

A Space of Fuzzy Orderings

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Abstract

In this paper the chain length of a space of fuzzy orderings is defined, and various properties of this invariant are proved. The structure theorem for spaces of finite chain length is proved.

Spaces of Fuzzy Orderings

Throughout $X = (X, A)$ denoted a space of fuzzy orderings. That is, A is a fuzzy subgroup of abelian group G of exponent 2. (see [1] (i.e. $x^2 = 1, \forall x \in G$), and X is a (non empty) fuzzy subset of the character group $\chi(A) = \text{Hom}(A, \{1, -1\})$ satisfying

1. X is a fuzzy closed subset of $\chi(A)$.
2. $\exists$ an element $e \in A$ such that $\sigma(e) = -1 \forall \sigma \in X$.
3. $X^\perp := \{a \in A \mid \sigma(a) = 1 \forall \sigma \in X\} = 1$.
4. If f and g are forms over A and if $x \in D(f \oplus g)$ then $\exists y \in D(f)$ and $z \in D(g)$ such that $x \in D\langle y, z \rangle$.

Observe, by 3, that the element $e \in A$ whose existence is asserted by 2 is unique.

Also, $e \neq 1$ (since $\sigma(1) = 1 \forall \sigma \in X$).

Notice that for $a \in A$, the set $X(a) := \{\sigma \in X \mid \sigma(a) = 1\}$ is clopen (i.e. both closed and open) in X . Moreover, $\sigma(a) = -1 \Leftrightarrow \sigma(-a) = 1$ holds for any $\sigma \in X$ (by 2).

Definition 1

A forms f and g are said to be isometric (over X) if they have the same dimension and $\sigma(f) = \sigma(g) \forall \sigma \in X$. This is denoted by writing $f \cong g$ or $g \cong f$ (over X).

Note A form f is said to represent the element $X \in A$ (over X) if $\exists$ elements $x_1, \dots, x_n \in A$ such that $f \cong \langle x, x_2, \dots, x_n \rangle \cdot D(f)$ or $D(f, X)$ will be used to denote the set of elements of A which are represented by f in this sense.

Definition 2

A form f is said to be isotropic if $\exists x_3, \dots, x_n \in A$, such that $f \cong \langle 1, -1, x_3, \dots, x_n \rangle$. Notice, in particular, this implies $\dim(f) \geq 2$. A form which is not isotropic is said to be anisotropic, for any $x \in A$, $\langle x, -x \rangle \cong \langle 1, -1 \rangle$. any such form will be called a hyperbolic plane.

Theorem 1

The following are equivalent

- (i) $\forall x \in G, x \neq -1 \Rightarrow D\langle 1, x \rangle = \{1, x\}$.
- (ii) $X = \{\alpha \in \chi(A) \mid \alpha(-1) = -1\}$.

Proof: see [3].

A space of fuzzy ordering satisfying either of the equivalent conditions in theorem 1 will be referred to as a fan.

Corollary 1

Suppose X is a fan. Then every subspace of X is also a fan.

Proof: compare [3].

Recall, a space of fuzzy orderings (X, A) is said to be finite if X (or equivalently A) is finite fuzzy set; and two spaces of fuzzy orderings (X, A) and (X', A') are said to be isomorphic if there exists a group isomorphism $\alpha: A \longrightarrow A'$ such that the dual isomorphism $\alpha^*: \chi(A') \longrightarrow \chi(A)$ maps X' on to X .

Definition 3

The chain length of X (denoted $C1(X)$) is the maximum integer $k \geq 1$ such that $\exists a_0, \dots, a_k \in A$ satisfying $X(a_{i-1}) \subset X(a_i)$, $i = 1, \dots, k$ (or $C1(X) = \infty$ if no such maximum exists).

Remark 1

It is easily verified that $C1(X) = 1$ if and only if $|X| = 1$, and $C1(x) \leq 2$ if and only if X is a fan.

Recall that X is said to be decomposable if there exist non-empty subspaces X_i of X , $i = 1, 2$ such that $X = X_1 \oplus X_2$.

Let us denote by $\text{gr}(X)$ the translation fuzzy group of X , i.e.,

$$\text{gr}(X) = \{T \in \chi(x) \mid TX = X\}.$$

Thus $\text{gr}(X)$ is a closed fuzzy subgroup of $\chi(A)$.

Let the residue space of X be defined to be $X' = (X', A')$ where $A' = \text{gr}(X)^\perp \subseteq A$, and where X' denotes the image of X in $\chi(A')$ via restriction, X' is a space of fuzzy orderings. Moreover $\text{gr}(X') = 1$, and X is a fuzzy group extension of X' .

We can state the main theorem concerning spaces of finite chain length.

Theorem 2

Suppose $C1(X) < \infty$. Then either $|X| = 1$, or $\text{gr}(X) \neq 1$, or X is decomposable.

The proof of this key result is found in [4]. For now we concentrate on giving two important applications.

Theorem 3

Suppose a form f is anisotropic over a space of fuzzy ordering X_0 . Then there exists a finite subspace $X \subset X_0$ such that f is an isotropic over X .

Proof: Let $X = (X, A)$ be a subspace of X_0 chosen minimal subject to f is anisotropic over X .

Let $a_0, \dots, a_k \in A$ satisfy: $D\langle 1, a_{i-1} \rangle \subset D\langle 1, a_i \rangle$, $i = 1, \dots, k$. Thus $\langle 1, a_i \rangle \cong \langle a_{i-1}, a_{i-1} a_i \rangle$ and $a_{i-1} \neq a_i$ for $i = 1, \dots, k$. We may assume $a_0 = 1$, $a_k = 1$. Let $b_i = a_{i-1} a_i$. Thus $b_i \neq 1$, so $X(b_i)$ is a proper subspace of X . Thus f is isotropic over $X(b_i)$, i.e. there exists a form g_i of dimension $n - 2$ (where n denotes the dimension of f) such that $f \sim g_i$ over $X(b_i)$. Thus:

$f \otimes \langle 1, b_i \rangle \sim g_i \otimes \langle 1, b_i \rangle$ over X , so by addition

$$f \otimes \left(\sum_{i=1}^k \langle 1, b_i \rangle \right) : \sum_{i=1}^k g_i \otimes \langle 1, b_i \rangle \text{ (over } X) \quad \dots(1)$$

But using the assumptions on $a_0, \dots, a_k$ we see that (over X) $\langle b_0, \dots, b_k \rangle \cong \langle a_0 a_1, a_1 a_2, \dots, a_{k-1} a_k \rangle \cong \langle a_1, a_1 a_2, \dots, a_{k-1} a_k \rangle \cong \langle 1, a_2, a_2 a_3, \dots, a_{k-1} a_k \rangle \cong \dots \cong \langle 1, \dots, 1, a_k \rangle \cong \langle 1, \dots, 1, 1 \rangle$.

Substituting this in (1) yields

$$(2k - 2)f : \sum_{i=1}^k g_i \otimes \langle 1, b_i \rangle$$

Now f (and hence $(2k - 2)f$, by (3, corollary 3.5(ii)) is anisotropic over X , so comparing dimensions, and using (3, lemma 2.4), $(2k - 2)n \leq k(n - 2)(2)$, i.e., $k \leq \frac{1}{2}n$. This

proves $CL(X) < \infty$.

Now, we apply theorem 2. If $|X| = 1$ we are done.

Suppose $X = X_1 \oplus X_2$ where $X_i = (X_i, A/\Delta_i)$ is a non-empty subspace of X , $i = 1, 2$. Thus there exist elements $a_{i3}, \dots, a_{in} \in A$ such that

$f \equiv \langle -1, 1, a_{i3}, \dots, a_{in} \rangle$ over X_i , $i = 1, 2$.

Since $X = X_1 \oplus X_2$, the natural injection $A \longrightarrow A/\Delta_1 \times A/\Delta_2$ is surjective, so there exist $a_3, \dots, a_n \in A$ such that $a_j \equiv a_{ij} \pmod{\Delta_i}$, $3 \leq j \leq n$, $i = 1, 2$.

Then clearly $f \equiv \langle 1, -1, a_3, \dots, a_n \rangle$ over X , a contradiction. Thus X is indecomposable, so $\text{gr}(X) \neq 1$. Let $X' = (X', A')$ denote the residue space of X and decompose f as $f \equiv \pi_1 f_1 \oplus \dots \oplus \pi_s f_s$ where $f_1, \dots, f_s$ are forms over A' , and $\pi_1, \dots, \pi_s \in A$ are distinct modulo A' .

The assertion that f is anisotropic over X is equivalent to the assertion that each $f_1, \dots, f_s$ is anisotropic over X' .

There are two cases to be considered.

Suppose $S = 1$. Let Δ be any fuzzy subgroup of A such that A is the direct product $A = \Delta \times A'$, and let $Y = \Delta^\perp \cap X$. Then one verifies easily that $Y = (Y, A/\Delta)$ is a subspace of X and that $(Y, A/\Delta) \sim (X', A')$, this equivalence being induced by the natural isomorphism $A/\Delta \cong A'$. Thus, since f_1 is anisotropic over X' , it (and then $f \equiv \pi_1 f_1$) is anisotropic over Y . But, on the other hand $\text{gr}(X) \neq 1$, i.e. $A' \neq A$, i.e. $\Delta \neq 1$, i.e., $Y \subset X$. This contradicts the minimal choice of X .

Thus $S \geq 2$. It follows that each f_i has strictly lower dimension than f so by induction on the dimension, there exist finite subspaces $Z_1, \dots, Z_s \subseteq X'$ such that f_i is anisotropic over Z_i . Thus $f_1, \dots, f_s$ are all anisotropic over the subspace of X' generated by $Z_1, \dots, Z_s$. Denote this space by $Z' = (Z', A'/\Delta')$. Note Z' is still finite $Z = \Delta'^\perp \cap X$. Then $Z = (Z, A/\Delta')$ is a subspace of X , and a fuzzy group extension of $Z' = (Z', A'/\Delta')$. Moreover, since $\pi_1, \dots, \pi_s$ are distinct modulo A' , f is anisotropic over Z . Thus, by minimal choice of X , $Z = X$, i.e. $\Delta' = 1$, i.e., $Z' = X'$ is finite. However, X itself could be infinite (since, a priori, $\text{gr}(x)$ could be infinite). Define A'' to be the fuzzy subgroup of A generated by A' and $\pi_1, \dots, \pi_s$, and let X'' denote the restriction of X to A'' .

Thus (X, A) is a fuzzy group extension see [2] of (X'', A'') which, in turn, is a fuzzy group extension of (X', A') . Moreover (X'', A'') is finite, and f is anisotropic over X'' . Finally, let Δ be fuzzy subgroup of A so that $A = \Delta \times A''$, and let $Y = \Delta^\perp \cap X$. Then $Y = (Y, A/\Delta)$ is a subspace of X naturally equivalent to (X'', A'') . Thus Y is finite, and f is anisotropic over Y . Thus $Y = X$ is finite.

Notice, the condition $X(a_{i-1}) \subset X(a_i)$ is equivalent to $D\langle 1, a_i \rangle \subset D\langle 1, a_{i-1} \rangle$.

Theorem 4

(i) Suppose $X_i = (X_i, A/\Delta_i)$, $i = 1, \dots, n$ are subspaces of X generating X . Then: $CL(X)$

$$= \sum_{i=1}^n CL(X_i).$$

(ii) If, in addition, $X = X_1 \oplus \dots \oplus X_n$, then: $CL(X) = \sum_{i=1}^n CL(X_i)$.

(iii) If X is a fuzzy group extension of X' , then $CL(X) = CL(X')$, except in the case $|X'| = 1$ (in which case X is a fan).

Proof:

- (i) Suppose $X(a_{j-1}) \subset X(a_j)$, $j = 1, \dots, k$. Then for each i , $1 \leq i \leq n$, $X_i(a_{j-1}) \subset X_i(a_j)$. Moreover, since $X(a_{j-1}) \neq X(a_j)$, there exists i , $1 \leq i \leq n$ such that $X_i(a_{j-1}) \neq X_i(a_j)$. (for if $X_i(a_{j-1}) = X_i(a_j)$ for all $i \leq n$, then $a_j a_{j-1} \in \bigcap_{i=1}^n \Delta_i = 1$, i.e., $a_j = a_{j-1}$ a contradiction). This holds for $j = 1, \dots, k$. Simple counting yields $k \leq \sum_{i=1}^n \text{CL}(X_i)$, i.e., $\text{CL}(X) \leq \sum_{i=1}^n \text{CL}(X_i)$.
- (ii) We are assuming $X = \bigcup_i X_i$ and the natural homomorphism from A into $\pi_i A / \Delta_i$ is an isomorphism. Suppose $X_i(a_{ij-1}) \subset X_i(a_{ij})$, $j = 1, \dots, k$, $i = 1, \dots, n$. We may as well assume $a_{i,0} = -1$, and $a_{i,k_i} = 1$. Choose elements $b_{ij} \in A$ such that: $b_{ij} = 1 \pmod{\Delta_k}$ for $k < i$. $b_{ij} \equiv a_{ij} \pmod{\Delta_i}$, and $b_{ij} \equiv -1 \pmod{\Delta_k}$, for $k > i$. Notice that $X(b_{ij}) = (\bigcup_{s < i} X_s) \cup X_i(a_{ij})$. It follows that $X(b_{10}) \subset \dots \subset X(b_{1k_1}) = X(b_{20}) \subset \dots \subset X(b_{nk_n})$. There are $\sum k_i$ inequalities in this chain, so $\text{CL}(X) \geq \sum k_i$, and hence $\text{CL}(X) \geq \sum \text{CL}(X_i)$. The other inequality follows from (i).
- (iii) Suppose $|X'| \neq 1$. Suppose $X'(a_{i-1}) \subset X'(a_i)$, $i = 1, \dots, k$, with $a_i \in A'$. Then clearly $X(a_{i-1}) \subset X(a_i)$, $i = 1, \dots, k$. Thus $\text{CL}(X) \geq \text{CL}(X')$. Now suppose $D < 1, a_i > \subset D < 1, a_{i-1} >$, $i = 1, \dots, k$, with $a_1, \dots, a_k \in A$. We may assume $a_0 = -1$, $a_k = 1$. Then $a_1 \neq -1$. There are two cases to be considered
- 1st Case: Suppose $a_1 \notin A'$. It follows (from the definition of fuzzy group extension) that $D < 1, a_1 > = \{1, a_1\}$. Thus $K \leq 2$ in this case. Thus, since $|X'| \neq 1$, $\text{CL}(X') \geq 2 \geq k$.
- 2nd Case: Suppose $a_1 \in A'$. Then $D < 1, a_1 > \subset A'$ (e.g. by (5, lemma 4.9); notice $a_1 \neq -1$. Thus $a_1, \dots, a_k$ are all in A' , and $X'(a_{i-1}) \subset X'(a_i)$, $i = 1, \dots, k$. Thus $\text{CL}(X') \geq K$. Thus, in any case $\text{CL}(X') \geq K$, so $\text{CL}(X') \geq \text{CL}(X)$.

Lemma 1

Suppose $b, a_0, \dots, a_k \in A$ satisfy $D < 1, b > = \{1, b\}$, and $D < 1, a_{i-1} > \subset D < 1, b > \subseteq D < 1, a_i > \subset D < 1, b >$, $i = 1, \dots, k$. Then there exists $a'_i \in D < a_i, a_i b > = \{a_i, a_i b\}$ such that $D < 1, a'_{i-1} > \subseteq D < 1, a'_i >$, $i = 1, \dots, k$.

Proof: compare [6].

We now proceed to prove a deeper property of chain length.

Theorem 5

Suppose Y is a subspace of X . Then $\text{C1}(Y) \leq \text{C1}(X)$.

Proof: Suppose, to the contrary, $\text{cl}(Y) > \text{cl}(X)$. Then, in particular, $\text{cl}(X) < \infty$. Choose a subspace $Z \subseteq X$ minimal subject to (1) $Z \supseteq Y$ and (2) $\text{C1}(Z) \leq \text{C1}(X)$. To show such Z exists. Suppose $\{Z_i\}$ is a collection of subspaces of X satisfying (1) and (2) and linearly ordered by inclusion. Let $z' = \bigcap_i Z_i$. Then z' is a subspace of X satisfying (1). To show z' satisfies (2) suppose $a_0, \dots, a_k \in A$ satisfy $z'(a_j) \subset z'(a_{j-1})$, $j = 1, \dots, k$. Thus the set $M = \{\sigma \in X \mid \sigma < 1, a_j \geq \sigma < a_{j-1}, a_{j-1} a_j >, j = 1, \dots, k\}$ is open in X and contains Z' . By compactness, $Z_i \subseteq M$ for some i , so $Z_i(a_j) \subset Z_i(a_{j-1})$, $j = 1, \dots, k$. These inclusions must be strict, since $Z' \subseteq Z_i$. Thus $k \leq \text{CL}(Z_i) \leq \text{CL}(X)$, so $\text{CL}(Z') \leq \text{CL}(X)$. So Z exists as asserted. To simplify notation, we may assume $X = Z$. Let $Y = (Y, A/\Delta)$, since $Y \neq X$ ($\text{CL}(Y) > \text{CL}(X)$). It follows that $\Delta \neq 1$, so there exists $a \in \Delta$, $a \neq 1$. Thus $Y \subseteq X(a) \subset X$. Since $\text{CL}(X) < \infty$, there exists $b \in A$, $b \neq 1$, such that

$X(a) \subseteq X(b) \subseteq X$, $X(b)$ maximal. Thus $D\langle 1, b \rangle$ is minimal, i.e., $D\langle 1, b \rangle = \{1, b\}$. By the minimal choice of $X (=Z)$, it follows that $CL(X(b)) > CL(X)$. On the other hand it follows from lemma (1) that $CL(X(b)) \leq CL(X)$. This is a contradiction.

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الفضاء الضبابي الترتيب

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الخلاصة

يعرض البحث تعريف طول سلسلة في فضاء ضبابي الترتيب ومن ثم عرض خواص وبرهنتها، ولقد تم برهان المبرهنة الأساسية لطول السلسلة المنتهية وعرض بعض النتائج المتعلقة بالموضوع.