to three regions in Figure. Where from the table1, region(1) is restricted depending on rescaling technique and we get the p-Lapacian type diffusion dominates with expanding interface. Similarly, regions (2) is restricted and we get the p-Lapacian type diffusion and advection in balance with expanding interface. The regions (3) is restricted and we get advection force dominated with expanding interface. Finally, the sub-regions (4a),(4b) and (4c) are restricted and we get stationary solution with WT interface.

Table 1. Description of the Regions where *p*-Laplacian type diffusion dominates.

Region	**Restrictions**	**Interface**
(1)	$\alpha<q(q-\min\{\lambda,(q+1)/2\})^{-1}$, $1\leq\lambda$;	**Expanding**
(2)	$\alpha=(q-\lambda)^{-1}q,\ \ 0<\lambda<(q+1)/2;$	**Expanding**
(3)	$(q-\lambda)^{-1}q<\alpha<(\lambda-1)^{-1}q,$ $1\leq\lambda<(q+1)/2;$	**Expanding**
(4a)	$\alpha>q(\lambda-1)^{-1},\ 1<\lambda<(q+1)/2;$	**WT**
(4b)	$\alpha>(q-\lambda)^{-1}q,\ \ (q+1)/2<\lambda<q;$	**WT**

(4c)	$2q(q-1)^{-1}<\alpha<(q-\lambda)^{-1}q,$ $(q+1)/2<\lambda<q;$	**WT**

The weak solution to equation (2) is existed from the definition in [5], and we can get some significant definitions and preliminary results, such as super or sub solutions.

Let us consider the identity as follows

$$I(u,f,\Psi)=\int_{\tau_0}^{\tau_1}\int_{\eta_1(t)}^{\eta_2(t)}\left(-u\frac{\partial f}{\partial x}+\left|\frac{\partial u}{\partial x}\right|^{q-1}\frac{\partial u}{\partial x}\frac{\partial f}{\partial x}-cu^{\lambda}\frac{\partial f}{\partial x}\right)dxdt+\int_{\eta_1(t)}^{\eta_2(t)}uf\,dx\Big|_{\tau_0}^{\tau_1}=0, \quad (6)$$

where $f\in C^{2,1}_{x,t}(\bar{\Psi}),\ f\,|_{x=\eta_1(t)}=0$, and

$$\Psi=\{(x,t):\ \eta_1(t)<x<\eta_2(t),\ \tau_0<t<\ \tau_1\}.$$

Additionally, u is called local sub-solution (or super-solution) to the parabolic PDEs (2) if $I(u,f,\Psi)\le 0$ (resp. ≥ 0). The general theory of the IVP for (1) was studied in several papers [18, 20, 21, 22, 27]. The authors proved the qualitative properties using the energy and the optimal growth rate is a feature of equation (2) that can be achieved with c = 0.

Also, we need to consider some concepts that are significant in proof our results. Let $\{\phi_n\}$ be a sequence of functions defined on a set E. It is said to be uniformly convergent to $\phi(t)$ on $E_1\subset E$ if for each $\varepsilon>0$; there exists $k(\varepsilon)$ such that $|\phi(t)-\phi_n(t)|<\varepsilon.$

From the Arzela-Ascoli theorem, let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of real-valued continuous functions on a compact set E that is $\phi_n\in C(E)$, for all n, $\{\phi_n\}$ is point-wise bounded on E. and $\{\phi_n\}$ is equi-continuous, then $\{\phi_n\}$ is uniformly bounded and it has a uniformly convergent subsequence, see [30].

Governing of p-Lapacian **Type Diffusion**

Based on the above preparation and preliminaries in previous article in [5], we keep going and formulate our problem to evaluate the solutions near interfaces in different regions. The advection term with $c>0$ is dominated over the p-Laplacian type diffusion factor with shrinking interface as shown in region (1), Figure 1. We will prove that situation in the theorem below as follows:

Theorem 2.1. If $\alpha<(q-\min\{\lambda,(q+1)/2\})^{-1}q$ $\lambda\ge 1$ and the interface is initially expanding as follows

$$\eta \sim t^{\frac{1}{q-\alpha(q-1)+1}}\zeta_*,\ as\ \ t\to 0^+, \quad (7)$$

where

$$\zeta_*=A^{(q-1)/(q-\alpha(q-1)+1)}\ \zeta'_*, \quad (8)$$

and $\zeta_*=\zeta_*(A,q,\alpha)>0.$ Then the p-Laplacian diffusion force dominates over the advection force and the local solution can be presented as follows

$$u\sim t^{\frac{\alpha}{q-\alpha(q-1)+1}}S(\rho), \quad (9)$$

along $x=\eta_\rho(t)=\rho t^{\frac{1}{q-\alpha(q-1)+1}}$, where S is a shape function that depends on A. To prove theorem 2.1, we have to consider the auxiliary results in two lemmas as below.

Lemma 2.1. The CP (2)-(4) has a solution u with $q>1$, $0<\alpha<(q+1)(q-1)^{-1}$, then it provides a self-similar solution

$$u_*(x,t)\sim t^{\frac{\alpha}{q-\alpha(q-1)+1}}S(\zeta),\quad \zeta=\ t^{\frac{-1}{q-\alpha(q-1)+1}}x, \quad (10)$$

$$S(\rho)=S_0\left(\frac{1-q}{q-\alpha(q-1)+1}\rho\right)A^{\frac{q+1}{1+q-\alpha(q-1)}}, \quad (11)$$

$$S_0(\rho)=w(1,\rho)\ ,\quad \zeta'_*=\sup\{\rho:\ S_0(\rho)>0\}>0. \quad (12)$$

where (10) satisfies the shape function S and w is a solution in special case. Additionally, if u_0 satisfies (3), the CP(2)-(3) satisfies the results (7)-(9).

Lemma 2.2. If CP(2)-(5) has a solution u, to equation (9) that is satisfied under the following restrictions

a. $0<\alpha<q(q-\lambda)^{-1},\ 0<\lambda<(q+1)/2.$
b. $0<\alpha<2q(q-1)^{-1},\ (q+1)/2\le\lambda<q.$
c. $0<\alpha<2q(q-1)^{-1},\ \lambda\ge q.$

Proof of lemma 2.2. From our previous techniques in proof of lemma 3.2 in [5]. $\exists x_{\varepsilon}<0$, then

$$(A-_{\varepsilon})\le u_0(x)/(-x)_+^{\alpha}\le(A+_{\varepsilon}),\ \ x_{\varepsilon}\le x. \quad (13)$$

By the continuity of solution to the parabolic PDEs(2) with initial data (3), $x_{\varepsilon}\le x<\infty$, $\exists\sigma>0$, depends on ε and $0\le t\le\sigma$ we have

$$u_{-\varepsilon}(x,t)\le u(x,t)\le u_{\varepsilon}(x,t); \quad (14)$$

From lemma 2.1 in [5], and (13)-(14), then

$$u_{-\varepsilon}\le u\le u_{\varepsilon}\quad \text{for } 0\le t\le\sigma,\ x_{\varepsilon}\le x<\infty \quad (15)$$

Now, let us rescale the functions $u_{\pm\varepsilon}$ as follows,

$$u_k^{\pm} = k u_{\pm}(k^{-1/\alpha} x, k^{-(q-\alpha(q-1)+1)/\alpha} t) \quad (16)$$

$u_{\pm}$ solves the following Cauchy Problem:

$$\mathrm{L}u = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{q-1}\frac{\partial u}{\partial x}\right) + ck^{\frac{\alpha(q-\lambda)-q}{\alpha}}\frac{\partial u^{\lambda}}{\partial x} = 0, \quad (17)$$

$$u(x,0) = (\mathrm{A} \pm)(-x)_+^{\alpha}. \quad (18)$$

Under the conditions of the definition 2.1 in [5], the local weak solution to the CP (17)-(18) is existed. Under the restriction $\alpha(q-\lambda)-q<0$, the following formula should be true

$$\lim_{k\to+\infty} u_k^{\pm} = v_{\pm};\ (x,t)\in R\times[0,\infty) \quad (19)$$

Let $v_{\pm}$ solve

$$\mathrm{L}u = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{q-1}\frac{\partial u}{\partial x}\right) + c\frac{\partial u^{\lambda}}{\partial x} = 0, \quad (20)$$

$$u(x,t) = (\mathrm{A}\pm)(-x)_+^{\alpha};\quad u(-x,t)=0, \quad (21)$$

$$u(x,0) = (\mathrm{A}\pm)(-x)_+^{\alpha},\ |x| \geq |x|, \quad (22)$$

where, $0<t<\sigma$, $|x| \geq |x|$, and $u_k^{\pm}$ solve the following problem

$$\mathrm{L}u = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{q-1}\frac{\partial u}{\partial x}\right) + ck^{\frac{\alpha(q-\lambda)-q}{\alpha}}\frac{\partial u^{\lambda}}{\partial x} = 0, \quad (23)$$

$$u(k^{\frac{1}{\alpha}}x, t) = k(\mathrm{A}\pm)(-x)^{\alpha},\ u(-k^{\frac{1}{\alpha}}x, t) = 0, \quad (24)$$

$$u(x,0) = (\mathrm{A}\pm)(-x)_+^{\alpha},\ k^{\frac{1}{\alpha}}|x| \geq |x|. \quad (25)$$

where $0\le t\le \sigma k^{\frac{q-\alpha(q-1)+1}{\alpha}}$, and

$$\mathrm{D}^k = \{(x,t);\ -k^{\frac{1}{\alpha}}|x| < x < k^{\frac{1}{\alpha}}|x|,\ t\le \sigma k^{\frac{q-\alpha(q-1)+1}{\alpha}}\}$$

The DPs (20)-(22) and (23)-(25) have unique solutions. Since the finite speed propagation property and $\sigma>0$ that is chosen such that $0\le t\le\sigma$, then we choose $u(-x,t)=0$. Applying lemma 2.1 in [5], and from (13),(14) and (15),(16) follows. Let us verify the sequence $\{u_k^{\pm}\}$ converges by the assumption

$$h = (\mathrm{A}+1)e^t(1+x^2)^{\alpha/2},\ x\in\mathrm{R},\ \mathbf{0}\le t\le\sigma.$$

Then we have

$$\mathrm{L}_k u \equiv \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{q-1}\frac{\partial u}{\partial x}\right) + ck^{\frac{\alpha(q-\lambda)-q}{\alpha}}\frac{\partial u^{\lambda}}{\partial x}$$

$$\equiv (\mathrm{A}+1)e^t(1+x^2)^{\frac{\alpha}{2}}\mathrm{T}\quad in\ \mathrm{D}^k$$

where $\mathrm{T} = 1-\mathrm{H}(x)+R$, and

$$\mathrm{H}(x) = k^{\frac{\alpha(q-\lambda)-q}{\alpha}}\alpha^q(1+x^2)^{\frac{\alpha(q-\lambda)-2q}{2}}x^{q-1}(\mathrm{A}+1)^{q-1}e^{t(q-1)}\times((\alpha-2)x^2(x^2+1)^{-1}+1)$$

is a continuous function on D^k and for $|x|<k^{\frac{1}{\alpha}}|x|$, thus $\mathrm{H}(x)$ is uniformly convergesent where $k\to\infty$. Also, $R = ce^{(\lambda-1)t}(1+x^2)^{\frac{\alpha(\lambda-1)-2}{2}}x(\mathrm{A}+1)^{\lambda-1}\alpha\lambda$ and hence

$$\mathrm{L}_k u = (\mathrm{A}+1)(1+x^2)^{\frac{\alpha}{2}}e^t\mathrm{T} \ge h$$

where $\mathrm{D}_0^k = \mathrm{D}^k \bigcap \{(x,t):\ 0<t\le\sigma\}$. Then we have $R = O(k^{\frac{(q-\lambda)-q}{\alpha}})$ uniformly converges on D_0^k as $k\to\infty$. Thus for $0<\varepsilon \ll 1$

$$u_k^{\pm}(x,0) \le h(x,0)\quad on\ k^{\frac{1}{\alpha}}|x| \geq |x|,$$

$$u_k^{\pm}(\pm k^{\frac{1}{\alpha}}x, t) \le h(\pm k^{\frac{1}{\alpha}}x, t),\ 0\le t\le\sigma.$$

Since, $\exists k_0 = k_0(\alpha,\lambda)$ thus $\forall k\ge k_0$ and lemma 2.1 in [5] implies

$$u_k^{\pm} \geq h \text{ in } \bar{\mathrm{D}}_0^k. \quad (26)$$

Suppose that $\mathrm{P} = \{(x,t): x\in\mathrm{R},\ t\le t_0\}$, and $\mathrm{B}\subset\mathrm{P}$ where B is a compact region. If k_0 is a large and $k\ge k_0$, such that $\mathrm{D}_0^k \supset \mathrm{B}$. Inequality (26) implies that $\{u_k^{\pm}\}$ is uniformly bounded in B, $\exists v_{\pm}$ is a subsequence $u_k^{\pm}$ converges such that

$$\lim_{k'\to+\infty} u_{k'}^{\mp} = v_{\mp},\ \text{on P}.$$

so, when $\lambda<1$, advection term drops out as $k\to\infty$, and $v_{\mp}$ solves (9).

Proof of theorem 2.1. By assumption of the theorem, let the restriction of α and λ be satisfied. Since the formula (7) comes from lemma 2.1. The lower bound of the interface satisfies as follows

$$\zeta_* \le \liminf_{t\downarrow 0^+} \eta(t)t^{\frac{1}{\alpha(q-1)-(q+1)}}. \quad (27)$$

in order to get the interface's upper bound, on the other hand assume the arbitrary sufficiently small $\varepsilon>0$ and the CP(2) and (5) has a solution u with $c=0$. We will change A to A +, and first and second inequality of (14) and (13) respectively, that will be considered. We must now demonstrate that $\bar{u}$ supersolution of (2) is satisfied with $c>0$, then we have

$L\bar{u} = c\frac{\partial \bar{u}}{\partial x}$. Now we should prove $c\frac{\partial \bar{u}}{\partial x} \geq 0$. Since the advection coefficient c is positive so only proving $\frac{\partial \bar{u}}{\partial x} \geq 0$ is required. To solve CP(2)-(3), the technique of regularization should be considered with $c = 0$ and $u = (A +)(-x)_+^{\alpha}$. Now, put $\omega = u$ to prove $\frac{\partial \bar{u}}{\partial x} \geq 0$ and assume that

$$\bar{u} = \max\left(\omega\left(xe^{ct}, (e^{ct(q+1)} - 1)/c(q+1)\right), 0\right) = \max\left(\omega(\zeta,\tau), 0\right)$$

$$L\bar{u} = e^{ct(q+1)} L\omega + c\omega_{\zeta}\left[\lambda\bar{u}^{\lambda-1}e^{ct} + \zeta\right]$$

$$= c\omega_{\zeta}\left[\lambda\bar{u}^{\lambda-1}e^{ct} + \zeta\right] \geq 0,$$

$$L\omega = \frac{\partial\omega}{\partial\tau} - \frac{\partial}{\partial\zeta}\left(\left|\frac{\partial\omega}{\partial\zeta}\right|^{q-1}\frac{\partial\omega}{\partial\zeta}\right) = 0$$

It is clear that $\partial\omega/\partial\zeta > 0$ and the reason is that $c > 0$ and $\lambda\bar{u}^{\lambda-1}e^{ct} + \zeta > 0$. In addition, since ω is a classical solution, using the maximum principle to obtain $\bar{u} < 1$ in D, by reducing $|x|$ and σ to such small values and so $\partial u/\partial x \geq 0$. Then it must be $\partial u^{\lambda}/\partial x \geq 0$, so we get $L\bar{u} \geq 0$ in D, $D = \{(x,t);\ x \leq x,\ \sigma > t\}$.

The r-h-s inequality of (15) holds, from a comparison principle along with (13), (14). And so we get

$$\eta(t)t^{-\frac{1}{\alpha(q-1)-(q+1)}} \leq (A +)^{\frac{1-q}{\alpha(q-1)-(q+1)}}\zeta_*';\ 0 \leq t \leq \sigma$$

$$\zeta_* \geq \limsup_{t\downarrow 0^+} \eta(t)t^{\frac{1}{\alpha(q-1)-(q+1)}}. \tag{28}$$

Therefore, $\eta(t) \sim t^{\frac{1}{(q+1)-\alpha(q-1)}}\zeta_*,\ t \to 0^+$ is valid, from (27) and (28).■

Example1. We consider the solution of CP (1)-(2) over the interface function $\eta(t) \approx \zeta_* t^{1/(p-\alpha(p-2))}$, with $p = 3, \alpha = 0.5$. This example to describe the situation where the diffusion dominates. Then we get three cases. If ζ_*=0, 1.8, 1.4. Firstly, $\eta(t) = 0$, the interface has waiting time for the initial function (2). In second case, $\eta(t) = 0.2$ and third case $\eta(t) = 0.4$ with the constant $C = 8$ and the interface is expending.

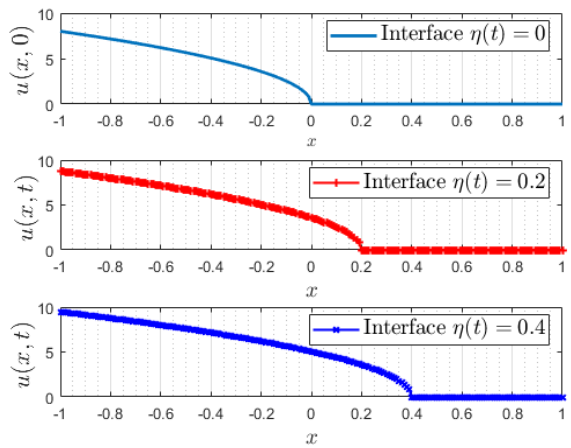

Figure 2. description of the solution where the diffusion dominates with expending interface.

Let us take some values where $\zeta_* = 0, 0.5, 1, 1.5$; to show behavior of the solution $u(x,t)$ near the interface function $\eta(t) \approx t^{1/(p-\alpha(p-2))}\zeta_*$, with $p = 3, \alpha = 0.5$; in the Figure 2.

Balance Situation between p-Laplacian Type Diffusion and Advection

In the previous study in [5], authors discussed the balance of the p-Laplacian type diffusion and advection forces in the case where the diffusion force stronger than advection force with the restriction $(A > A_*)$ and no-positive advection coefficient. On the other hand, we identify also in this section the same case but when the advection coefficient is positive, and under the restrictions of the parameters as shown in Figure 1 (region(2) in table 1).

Theorem 3.1. If $\alpha = q(q-\lambda)^{-1}$ with $\lambda < (q+1)/2$ and

$$A_* = [(q-\lambda)q^{-1}(-c)^{\frac{1}{q}}]^{\frac{q}{q-\lambda}}. \tag{29}$$

And if the initial data u_0 is satisfied (3) then the interface has the formula $\eta \sim t^{\frac{q-\lambda}{\vartheta}}\xi_*, t \to 0^+$ which is expanding according as $A > A_*$ and ξ_* is a positive value. Also, the self-similar solution

$$u \sim t^{\frac{q}{\vartheta}} S(\rho),\ \text{as } t \to 0^+,\ x = \xi_\rho(t) = t^{\frac{1}{(q+1)-\alpha(q-1)}}\rho. \tag{30}$$

where $S(\rho) > 0$ is a shape function for $\rho < \xi_*$ and $\vartheta = (q+1)(q-\lambda) - q(q-1) > 0$. It is evident from lemma 3.1 that global solution to CP (2),(4) satisfies

$$u(x,t)=t^{\frac{q}{\vartheta}}S(\xi),\quad \xi=t^{\frac{-(q-\lambda)}{\vartheta}}x \tag{31}$$

$$\eta=\xi_* t^{\frac{(q-\lambda)}{\vartheta}},\quad 0<t<+\infty, \tag{32}$$

where, $\eta(t)$ is expanding, $S(0)>0$ (see lemma 3.1), and for $0\le x<+\infty,\ 0<t<+\infty$, such that

$$A_2 t^{\frac{q}{\vartheta}}(\xi_2-\xi)_+^{\frac{q}{q-\lambda}}\le u\le A_1 t^{\frac{q}{\vartheta}}(\xi_1-\xi)_+^{\frac{q}{q-\lambda}}, \tag{33}$$

which indicates (34)

observing the r-h-s of (33) (respectively (34)) relates to (31) (respectively (32)).

Lemma 3.1. Under the restrictions of parameters $\alpha=q(q-\lambda)^{-1}$ and $0<\lambda<(q+1)/2$; then the CP(2),(5) has the self-similar form as follows

$$u(x,t)=\delta(\xi)t^{\frac{q}{\vartheta}},\ \xi=xt^{-\frac{(q-\lambda}{\vartheta}} \tag{35}$$

with $S(0)>0$ under the restriction

Proof of lemma 3.1. The same technique in [5], we assume that rescaling function for $k>0$,

$$u_k=ku(xk^{\frac{\lambda-q}{q}},\ tk^{\frac{-\vartheta}{q}}), \tag{36}$$

that satisfies CP(2),(5). Then the global solution of CP(2),(5) is existed and unique. Thus,

$$u=ku(xk^{\frac{\lambda-q}{q}},\ tk^{\frac{-\vartheta}{q}}). \tag{37}$$

By choosing $k=t^{\frac{q}{\vartheta}}$ and assuming the shape function $S(\xi)=u(\xi,1)$ solves the boundary value problem

$$(|S'|^{q-1}S')'-\frac{\alpha S}{(q+1)-(q-1)\alpha}+\frac{\xi S'(\xi)}{(q+1)-(q-1)\alpha}-c(S^{\lambda})'=0, \tag{38}$$

$$S(\xi)\sim A(-\xi)^{\alpha},\xi_*\ge\xi;S(\xi)=0,\xi_*\le\xi \tag{39}$$

where S satisfies (38)-(39). Thus, (31) and (32) are valid. Let us consider the initial data (4) and similar technique as proof of lemma 2.2, then (13), (14) are easily satisfied from (4). Therefore, from previous results, implies

$$u_{\pm}(\xi_{\rho}(\tau),\ \tau)\sim\tau^{\frac{q}{\vartheta}}S(\rho,A),\quad as\ \ \tau\to 0^{+} \tag{40}$$

where $\tau=k^{\frac{-\vartheta}{q}}t$, so from (40) and (15) , the formula (35) follows for the arbitrary value $_{\xi}>0$.

Proof of theorem 3.1. Under the condition $A>A_*$, the CP(2),(5), has a unique solution and the self-similar form (31) in lemma 3.1 is satisfied. Assume that

$$\hbar\sim t^{\frac{q}{\vartheta}}S_1(\xi),\quad in\ \ U. \tag{41}$$

To estimate $L\hbar$ in $U=\{(x,\ t):0<x<\xi_1 t^{\frac{q-\lambda}{\vartheta}},\ t>0\}$, where $S_1(\xi)=A_0(\xi_0-\xi)_+^{\alpha_0}$ with some constants $A_0\geqslant 0$, $\xi_0\geqslant 0$, $\alpha_0=q(q-\lambda)^{-1}>\xi_0$. Calculating $L\hbar$ in U as follows

$$L\hbar=t^{\frac{(q+1)(\lambda-1)+1}{\vartheta}}L^0S_1, \tag{42}$$

$$L^0S_1=\frac{q}{\vartheta}S_1(\zeta)-\frac{(q-\lambda)\zeta}{\vartheta}S_1'(\zeta)-(|S_1'(\zeta)|^{q-1}S_1'(\zeta))'+c(S_1^{\lambda}(\xi))' \tag{43}$$

By substituting the value $S_1(\xi)$ in (43),

$$L^0S_1=\frac{-c\lambda q}{\vartheta}A_0^{\lambda}(\xi_0-\xi)_+^{\frac{(q+1)(\lambda-1)+1}{q-\lambda}}\left\{1-(A_0/A_*)^{q-\lambda}+\frac{q-\lambda}{(-c\lambda)\vartheta}\xi_0(\xi_0-\xi)_+^{\frac{q(1-\lambda)}{q-\lambda}}\right\}$$

if $\lambda>1,\ c>0$. To estimate the r-h-s of (43), we choose $\xi_0=\xi_1,\ A_0=A_1$, then it becomes

$$L^0S_1=\frac{-c\lambda q}{\vartheta}A_1^{\lambda}(\xi_1-\xi)_+^{\frac{(q+1)(\lambda-1)+1}{q-\lambda}}\left\{1-(A_1/A_*)^{q-\lambda}+\frac{q-\lambda}{(-c\lambda)\vartheta}\xi_1^{\frac{2q-\lambda(q+1)}{q-\lambda}}\right\}$$

since $A_1>A_*$ where

$$A_1=A_*\left(\frac{q-\lambda}{(-c\lambda)\vartheta}\right)^{\frac{1}{q-\lambda}}\xi_1^{\frac{2q-\lambda(q+1)}{q-\lambda}},$$ and $0\le\xi\le\xi_1$, then

$$L^0S_1\ge\frac{-c\lambda q}{\vartheta}A_1^{\lambda}(\xi_1-\xi)_+^{\frac{q(\lambda-1)+\lambda}{q-\lambda}}\left\{1-(A_1/A_*)^{q-\lambda}\right\}. \tag{44}$$

Therefore, from (42)-(44) then

$$\begin{cases}L\hbar(x,\ t)=0,\ \ for\ x>\xi_1 t^{\frac{q-\lambda}{\vartheta}},t>0.\\ L\hbar(x,\ t)\ge 0\ \ for\ \ x<\xi_1 t^{\frac{q-\lambda}{\vartheta}},t>0;\end{cases} \tag{45}$$

$$\hbar(x,\ 0)=u(x,\ 0)=0\ \ in\ \ \ 0\le x<x_0, \tag{46}$$

$$\hbar(0,\ t)=u(0,\ t)=0\ \ in\ \ \ 0\le t<+\infty, \tag{47}$$

where $x_0>0$ is arbitrary. We use (45)-(47) and by the comparison technique (lemma 2.1 in [5]) in the domain U. Thus the desired lower estimate in (34) follows. To estimate the l-h-s of (33), we choose $\hbar$ as in previous situation $\lambda<1,\ \alpha_0=q(q-1)^{-1}$. Choose $\xi_0=\xi_2, A_0=A_2$, then

$$L^0S_1=\frac{-c\lambda q}{\vartheta}A_2^{\lambda}(\xi_2-\xi)_+^{\frac{(q+1)(\lambda-1)+1}{q-\lambda}}\left\{1-(A_2/A_*)^{q-\lambda}+\frac{q-\lambda}{(-c\lambda)\vartheta}\xi_2^{\frac{2q-\lambda(q+1)}{q-\lambda}}\right\}.$$

It implies that

$$L^0S_1\le\frac{q}{\vartheta}A_2^{\lambda}(\xi_2-\xi)_+^{\frac{1}{q-1}}\left\{\xi_2-A^{q-1}\frac{\vartheta q^q}{(q-1)^{q+1}}+(-c)\lambda(\vartheta)A_2^{\lambda-1}(q-1)^{q-1}\xi_2^{\frac{q(\lambda-1)}{q-\lambda}}\right\}=0$$

is satisfied if $A_2>A_*$, where

$$A_2=\frac{\xi_2^{\frac{q(1-\lambda)}{q-\lambda}}}{(-c)\lambda(\vartheta)(q-1)^{q-1}q}\left(A_*^{q-1}\vartheta(q-1)^{q+1}q^q-\xi_2\right)^{\frac{1}{\lambda-1}}.$$

Thus, we obtain that

$$L\hbar \le 0 \ \ in \ \ x < \xi_2 t^{\frac{q-\lambda}{9}}, t > 0;$$

$$L\hbar = 0 \ in \ \ x > \xi_2 t^{\frac{q-\lambda}{9}}, t > 0;$$

with initial and boundary values conditions (46),(47). Moreover, the value ξ_2 is arbitrary to estimate u. So, we have proved the lower estimation under $A_2 > A_* > 0$ for $\lambda < 1$. Thus $\eta \sim t^{\frac{q-\lambda}{9}} \xi_*, \ as \ \ t \to 0^+$ is valid, from the upper and lower estimations and comparison theorem.

Governing of Advection Force

Based on the above techneque and preliminaries and from our previous study in [5], we keep going and formulate our problem to evaluate the solutions near interfaces in different regions. The advection term is dominated over the p-Laplacian type diffusion factor with expanding interface as shown in region (3), Figure 1.Now, let us prove the situation in the following theorem as follows:

Theorem 4.1. If $q(q-\lambda)^{-1} < \alpha < q(\lambda-1)^{-1}$, and $1 \le \lambda < (q+1)/2$ and the interface is initially shrinking as follows $\eta(t) \sim t^{\frac{1}{1-\alpha(\lambda-1)}} \rho_*; \ as \ \ t \to 0^+$, where

$$\rho_* = A^{\frac{\lambda-1}{1+\alpha(1-\lambda)}} (-c\lambda)^{\frac{1}{1-\alpha(\lambda-1)}} \{((1-\lambda)\alpha)^{\frac{1}{1-\alpha(\lambda-1)}} + ((1-\lambda)\alpha)^{\frac{\alpha(\lambda-1)}{1+\alpha(1-\lambda)}}\} \quad (48)$$

and $\rho_* = \rho_*(c, \ A, \ \lambda, \ \alpha) > 0$. Then the advection force dominates over the p-Laplacian type diffusion and the local solution

$$u \sim A[c\lambda G_\rho^{\lambda-1} t^{\frac{1}{1-\alpha(\lambda-1)}} + x]_+^\alpha, \quad (49)$$

along $x = \rho t^{\frac{1}{1-\alpha(\lambda-1)}}$; $c\lambda G_\rho^{\lambda-1} < \rho$. To prove theorem4.1, we have to consider the auxiliary results in the two lemmas as below:

Lemma 4.1. The CP(2)-(4) has a solution u with $1 \le \lambda < (q+1)/2, \ \alpha > 0$. Then, there exists $\xi_* > 0$ for all $\rho < \xi_*$, such that $\lim_{k\to\infty} u_k^{\pm} \doteq v_{\pm}$ solves a nonlinear advection equation

$$\frac{\partial v}{\partial t} + c\frac{\partial v^\lambda}{\partial x} = 0 \quad (50)$$

$$v = (A \pm)(-x)_+^\alpha \quad (51)$$

Lemma 4.2. Let CP(50)-(51) have a local weak solution u for $A > 0$ and if $q(q-\lambda)^{-1} < \alpha < q(\lambda-1)^{-1}$, $1 \le \lambda < (q+1)/2$, then as $t \downarrow 0$,

$$u(0, \ t) = o[t^{\frac{1}{1-\alpha(\lambda-1)}}]$$

for all $\rho_*(A, \ \alpha, \ \lambda) < \rho$; there exists $G_\rho > 0$, such that

$$u = A[c\lambda G_\rho^{\lambda-1} t^{\frac{1}{1-\alpha(\lambda-1)}} + x]_+^\alpha, \ \ 0 < t \le \sigma \quad (52)$$

along $x = \eta_\rho(t) = \rho t^{\frac{1}{1-\alpha(\lambda-1)}}$. In particular, if $\rho = \rho_*$, then (52) is satisfied $G_{\rho_*} = [c\lambda(\lambda-1)\alpha A^{1/\alpha}]^{\alpha/(1-\alpha(\lambda-1))}$, and $G_{\rho_*}^{\lambda-1} < \rho_*/(c\lambda)$.

Proof of Lemma 4.1. Firstly, from the previous techniques in proof of lemma 3.2 in [5]; $x \le 0$, then

$$(A -) \le u_0/(-x)_+^\alpha \le (A -), \ x \le x < +\infty \quad (53)$$

By the continuity of solution to the parabolic PDEs(2) with initial data(3), $\exists \sigma > 0$, depends on and $0 \le t \le \sigma$ we have

$$u_-(x, t) \le u(x, t) \le u(x, t); \quad (54)$$

From lemma 2.1 in (Aal-Rkhais et al., 2021), and (53)-(54), then

$$u_- \le u \le u \ , \ \ \text{for} \ \ 0 \le t \le \sigma, \ x \le x < \infty \quad (55)$$

Now, let us rescale the functions $u_\mp$ as follows,

$$u_k^{\pm} \doteq k u_\pm (k^{-1/\alpha} x \ , k^{(\alpha(\lambda-1)-1)/\alpha} t), \quad (56)$$

$u_\mp$ solves the following Cauchy Problem:

$$Lu = \frac{\partial u}{\partial t} - k^{\frac{\alpha(\lambda-q)+q}{\alpha}} \frac{\partial}{\partial x}(|\frac{\partial u}{\partial x}|^{q-1} \frac{\partial u}{\partial x}) + c\frac{\partial u^\lambda}{\partial x} = 0, \quad (57)$$

$$u(x, 0) = (A \pm)(-x)_+^\alpha. \quad (58)$$

Under the conditions of the definition 2.1 in (Aal-Rkhais et al., 2021), the local weak solution to the CP (57)-(58) is existed. Under the restriction $\alpha(\lambda-q)+q < 0$ the following formula should true

$$\lim_{k\to+\infty} u_k^{\pm} \doteq v_\pm ; (x, \ t) \in R \times [0, \ \infty) \quad (59)$$

Let $u_\mp$ solves the problem

$$Lu = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(|\frac{\partial u}{\partial x}|^{q-1} \frac{\partial u}{\partial x}) + c\frac{\partial u^\lambda}{\partial x} = 0, \quad (60)$$

$$u(x, \ t) = (A \pm)(-x)_+^\alpha \ ; \ u(-x, t) = 0, \quad (61)$$

$$u(x, \ 0) = (A \pm)(-x)_+^\alpha, \ |x| \ge |x|. \quad (62)$$

Also $u_k^{\pm}$ solves the following problem

$$Lu = \frac{\partial u}{\partial t} - k^{\frac{\alpha(\lambda-q)+q}{\alpha}} \frac{\partial}{\partial x}(|\frac{\partial u}{\partial x}|^{q-1} \frac{\partial u}{\partial x}) + c\frac{\partial u^\lambda}{\partial x} = 0, \quad (63)$$

$$u(k^{\frac{1}{\alpha}}x, t) = k(A\pm)(-x)_+^{\alpha};\ u(-k^{\frac{1}{\alpha}}x, t) = 0. \quad (64)$$

$$u(x,0) = (A\pm)(-x)_+^{\alpha},\ |x| < k^{\frac{1}{\alpha}}|x|, \quad (65)$$

where

$$D^k = \{(x,\ t);\ -k^{\frac{1}{\alpha}}|x| < x < k^{\frac{1}{\alpha}}|x|,\ 0 < t \le \sigma k^{\frac{1-\alpha(\lambda-1)}{\alpha}}\}$$

The DPs(60)-(62) and (63)-(65) have unique solutions. The finite speed propagation property, and let $\sigma > 0$ be chosen such that $0 \le t \le \sigma$ where $u(-x,\ t) = 0$. Applying lemma 2.1 in [5], and from (53),(54) and (55),(59) follows. Let us verify the sequence $\{u_k^{\pm}\}$ converges by the assumption

$$\hbar = (A+1)e^t(1+x^2)^{\alpha/2},\ x\in R,\ 0 \le t \le \sigma.$$

Then we have

$$L_k\hbar \equiv \frac{\partial\hbar}{\partial t} - k^{\frac{\alpha(\lambda-q)-q}{\alpha}}\frac{\partial}{\partial x}(|\frac{\partial\hbar}{\partial x}|^{q-1}\frac{\partial\hbar}{\partial x}) + c\frac{\partial\hbar^{\lambda}}{\partial x}$$

$$\equiv (A+1)e^t(1+x^2)^{\alpha/2}T;\ in\ D^k$$

where $T = 1 + H(x) + E$, and

$$H(x) = -k^{\frac{\alpha(\lambda-q)+q}{\alpha}}\alpha^q(1+x^2)^{\frac{\alpha(q-1)-2q}{2}}x^{q-1}(A+1)^{q-1}e^{t(q-1)}q\times((\alpha-2)x^2(1+x^2)^{-1}+1)$$

is continuous and for $k^{\frac{1}{\alpha}}|x| \ge |x|$, thus $H \approx O(k^{\frac{\alpha(\lambda-q)+q}{\alpha}})$ uniformly converges where $k\to\infty$. Also, $E = ce^{(\lambda-1)t}(1+x^2)^{\frac{\alpha(\lambda-1)-2}{2}}x(A+1)^{\lambda-1}\alpha\lambda$ and hence

$$L_k\hbar = (A+1)(1+x^2)^{\frac{\alpha}{2}}e^tT \ge \hbar$$

where $D_0^k = D^k \cap \{(x,\ t):\ 0 < t \le \sigma\}$. Then we have $R = O(k^{\frac{\alpha(\lambda-1)-1}{\alpha}})$ uniformly on D_0^k as $k\to\infty$. Thus, for $0 < \square 1$,

$$\hbar(x,\ 0) \ge u_k^{\pm}(x,\ 0)\ on\ |x| < k^{\frac{1}{\alpha}}|x|,$$

$$\hbar(\pm k^{\frac{1}{\alpha}}x,\ t) \ge u_k^{\pm}(\pm k^{\frac{1}{\alpha}}x,\ t),\ 0 \le t \le \sigma.$$

Since, $\exists\ k_0 = k_0(\alpha,\ \lambda)$ thus $\forall k \ge k_0$ and lemma 2.1 in [5] implies

$$u_k^{\pm} \le \hbar\ \ in\ \bar{D}_0^k.\ \ in\ \bar{D}_0^k. \quad (66)$$

Suppose that $P = \{(x,t):\ x\in R,\ 0 \le t \le t_0\}$, and $B \subset P$ where B is a compact region. If k_0 is a large and $k \ge k_0$, such that $D_0^k \supseteq B$. Inequality (66) implies that $\{u_k^{\pm}\}$ is uniformly bounded in B, such that there exists a subsequence $u_k^{\mp}$ converges to $v_{\pm}$ that means $\lim_{k'\to+\infty} u_k^{\mp} = v_{\mp}$ on P..

So, when $\lambda < 1$, p-Laplacian type diffusion term drops out as $k\to\infty$, and $v_{\pm}$ solves (50)-(51).

Proof of Lemma 4.2. Let u solve (50)-(51) and assume that $v = u^{\lambda-1}$, then v solves

$$\frac{\partial v}{\partial y} + c\lambda v\frac{\partial v}{\partial x} = 0,\ v(x,\ 0) = A^{\lambda-1}(-x)_+^{\alpha(\lambda-1)}. \quad (67)$$

Applying the characteristic method for (67). Let $z = v(x,\ y)$ and consider the characteristic system

$$\frac{dx}{dt} = c\lambda z,\ \ x(s,0) = s;$$

$$\frac{dy}{dt} = 1,\ \ y(s,0) = 0;$$

$$\frac{dz}{dt} = 0,\ \ z(s,0) = A^{\lambda-1}(-s)_+^{\alpha(\lambda-1)}.$$

Since the Jacobian determinant is nonzero at the point (s,0) then we obtain existence of local C^1-solution. The system of ODEs has the following solutions

$$x(s,\ t) = c\lambda z(s,\ t)t + s;\qquad y(s,\ t) = t;$$

$$z(s,t) = A^{\lambda-1}(-s)_+^{\alpha(\lambda-1)},$$

then we get the implicit solution of (50) that is

$$u = A[x + c\lambda u^{\lambda-1}t]_+^{\alpha} \quad (68)$$

then the solution along $x = \eta_\rho(t) = t^{\frac{1}{1-\alpha(\lambda-1)}}\rho$ is

$$u(\eta_\rho(t),\ t) = At^{\frac{\alpha}{1-\alpha(\lambda-1)}}[c\lambda t^{\frac{-\alpha(\lambda-1)}{1-\alpha(\lambda-1)}}u^{\lambda-1}(\eta_\rho(t),\ t) + \rho]_+^{\alpha}. \quad (69)$$

By assuming that $u(\eta_\rho(t),\ t) = t^{\frac{\alpha}{1-\alpha(\lambda-1)}}G_\rho$ is a solution of the implicit equation (69), then we get the following new implicit equation depends on a variable $G_\rho > 0$, where

$$G_\rho = A[\rho + c\lambda G_\rho^{\lambda-1}]^{\alpha} > 0,\ \ G_\rho > (\rho/(c\lambda))^{1/(\lambda-1)}. \quad (70)$$

To find the optimal value of (70), we assume that

$$M(G_\rho) = A^{\frac{-1}{\alpha}}G_\rho^{\frac{1}{\alpha}} - c\lambda G_\rho^{\lambda-1} = -\rho, \quad (71)$$

and under the restriction $1 \le \lambda < (q+1)/2, \alpha > 0$; and $M(0) = 0,\ \ M(+\infty) = +\infty$ then there exists a minimum value $G_{\rho_*} > 0$ such that $M'(G_{\rho_*}) = 0$, where $G_{\rho_*} = [c\lambda(\lambda-1)\alpha A^{\frac{1}{\alpha}}]^{\frac{\alpha}{1-\alpha(\lambda-1)}}$. The value of ρ_* is coming directly from (71). Then for $\rho < c\lambda G_\rho^{\lambda-1}$ the function (69) along $\eta_\rho(t) = \rho t^{\frac{\alpha}{1-\alpha(\lambda-1)}}$ solves the implicit equation (68) as well as G_ρ solves (70), therefore (52) holds.

Proof of theorem 4.1. The initial u_0 satisfies (4) and assume that $\varepsilon > 0$ is arbitrary small, $\exists x_\varepsilon < 0$ such that (52) is satisfied for $x_\varepsilon \le x < +\infty$. Then let us consider a function

$\hbar(x,t) = (A+\varepsilon)[Gt^{\frac{1}{1-\alpha(\lambda-1)}}(A+\varepsilon)^{\frac{\lambda-1}{1-\alpha(\lambda-1)}} + x]_+^{\alpha}$, where $G = c\lambda[c\lambda(\lambda-1)\alpha]^{\frac{\alpha(\lambda-1)}{1-\alpha(\lambda-1)}} > 0$. To estimate $L\hbar_\varepsilon$ in $D_1 = \{(x,\ t):\ x_\varepsilon < x < \eta_\rho(t),\ 0 < t < \sigma_1\}$ and

$\eta_\rho(t) = \rho t^{\frac{1}{\alpha(1-\lambda)}}$, where $\rho < \rho_*$ and ρ_* satisfies (48). Also, for $\sigma > 0$ is chosen such that $\eta_{\rho(\varepsilon)}(\sigma) = x_\varepsilon$. We get

$$L\hbar_\varepsilon = \hbar^{\lambda-\frac{1}{\alpha}}_{\varepsilon}\{T\};$$

$$T = -\alpha^q(\alpha-1)q(A+\varepsilon)^{\frac{\alpha(q-\lambda)+1}{\alpha}} S_+^{\alpha(q-\lambda)-q} - c\alpha\lambda(A+\varepsilon)^{\frac{1}{\alpha}}$$

$$+\alpha G(1-\alpha(\lambda-1))^{-1}(A+\varepsilon)^{\frac{(\alpha\lambda-\alpha)^2-\alpha(\lambda-1)+1}{1-\alpha(\lambda-1)}} t^{\frac{1}{1-\alpha(\lambda-1)}-1} M_+^{\alpha(1-\lambda)}$$

where $M = x + Gt^{\frac{1}{1-\alpha(\lambda-1)}}(A+\varepsilon)^{\frac{\lambda-1}{1-\alpha(\lambda-1)}}$ Choose $x_\varepsilon < 0$ with $|x_\varepsilon|$ sufficiently small, such that $|T| \ge \varepsilon/2$ in D_1 that implies $L\hbar_\varepsilon \ge \hbar^{\lambda-\frac{1}{\alpha}}_{\varepsilon}(\varepsilon/2)$. Moreover, let us prove a relevant upper estimate by considering

$\hbar(x,\ t) = A_1(t^{\frac{1}{\alpha(1-\lambda)}}\xi_1 + x)_+^{\alpha}$ in $N_{\sigma,\rho}$,

where, $N_{\sigma,\rho} = \{(x,\ t):\ 0 < t < \sigma;\ \eta_\rho < x < +\infty\}$, with $\rho \in [\rho_*, +\infty)$ and from (49) for arbitrary $\rho \ge \rho_*$ and for $\varepsilon > 0$, $\exists\sigma = \sigma(\varepsilon, \sigma) > 0$ such that

$$u(\eta_\rho(t),\ t) \le t^{\frac{\alpha}{1-\alpha(\lambda-1)}}(A+\varepsilon)[\rho-(A+\varepsilon)^{\frac{\lambda-1}{1-\alpha(\lambda-1)}}G]_+^{\alpha}; \quad (72)$$

Calculating $L\hbar$ in

$$N^+_{\sigma,\rho} = \{(x,\ t):\ \eta_\rho(t) < x < \xi_1 t^{\frac{1}{1-\alpha(\lambda-1)}},\ t < \sigma\},$$

$$L\hbar = t^{\frac{\alpha\lambda-1}{1-\alpha(\lambda-1)}}T;;$$

$$T = \Big[A_1\alpha(1-\alpha(\lambda-1))^{-1}(\rho-\xi_1)_+^{\frac{1-\lambda}{q-\lambda}}\left((1+\lambda-q)(q-\lambda)^{-1}\rho\right)-\xi_1\Big)$$

$$-A_1^{\rho-1}\left(\frac{q\lambda}{(q-\lambda)^2}\right)t^{\frac{\alpha(q-\lambda)-1}{1-\alpha(\lambda-1)}} - A_1^{\lambda}\frac{c\lambda}{q-\lambda}\Big](\rho-\xi_1)_+^{\frac{2\lambda-q}{q-\lambda}}$$

Let $\sigma = \sigma(\varepsilon)$, be so small, then we have

$$T \ge \left[t^{\frac{q\lambda}{1-\alpha(\lambda-1)}}A_1^q\left(\frac{q\lambda}{(q-\lambda)^2}\right) - A_1^{\lambda}\frac{c\lambda}{q-\lambda}\right](\rho-\xi_1)_+^{\frac{2\lambda-q}{q-\lambda}} \ge 0 \ in\ N^+_{\sigma,\rho}$$

hence

$$L\hbar \ge 0 \quad in \quad N^+_{\sigma,\rho}. \quad (73)$$

By using (73), we can apply [5, Lemma 2.1], in $N'_{\rho,\sigma} = N_{\rho,\sigma} \cap \{x < x_0\}$ for $\forall x_0 > 0$, such that

$$L\hbar = 0 \ in \ N_{\rho,\sigma}/\bar{N}^+_{\rho,\sigma}, \quad (74)$$

$$u(\eta_\rho(t),\ t) \le (A+\varepsilon)t^{\frac{\alpha}{1-\alpha(\lambda-1)}}[\rho-(A+\varepsilon)^{\frac{\lambda-1}{1-\alpha(\lambda-1)}}]_+^{\alpha}, \quad (75)$$

$$u(x,0) = \hbar(x,0) = 0;\ \ u(x_0,0) = \hbar(x_0,0) = 0 \quad (76)$$

since $x_0 > 0$ is arbitrary, from (73)-(76) and comparison theorem $\forall\rho \ge \rho_*,\ \varepsilon > 0$ there exists $\sigma = \sigma(\varepsilon,\ \rho)$ such that

$$u(x,t) \le A_1(\xi_1 t^{\frac{1}{1-\alpha(\lambda-1)}} + x)_+^{\alpha} \quad in \quad \bar{N}^+_{\rho,\sigma}. \quad (77)$$

Now, we shall prove the lower estimation by considering

$$\hbar_{-\varepsilon}(x,t) = (A-\varepsilon)[x + G(A-\varepsilon)^{\frac{\lambda-1}{1-\alpha(\lambda-1)}} t^{\frac{1}{1-\alpha(\lambda-1)}}]_+^{\alpha}.$$

From lemma 4.2, the formula (49) is satisfied along the optimal curve $\eta_\rho(t) = \rho t^{\frac{1}{1-\alpha(\lambda-1)}}$ where $\rho(-\varepsilon)$ satisfies (48). Estimate $L\hbar_{-\varepsilon}$ in

$$D_1 = \left\{(x,\ t): x_\varepsilon < x < \eta_{\rho(-\varepsilon)}(t); \eta_{\rho(-\varepsilon)}(t) = t^{\frac{1}{1-\alpha(\lambda-1)}}\rho(-\varepsilon)\right\}$$

Now to calculate $Lh_{-\varepsilon}$:

$$L\hbar_{-\varepsilon} = (\hbar_{-\varepsilon})_t - (|\hbar_{-\varepsilon}|^{q-1}(\hbar_{-\varepsilon})_x)_x + c(\hbar^{\lambda}_{-\varepsilon})_x$$

$$= -\hbar^{\lambda-\frac{1}{\alpha}}_{-\varepsilon}\Big[\alpha A((A-\varepsilon)^{\frac{1}{1-\alpha(\lambda-1)}}(1-\alpha(\lambda-1))^{-1}t^{\frac{1}{1-\alpha(\lambda-1)}-1}\hbar^{1-\lambda}_{-\varepsilon}$$

$$+\alpha^q q(1-\alpha^{-1})(A-\varepsilon)^{\frac{q}{\alpha}}\hbar_{-\varepsilon}^{(q-\lambda-1)-(\frac{q-1}{\alpha})} - (-c)\alpha\lambda(A-\varepsilon)^{\frac{1}{\alpha}}\Big].$$

If $|T| < \varepsilon/2$ in D_1 so

$L\hbar_\varepsilon > (D/2)\hbar^{\lambda-\frac{1}{\alpha}}$ (respectively) $L\hbar_{-} < -(3/2)\hbar^{\lambda-\frac{1}{\alpha}}_{-\varepsilon}$ in D_1,

$u(x,0) \le \hbar_\varepsilon(x,0)$ (respectively) $u_0(x) \ge \hbar_{-}(x,0),\ x \ge x_\varepsilon$.

Because of the continuity of functions $u,\ \hbar_{\mp\varepsilon}$; and $\sigma = \sigma(\varepsilon) \in (0, \sigma_1]$ such that $\hbar_\varepsilon(x,\ t) \ge u(x_\varepsilon, t)$ and since $\rho \ge \rho_*$ and $\varepsilon > 0$ are arbitrary numbers. (respectively) $\hbar_{-\varepsilon}(x,\ t) \le u(x_\varepsilon, t),\ 0 \le t \le \sigma.$ From lemma 2.1 in [5], it follows that

$$\hbar_{-\varepsilon} \le u \le \hbar,\ x \ge x_0$$

$$\eta_{\rho(-\varepsilon)}(t) \le \eta(t) \le \eta_{\rho(\varepsilon)}(t).$$

which implies (48). Evidently, (49) is valid along $\eta(t)$ such that

$$t^{\frac{1}{1-\alpha(\lambda-1)}}\rho(-\varepsilon) \le \eta(t) \le t^{\frac{1}{1-\alpha(\lambda-1)}}\xi_1,\ 0 \le t \le \sigma \quad (78)$$

with $\xi_1 = \rho(\varepsilon)$

Stationary Situation with WT Interface

In this part, we consider the parameters when p-Laplacian type diffusion and convection are in equilibrium. This parameters as shown in Figure 1 (region(4) in table 1) and it is presented in the following theorem.

Theorem 5.1. Let $\alpha < (\lambda-1)^{-1}q,\ 1<\lambda<(q+1)/2,$ and $\alpha > (q-1)^{-1}q,\ (q+1)/2 \le \lambda < q,$ then the advection force and the p-Laplacian type diffusion are in balance and the interface is in WT.

Proof of theorem 5.1. To estimate the upper-solution and sub-solution by assuming the functions

$\hbar_{+_\varepsilon} = (A+\varepsilon)(-x)_+^{\frac{q}{q-\lambda}}$ and $\hbar_{-_\varepsilon} = (A+\varepsilon)((\tau-t)/\tau)^{\lambda}(-x)_+^{\alpha_0}$, where $0<t<\tau$, and $\lambda>0$. To estimate upper case, let assume that $(\lambda-1)^{-1}q<\alpha$, then we directly get

$\hbar_{+_\varepsilon} = (A+\varepsilon)(-x)_+^{\frac{q}{q-\lambda}} \ge u(x,t),\ if\ \ |x| \ll 1.$

On the other hand, to get the lower estimate for u let $(\lambda-1)^{-1}q>\alpha$ such that,

$$\hbar_{-_\varepsilon} = ((\tau-t)/\tau)^{\lambda}(A-\varepsilon)(-x)_+^{\frac{q}{\lambda-1}}$$

$$u_0(x) = A(-x)_+^{\alpha} \ge A(-x)_+^{\frac{q}{\lambda-1}},$$

$\exists x_\varepsilon < 0$, for $x_\varepsilon < x < 0$, and by continuity of the solution there exist $\sigma_\varepsilon > 0$ such that $u(x_\varepsilon, t) \ge \hbar_{-_\varepsilon}(x_\varepsilon, t)$. Also, we get

$L\hbar_{-_\varepsilon} \le 0,$ in $P^+ = \{(x,t): x_\varepsilon < x < 0,\ 0<t<\sigma_\varepsilon\}.$

To calculate $L\hbar_{-_\varepsilon}$ when $\alpha_0 = q(\lambda-1)^{-1}$, it implies $L\hbar_{-_\varepsilon} \le 0\ in\ P^+$. From the comparison technique and lemma 2.1 in [5], it implies that $\hbar_{-_\varepsilon}$ is the lower estimation of u. Thus, we get $\hbar_{-_\varepsilon} \le u \le \hbar_{+_\varepsilon}$.

Conclusion

The model of nonlinear parabolic p-Laplacian type diffusion equation with convection term under the condition of a non-negative convection coefficient was discussed by using self-similar form to local solutions in irregular domains. Classification of behavior of the interface and the local solution near the interface was clarified and estimated in four parts. Firstly, the p-Lapacian type diffusion dominates over convection force with expanding interface(table1, region(1)). Secondly, the p-Lapacian type diffusion and advection in balance with expanding interface(table1, region(2)). Thirdly, The convection force is dominated over p-Lapacian type diffusion with expanding interface(table1, region(3)). Finaly, stationary solution and WT behavior of the interface were considered (table1, region(4)). The interest of this study is that model can be used in a variety of fields, including chemical process design, biophysics, plasma physics, quantum physics, and others.

Nomenclature

To clarify the meaning of some notations to readers, we add the following notations.

Notation	**Meaning**
ODE	**Order differential equation**
PDE	**Partial differential equation**
PME	**Porous medium equation**
IVP	**Initial value problem**
CP	**Cauchy problem**
DP	**Dirichlet problem**
WT	**Waiting time interface**
$\bar{\Psi}$	$\partial\Psi \cup \Psi$
$C(\bar{\Psi})$	**Banach space of continuous functions on $\bar{\Psi}$ with the norm $\|u\|_{C(\bar{\Psi})} = \max_{(x,t)\in\bar{\Psi}} \|u(x,t)\|$.**
$C^{2,1}_{x,t}(\bar{\Psi})$	**Banach space of continuous on $\bar{\Psi}$ with x-derivatives up to the order 2 , and continuous t-derivative up to the order 1.**
$f = O(g)$	**There exists a constant C such that $\|f(x)\| \le C\|g(x)\|$ for all x sufficiently closed to x_0.**
$(f)_+$	$\max\{f, 0\}$

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دراسة سلوك الحلول التقريبية لنموذج الانتشار بصيغة الــ p–لابلاسيا مع الحمل الحراري

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الخلاصة:

تم عمل هذا البحث لتقديم طريقة إعادة القياس للسماح من خلالها إنشاء حلول محلية موجبة لمناقشة مسألة كوشي لمعادلة p–لابلاسيا المكافئة غير الخطية مع قوة الحمل الحراري التي تمثل قانون الحفظ في فضاء ذو بُعد أحادي. نناقش جزء محدد لهذا النموذج الرياضي و ضمن قيود محددة في نطاق المعلمات ومعامل الحمل الحراري غير السالب باستخدام حلول التماثل الذاتي الذي يمثل السمة الرئيسية في دراستنا. تم تحديد مناطق الدراسة بالاعتماد على نطاق المعلمات لتحليل و ايجاد دوال الانترفيس و الحلول الضعيفة المحلية حول تلك الدوال في مجالات غير منتظمة. إن حلول مسائل كوشي لمعادلات p–لابلاسيا و الحمل الحراري تكون متساوية بصورة تقريبية الى حلول معادلات p–لابلاسيا او الحمل الحراري على شكل منفرد ضمن قيود محددة. الطرق التي تم اعتمادها في مناقشة النتائج، تقنية التضخم وطريقة المقارنة و طريقة المميز لتقدير دوال الانترفيس و الحلول المحلية لـــ مسائل كوشي و مسائل القيم الحدودية . مكن استخدام نتائج هذه الورقة لحل المشكلات في صناعات النفط والغاز، مثل تقدير حجم موارد النفط والغاز والتحكم فيها مع تطورها بمرور الزمن.

الكلمات المفتاحية: *المعادلات التفاضلية الجزئية المكافئة، معادلة p–لابلاسياً، قوة الحمل الحراري، دوال الانترفيس، الحلول المحلية الضعيفة.*