Solution of Differential-Algebraic Equations(DAEs) by Variational Iteration Method

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Abstract:

In this paper, we use variational iteration method (VIM) to handle the differential-algebraic equations(DAEs) systems. The results reveal that the variational iteration method (VIM) is more efficient than the ADM and it is very effective, convenient and quite accurate to systems of linear partial differential equations. It is predicted that VIM can be found widely applicable in engineering.

Keywords: Variational iteration method, Differential-Algebraic, Equations(DAEs).

1. Introduction

The first proposed to solve non-linear problems in quantum mechanics is by a general Lagrange multiplier method in 1978 [12]. In 1998, the Lagrange multiplier method is modified by He [5-9] into an iteration method that is called variational iteration method (VIM). The VIM can be applied to obtain series solutions and closed-form solutions. Besides its mathematical importance and its links to other branches of mathematics, it is widely used in all parts of modern sciences. The VIM does not need small parameter or linearization, the solution procedure is very simple by means of variational theory, and only few iterations lead to high accurate solutions which are valid for the whole solution domain. The variational iteration method changes the differential equation to a recurrence sequence of functions, where the limit of that sequence is considered as the solution of the differential equations. The main advantage of the method is that it can be applied directly to all types of nonlinear differential and integral equations, homogeneous or inhomogeneous, with constant or variable coefficients [1, 14-16]. Moreover, the proposed method is capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution.

Modelling with differential-algebraic equations plays a vital role, among others, for constrained mechanical systems, mathematical models of physical phenomenons, such as electrical circuits or mechanical multibody-systems , chemical reaction kinetics, Constrained variational problems, e.g. optimal control problems and Euler-Lagrange equations. DAI is handled, in the context of optimal control problems where the inequality path constraints in the discretized optimal control problem are handled by the optimizer as inequality constraints at each mesh point .The solving of a DAI system needing in safety envelope [13] , voltage control of electrical equipments [18], and in robotic path planning [17]. for example , all these types of systems arise, in circuit analysis, chemical process simulation, power systems, and many other applications.

In this work we applied the variational iteration method to approximation solution of the system differential-algebraic equations(DAEs) that The most general form of it is given by

$$F(t,x,x') = 0 \qquad (1)$$

with initial values $x(t0) = x0$, $x'(t) = x1$

where $f : R \times Rm \times Rm \to Rm$ is a given function.

If $\partial f/\partial x$ is invertible then x is also determined by ODE: $x' = g(t, x)$:

In recent years, much research has been focused on the numerical solution of systems of differentia-lalgebraic equations (DAEs) say Brenan et al. [10], Hairer et al. [4], and Petzold et al. [11].

The aim of this paper is to apply the variational iterations method to solve nonlinear initial value problems that do not apply the method for solving it, by using VIM exact solution and approximate solutions of the problems have been obtained in terms of convergent series with easily computable components. The organization of this paper is as follows; section 2 gives brief ideas of VIM. In section 3, two examples are given to illustrate the effectiveness and the useful of the variational iteration method. In section 4, we presented discussion of our work. Conclusions are presented in the last section.

2. Variational Iteration Method:

The variational iteration method changes the differential equation to a recurrence sequence of functions[8,12]. The limit of that sequence is considered as the solution of the partial differential equation. Now consider the we consider the following linear differential equation:

$$L(u)+R(u)=g(x), \qquad \ldots (2a)$$

with specified initial condition:

$$u_0 = u(0) \qquad \ldots (2b)$$

where L is a linear operator and R is a linear operator, and g(x) is an inhomogeneous term. According to the VIM [2]. The variational iteration method changes the partial differential equation to a correction functional in t-direction in the following form:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)[L(u_n(\tau)) + R(\tilde{u}_n(\tau) - \tilde{g}(\tau)]d\tau \qquad \dots (3)$$

where $\lambda(\tau)$ is called general Lagrange multiplier [1,15,16], which can be identified optimally via the variational theory and integration by parts. The iterates u_n denote the nth order approximate solutions, where n refers to the number of iterates. $\tilde{u}_n$ is considered as restricted variations so that their variations are zero, $\delta\tilde{u}_n = 0$ [8]. The successive approximation u_{n+1}, $n \geq 0$ of the solution $u(x,t)$ will be obtained by using the determined Lagrange multiplier and any selective function u_0.

To find the optimal value of $\lambda(\tau)$, we applied the restricted variations of correction functional (3) and integrating by part, noticing that $\delta u(0) = 0$, in the following form:

$$\begin{aligned} \delta u_{n+1}(x) &= \delta u_n(x) + \delta \int_0^x \lambda(\tau)[(u_n(\tau))_\tau + \\ &\quad R(\tilde{u}_n(\tau)) + N(\tilde{u}_n(\tau)) - \tilde{g}(x)]d\tau \\ &= \delta u_n(x) + \lambda\delta u_n(\tau)\big|_{\tau=x} - \int_0^x \lambda'\delta u_n(\tau)d\tau = 0 \end{aligned}$$

yields the following stationary conditions:

$$\delta(u_n) : \lambda' = 0$$

$$\delta(u_n) : 1 + \lambda\big|_{\tau = x} = 0$$

So, the Lagrange multiplier in this case can be identified as follows:

$\lambda = -1$

Consequently, we can write the equation (3) as the follows

$$u_{n+1}(x) = u_n(x) + \int_0^x (-1)[L(u_n(\tau)) + R(\tilde{u}_n(\tau) - \tilde{g}(\tau)]d\tau$$

So on, where by finding the nth order approximation. Finally summing up iterates to yields,

$$U_M = \sum_{n=0}^{M} u_n, \quad M \geq 1$$

The general solution obtained by the VIM can be written as:

$u(x,t) = \lim_{M\to\infty} U_M$

The Applications:

We will present numerical and analytical solutions for two models systems of differential-algebraic equations(DAEs).These examples are somewhat artificial in the sense that the exact answer is known in advance and the initial and boundary conditions are directly taken from this answer.

Example 1. We consider the following system of differential-algebraic equations(DAEs)

$$u' - x v' + u - (1+x)v = 0$$

$$v = \sin(x) \qquad \dots (4a)$$

with the following initial conditions:

$$u(0)=1 \quad ,v(0)=0, \qquad \dots (4b)$$

Eq.(4a) can be written

$$u' = -u + x\ \cos(x) + (1+x)\sin(x) \qquad \dots (4c)$$

To solve the equation (4c) using VIM, the correct functional (3) is given as:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)[(u_n(\tau)) + \tilde{u}(\tau) - \tilde{g}(x)]\, d\tau \qquad \dots (5)$$

$g(x) = x\ \cos(x) + (1+x)\sin(x)$

The inhomogeneous term .

where λ is a general Lagrange multiplier. The value of λ can be found by considering $(\tilde{u}_n)$ and $\tilde{g}(x)$ as restricted variations (i.e. $\delta\tilde{u}_n = \delta\tilde{g}(x) = 0$) in equation (5) then integrating the result by part to obtain $\lambda = -1$. Then the correction functional (5) becomes in the following formula:

$$u_{n+1}(x) = u_n(x,t) - \int_0^x [(u_n(\tau)) + u(\tau) - g(x)]\, d\tau \qquad \dots (6)$$

Consequently, the following approximants are obtained by using the above iteration formulas (6) with the initial approximations (4b):

u1 (x) = $1 - x + x\sin(x) + \sin(x) - x\cos(x)$

u2 (x) = $-1 - x + 2x\sin(x) + 2\cos(x) + \frac{1}{2}x^2$

u3 (x) = $1 + x + x\sin(x) + \frac{1}{2}x^2 - 3\sin(x) + x\cos(x) - \frac{1}{6}x^3$ … (7)

So on. The solution in a closed form is readily found to be [3].

$$u(x) = e^{-x} + x\sin(x) \quad \dots (8)$$

We explained the Comparison between the variational iteration method and Adomian Decomposition Method [3] for examples 1 and 2 in tables 1 and 2a,b.

Table 1: Comparison between the VIM and the ADM solutions with exact solution.

x	u_e	$u_{7(VIM)}$	$\left\|u_e - u_{7(VIM)}\right\|$	$\left\|u_e - u_{9(ADM)}\right\|$
0.1	0.914820759	0.914820759	0.1015E-10	0
0.2	0.858464619	0.858464619	0.6031E-10	0.10000E-9
0.3	0.829474282	0.829474280	0.1692E-8	0.16000E-8
0.4	0.826087382	0.826087369	0.1347E-7	0.15800E-7
0.5	0.846243429	0.846243347	0.8109E-7	0.94400E-7
0.6	0.887597120	0.887596784	0.3353E-6	0.40700E-6
0.7	0.947537684	0.947536580	0.1105E-5	0.14033E-5
0.8	1.02321383	1.02321075	0.3086E-5	0.41130E-5
0.9	1.11156387	1.11155628	0.7593E-5	0.10650E-4
1.0	1.20935042	1.20933354	0.1688E-4	0.25030E-4

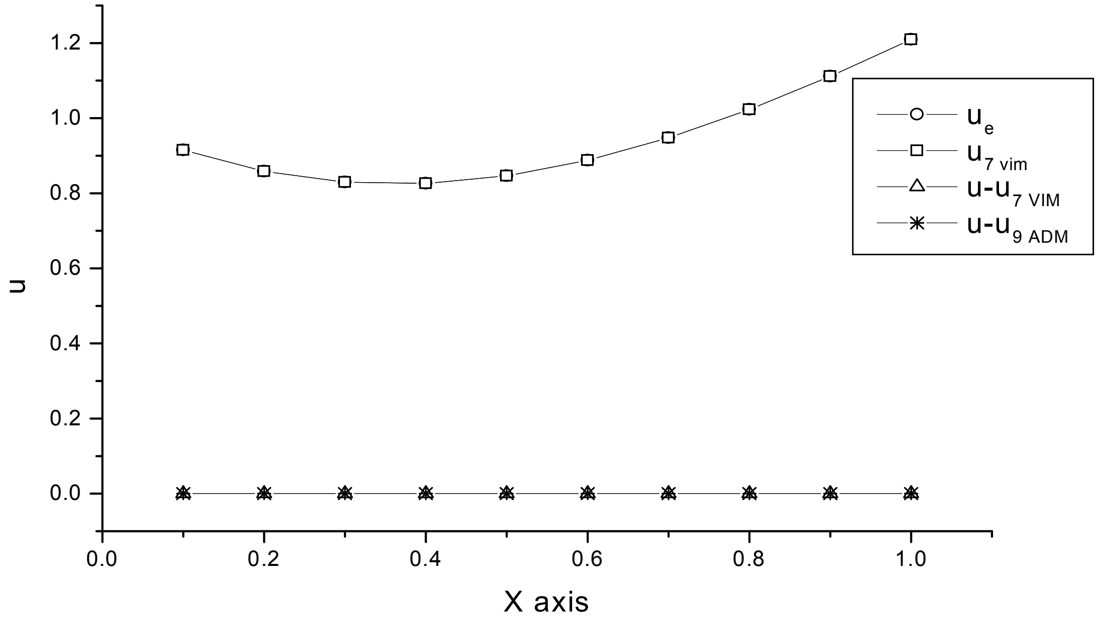

Figure (1) Comparison between exact solution and VIM solutions for u(x).

Example 2. We consider the following system of differential-algebraic equations(DAEs)

$$u' - x v' + x^2 w' + u - (1+x)v + (x^2+2x)w = 0$$
$$v' - xw' - v + (x-1)w = 0$$
$$w = \sin(x) \quad \ldots (9a)$$

with the following initial conditions:

$u(0)=1 \quad ,v(0)=1 \quad ,w(0)=0,$ … (9b)

Eq.(10)can be written

$$u' = -u + (2x+1)v - (2x^2+x)\sin(x)$$

$$v' = v + x\cos(x) - (x-1)\sin(x) \quad \ldots (9c)$$

To solve the system (9c) using VIM, the correct functional (3) is given as:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)[(u_n(\tau)) + \tilde{u}_n(\tau) - \tilde{g}_1(x)]\, d\tau$$

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\tau)[(v_n(\tau)) - \tilde{v}_n(\tau) - \tilde{g}_2(x)]\, d\tau \qquad \ldots(10)$$

Where g1(x) and g2(x) are the inhomogeneous term for (9 a,b)respectively where λ is a general Lagrange multiplier. The value of λ can be found by considering $(\tilde{u}_n)$, $(\tilde{v}_n)$ $\tilde{g}_1(x)$ and $\tilde{g}_2(x)$ as restricted variations (i.e. $\delta \tilde{u}_n = \delta \tilde{u}_n = \delta \tilde{g}_1(x) = \delta \tilde{g}_2(x) = 0$) in equation (10) then integrating the result by part to obtain $\lambda = -1$. Then the correction functional (10) becomes in the following formula:

$$u_{n+1}(x) = u_n(x) - \int_0^x [(u_n(\tau)) + u_n(\tau) - g_1(x)]\, d\tau$$

$$v_{n+1}(x) = v_n(x) - \int_0^x [(v_n(\tau)) - v_n(\tau) - g_2(x)]\, d\tau \qquad \ldots(11)$$

Consequently, the following approximants are obtained by using the above iteration formulas (11) with the initial approximations (9b):

u1 (x) = $5 + 2\cos(x)\, x^2 - 4\cos(x) - 4x\sin(x) - \sin(x) + x\cos(x) + x^2$

v1 (x) = $1 + x + x\sin(x) - \sin(x) + x\cos(x)$

u2 (x) = $1 + 6\sin(x) - 2x\cos(x) + \frac{3}{2}x^2 - 4x + \frac{1}{3}x^3$ … (12)

v2 (x) = $-1 + x + 2x\sin(x) + 2\cos(x) + \frac{1}{2}x^2$

u3 (x) = $-15 + 3\sin(x) - x\cos(x) + \frac{3}{2}x^2 - 2x + \frac{1}{3}x^3 - 2\cos(x)\, x^2 + 16\cos(x) + 10x\sin(x) + \frac{1}{6}x^4$

v3 (x) = $1 - x + x\sin(x) + \frac{1}{2}x^2 + 3\sin(x) - x\cos(x) + \frac{1}{6}x^3$

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So on. The solution in a closed form is readily found to be [3].

$$u(x) = e^{-x} + x e^{x}$$

$$v(x) = e^{x} + x\sin(x) \qquad \dots (13)$$

Table 2a: Comparison between the VIM and the ADM solutions with exact solution of the component u .

x	u_e	$u_{7(VIM)}$	$\lvert u_e - u_{7(VIM)}\rvert$	$\lvert u_e - u_{9(ADM)}\rvert$
0.1	1.01535451	1.01535451	0.6600E-8	0.1100E-7
0.2	1.06301130	1.06301129	0.3990E-8	0.1050E-6
0.3	1.14577586	1.14577585	0.2982E-8	0.1100E-5
0.4	1.26704992	1.26704990	0.1796E-7	0.6300E-5
0.5	1.43089129	1.43089131	0.1699E-6	0.2495E-4
0.6	1.64208291	1.64208216	0.7594E-6	0.7771E-4
0.7	1.90621219	1.90620939	0.2809E-5	0.2052E-3
0.8	2.22976170	2.22975279	0.8911E-5	0.4805E-3
0.9	2.62021246	2.62018773	0.2472E-4	0.1027E-2
1.0	3.08616126	3.08609922	0.6204E-4	0.2042E-2

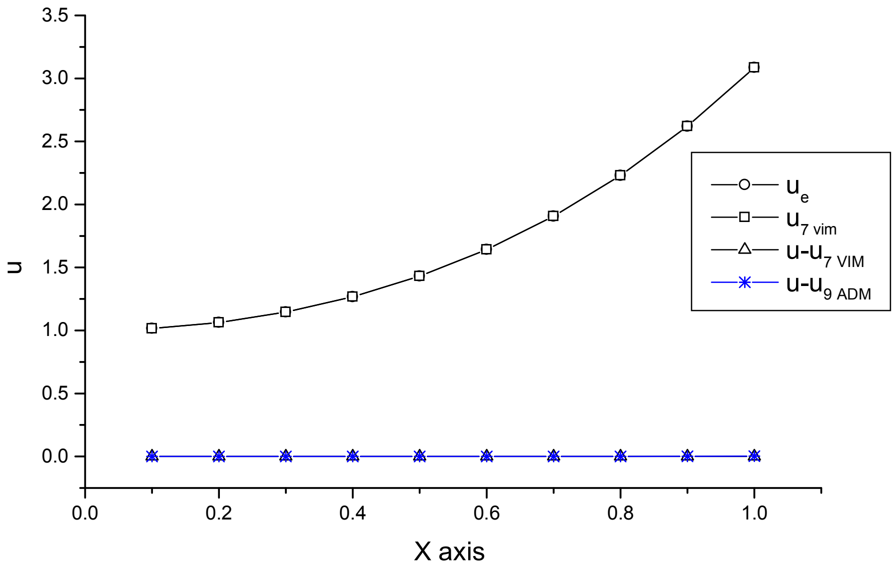

Figure (2a) Comparison between exact solution and VIM solutions for u(x).

Table 2b: Comparison between the VIM and the ADM solutions with exact solution of the component v .

x	v_e	$v_{7(VIM)}$	$\lvert v_e - v_{7(VIM)}\rvert$	$\lvert v_e - v_{9(ADM)}\rvert$
0.1	1.11515426	1.11515425	0.1898E-9	0.2E-8
0.2	1.26113662	1.26113662	0.9396E-9	0.92E-7
0.3	1.43851487	1.43851486	0.2307E-8	0.1061E-5
0.4	1.64759203	1.64759201	0.1852E-7	0.6065E-5
0.5	1.88843404	1.88843392	0.1139E-6	0.23551E-4
0.6	2.16090428	2.16090378	0.50061E-6	0.71650E-4
0.7	2.46470508	2.46470331	0.1769E-5	0.184228E-3
0.8	2.79942580	2.79942050	0.5293E-5	0.418878E-3
0.9	3.16459733	3.16458337	0.1395E-4	0.867168E-3
1.0	3.55975281	3.55971954	0.3327E-4	0.1667496E-2

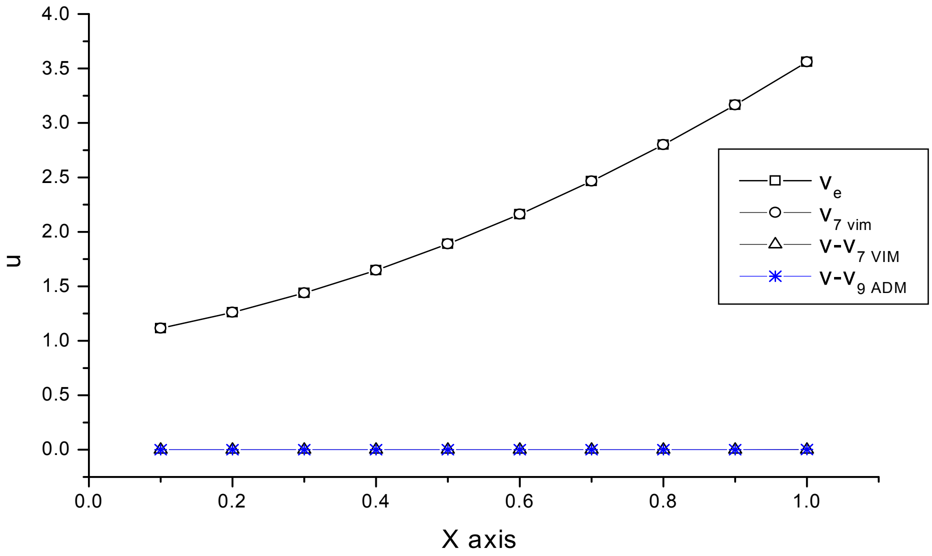

Figure (2b) Comparison between exact solution and VIM solutions for v(x).

4. **Discussion**

In this paper, we have used the VIM for solving the system of differential-algebraic equations(DAEs). The initial condition as a function of x solution region of this problem is bounded by $0 \leq x \leq 1$. We should be note that only 7 iterations were needed to obtain the approximately accurate solutions for the examples 1 and 2, i.e. when $n \geq 7$ the results are converging to the exact solution. The obtained results by using VIM are compared with the exact solution, which correspond to the various values of x for u and v. Also, we presented the absolute errors for the solution in several iterations and compared it with absolute errors of Adomian Decomposition Method for example 1 and example 2,all these errors are listed in Tables (1,2) and represented graphically in Figures (1,2a,b). The results show that the iterate approximation solutions obtained by using first seventh terms of this method are very well converged to the exact solution. From The tables, one can also see that the accuracy of this method increases with increasing the iterations. With other means, the errors are decreasing with increasing the number of iterations. The results we got from the VIM were better than the results obtained by (ADM) in accuracy.

5. Conclusions

In this paper, the variation iteration method has been successfully employed to obtain the approximate analytical solutions of differential-algebraic equations(DAEs) systems. The method has been applied directly without using linearization or any restrictive assumptions. The comparison of the numerical results of VIM with other solutions by using other methods show that the variational iteration method is a powerful mathematical tool to solving this type of and problems faster in convergence to exact solution.

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