Operators Approximation by General Family of Summation Baskakov-Type Preserving the Exponential Functions

Sara Adel Hussein, Ali J. Mohammad*

Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basra, Iraq.

*Corresponding author E-mail: ali.mohammad@uobasrah.edu.iq

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Abstract

The present paper is defining and studding a modification of the general family sequence of summation Baskakov-type operators. This modification is preserved that the functions 1 and e^{2ax}, where $a > 0$ is fixed. We show that the uniform convergence theorem of this sequence by using the modulus of continuity to the function being approximated. Finally, we introduce the asymptotic formula for the Voronovskaya-type theorem

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1. Introduction

In 1957 [2], the well-known classical Baskakov sequence are defined as

$$M_n(f,x)=\sum_{k=0}^{\infty}p_{n,k}(x)f\left(\frac{k}{n}\right),$$

where $p_{n,k}(x)=\binom{n+k-1}{k}x^k(1+x)^{-n-k}$ and $x\in[0,\infty)$.

After that, many papers for new classes and modifications for the Baskakov sequence were defined and studied. Here, we refer to some of them [3, 6, 8]. In 2007 [5], Duman and Ozarslan were inserted in Szasz-Mirakyan type operators preserving 1, e^{ax}, $a>0$ on $[0,\infty)$. On the other side, in 2014 [11], Mohammad and et al. have defined a general family of Baskakov sequence as follows

$$B_{n,k,r}(f,x)=\frac{1}{(n)_r}\sum_{k=0}^{\infty}\beta_{n,k,r}(x)f\left(\frac{k}{n}\right), \qquad (1)$$

where $r\in\ \mathbb{N}_0=\{0,1,\dots\}$, $\beta_{n,k,r}(x)=\frac{(n+k)_r}{(1+x)^r}p_{n,k}(x)$ and $(n)_r$ is the Pochhammer symbol. Clearly, $B_{n,k,0}(f,x)=M_n(f,x)$ it reduced to the classical Baskakov sequence.

In 2016 [1], Acar et al. suggested the modification of the classical Szasz-Mirakyan operators which preserved 1 and e^{2ax} , $a>0$. Also, they have investigated the uniform convergence and preserving some properties of this sequence. Finally, in 2017 [7], Gupta, Yilmaz, and Aral modified the Baskakov sequence which is preserving $1, e^{2ax}, a>0$ on $[0,\infty)$ and achieved a better approximation for the generalization of the sequence under observation.

Many papers were studied the preserving some functions by some sequences of linear positive operators are done. We refer here to [12,13].

This paper is constructing and studying a new sequence preserving 1 and e^{2ax}. Here, the construction, uniform convergence, error, and Voronovskaya-type asymptotic formula have been given for this sequence in Banach space.

2. Construction of the Sequence and Preliminary Results

In this paper, we assume that $g_i(t) = t^i$, $g_i^x(t) = (t-x)^i, i = 0,1,2$, where $t, x \in [0,\infty)$ and x is arbitrary but fixed. Suppose that $\theta_n = \theta_n(x)$ where $\theta_n \geq 0\ \forall x \in [0,\infty)$. We introduce a new sequence preserving the functions 1 and e^{2ax} for $a > 0$ as $f \in C[0,\infty)$, one has

$$B_{n,k,r,\theta}(f(t),x)$$

$$= \frac{1}{(n)_r(1+\theta_n)^r}\sum_{k=0}^{\infty}(n+k)_r\binom{n+k-1}{k}\theta_n^k\,(1+\theta_n)^{-n-k}f\left(\frac{k}{n}\right), \qquad (2)$$

where (2) is converging and satisfying the condition that is preserving the exponential function e^{2at}

$$B_{n,k,r,\theta}(e^{2at},x) = e^{2ax}\,. \qquad (3)$$

The sequence $B_{n,k,r,\theta}\,(f,x)$ is of positive and linear operators and it is reducing to the sequence $B_{n,k,r}(f,x)$ whenever $\theta_n = x$.

Now, using equations (2), (3) and for sufficiently large n, one has

$$B_{n,k,r,\theta}(e^{2at},x)$$

$$= \frac{1}{(n)_r(1+\theta_n)^{r+n}}\sum_{k=0}^{\infty}(n+k)_r\binom{n+k-1}{k}\theta_n^k(1+\theta_n)^{-k}e^{\frac{2ak}{n}}$$

$$= \frac{(n-1)!}{(n+r-1)!\,(1+\theta_n)^{r+n}}$$

$$\times\sum_{k=0}^{\infty}\frac{(n+k+r-1)!}{(n+k-1)!}\frac{(n+k-1)!}{k!\,(n-1)!}\left(\frac{\theta_n\,e^{\frac{2a}{n}}}{1+\theta_n}\right)^k$$

$$= \frac{1}{(1+\theta_n)^{r+n}}\left(\frac{1+\theta_n-\theta_n e^{\frac{2a}{n}}}{1+\theta_n}\right)^{-n-r}.$$

Hence, $B_{n,k,r,\theta}(e^{2at},x) = \left(1+\theta_n-\theta_n e^{\frac{2a}{n}}\right)^{-n-r}$. (4)

By (3), one has

$$\theta_n = \frac{e^{\frac{-2ax}{n+r}}-1}{1-e^{\frac{2a}{n}}}. \qquad (5)$$

Lemma 2.1. *We have* $\lim_{n\to\infty} B_{n,k,r,\theta}(e^{at},x) = e^{ax}$.

Proof. By (4) and (5), one has

$$B_{n,k,r,\theta}(e^{at},x) = \lim_{n\to\infty}\left(\frac{e^{\frac{a}{n}} + e^{\frac{-2ax}{n+r}}}{1+e^{\frac{a}{n}}}\right)^{-n-r}$$

Hence, $\lim_{n\to\infty} B_{n,k,r,\theta}\,(e^{at},x) = e^{ax}$. ■

Lemma 2.2. *We have* $B_{n,k,r,\theta}(1,x) = 1$, $B_{n,k,r,\theta}(g_1(t),x) = \frac{(n+r)}{n}\,\theta_n$

and $B_{n,k,r,\theta}\,(g_2(t),x) = \frac{n\theta_n^2+\theta_n+\theta_n^2+2r\theta_n^2}{n} + \frac{r\theta_n+r^2\theta_n^2+r\theta_n^2}{n^2}$.

Proof. For the function $e^{\gamma t}, \gamma \in R$, then from (4) and Maclaurin's expansion of $e^{\gamma t}$, one has

$$B_{n,k,r,\theta}(e^{\gamma t},x) = 1 + \left(\frac{(n+r)}{n}\,\theta_n\right) + \left(\frac{n\theta_n^2+\theta_n+\theta_n^2+2r\theta_n^2}{n} + \frac{r\theta_n+r^2\theta_n^2+r\theta_n^2}{n^2}\right)\frac{\gamma^2}{2!} + O(\gamma^3).\ ■$$

Lemma 2.3. *For the sequence* $B_{n,k,r,\theta}$, *we have*

(i) $B_{n,k,r,\theta}\,(1,x) = 1$,

(ii) $B_{n,k,r,\theta}\,(g_1^x(t),x) = \frac{(n+r)}{n}\,\theta_n - x$,

(iii) $B_{n,k,r,\theta}\,(g_2^x(t),x) = \left(\frac{(n+r)}{n}\theta_n - x\right)^2 + \frac{(n+r)\theta_n^2+(n+r)\theta_n}{n^2}$,

(iv) $\lim_{n\to\infty} n\,B_{n,k,r,\theta}(g_1^x(t),x) = -ax(1+x)$, (6)

(v) $\lim_{n\to\infty} n\,B_{n,k,r,\theta}(g_2^x(t),x) = x(x+1)$. (7)

Proof. By Lemma (2.2.) and Maclaurin's expansion of $e^{\gamma t}$,

one gets the properties *(i)*-*(iii)*.

The proof of consequence *(iv)*, is given using (5) and L'Hospital's rule, as

$$= \lim_{n\to\infty} n\left(\left(\frac{e^{\frac{-2ax}{n+r}}-1}{1-e^{\frac{2a}{n}}}\right) + \frac{r}{n}\left(\frac{e^{\frac{-2ax}{n+r}}-1}{1-e^{\frac{2a}{n}}}\right) - x\right) = -ax(1+x).$$

Using the same manner, one can evaluate the value of (*v*). ■

Now, suppose that $C^*[0,\infty)$ denotes to the Banach space containing every real-valued function that is continuous functions on $[0,\infty)$ with the property $\lim_{x\to\infty} f(x)$ exists and finite. This space is normed by the uniform norm $\|.\|_{C^*[0,\infty)}$.

Theorem 2.4. [4]. *Suppose that $A_n: C^*[0,\infty) \to C^*[0,\infty)$ be a sequence of linear positive operators satisfying $\lim_{n\to\infty} A_n(e^{-kt},x) = e^{-kx}$ uniformly convergence in the interval $[0,\infty)$ for $f \in C^*[0,\infty)$ and $k = 0,1,2$. Then $\lim_{n\to\infty} A_n(f(t),x) = f(x)$ uniformly in $[0,\infty)$. After that, Holhos* [9] *expanded Theorem (2.4.) and decided the result for such exponential functions as in the next Theorem.*

Theorem 2.5. [10]. *Let $A_n^*: C^*[0,\infty) \to C^*[0,\infty)$ be a sequence of positive linear operators satisfying that*

$\|A_n(1,x) - 1\|_{C^*[0,\infty)} = a_n$, $\|A_n(e^{-t},x) - e^{-x}\|_{C^*[0,\infty)} = b_n$ *and*

$\|A_n(e^{-2t},x) - e^{-2x}\|_{C^*[0,\infty)} = c_n$, *where a_n, b_n, c_n are sequences tend to zero as $n \to \infty$.*

$$\Rightarrow \|A_n(f,x) - f(x)\|_{C^*[0,\infty)} \le \|f\|_{C^*[0,\infty)} a_n + (2 + a_n)\omega^*\left(f, \sqrt{a_n + 2b_n + c_n}\right), f \in C^*[0,\infty),$$

where $\omega^*(f,\delta) = \max_{\substack{x,t>0 \\ |e^{-x}-e^{-t}|\le\delta}} |f(x) - f(t)|, \delta > 0$ *is the modulus of continuity. Further,*

$$|f(t) - f(x)| \le \left(1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2}\right)\omega^*(f,\delta).$$

3. Main Results

This section deals with the uniform convergence of the sequence $B_{n,k,r,\theta}(f,x)$ to the function $f \in C^*[0,\infty)$. Also, we describe the error of this sequence by the uniform norm $\|.\|_{C^*[0,\infty)}$ and $\omega^*(f,\delta)$.

Theorem 3.1. *For $f \in C^*[0,\infty)$. We have*

$$\left\|B_{n,k,r,\theta}(1,x) - 1\right\|_{C^*[0,\infty)} = a_n, \left\|B_{n,k,r,\theta}(e^{-t},x) - e^{-x}\right\|_{C^*[0,\infty)} = b_n,$$

and $\left\|B_{n,k,r,\theta}(e^{-2t},x) - e^{-2x}\right\|_{C^*[0,\infty)} = c_n$, *where a_n, b_n, c_n are sequences tend to zero as $n \to \infty$. Then, $B_{n,k,r,\theta}(f,x)$ converges uniformly to the function f and* $\left\|B_{n,k,r,\theta}(f,x) - f(x)\right\|_{C^*[0,\infty)} \le 2\omega^*\left(f, \sqrt{2b_n + c_n}\right)$.

Proof. From Lemma (2.2.), one has

$$\left\|B_{n,k,r,\theta}(1,x) - 1\right\|_{C^*[0,\infty)} = a_n = 0, \text{ therefore } a_n \to 0 \text{ as } n \to \infty.$$

Using (4), (5), and Maclaurin's expansion for e^{-t}, it follows that

$$B_{n,k,r,\theta}(e^{-t},x)$$

$$= e^{-x} + \frac{e^{-x}}{n}\left(\frac{(1+2a)x(1+x)}{2}\right) + \frac{e^{-x}}{24n^2}\begin{pmatrix}12a^2x^3 + 8a^2x^2 + 12ax^3 - 12xa^2 - 24axr + 3x^3 - 8a^2 \\ -24ax - 12rx - 2x^2 - 12a - 9x - 4\end{pmatrix} + O(n^{-3}).$$

Hence,

$$\left\|B_{n,k,r,\theta}(e^{-t},x) - e^{-x}\right\|_{C^*[0,\infty)}$$

$$= \left\|\frac{e^{-x}}{n}\left(\frac{(1+2a)x(1+x)}{2}\right) + \frac{1}{24n^2}e^{-x}\begin{pmatrix}12a^2x^3 + 8a^2x^2 + 12ax^3 - 12xa^2 - 24axr + 3x^3 - 8a^2 \\ -24ax - 12rx - 2x^2 - 12a - 9x - 4\end{pmatrix} + O(n^{-3})\right\|_{C^*[0,\infty)} = b_n,$$

Using the same manner, one can evaluate the images of e^{-2t}, as

$$\left\|B_{n,k,r,\theta}(e^{-2t},x) - e^{-2x}\right\|_{C^*[0,\infty)}$$

$$= \left\|\frac{e^{-2x}}{n}\Big(\big(2x(1+a)(1+x)\big)\Big) + \frac{2}{3n^2}e^{-2x}\begin{pmatrix}3a^2x^3 + 4a^2x^2 + 6ax^3 - 3axr + 6x^2a + 3x^3 - a^2 \\ -3ax - 3rx + 2x^2 - 3a - 3x - 2\end{pmatrix} + O(n^{-3})\right\|_{C^*[0,\infty)} = c_n.$$

That is b_n, c_n tend to zero as $n \to \infty$, so that $B_{n,k,r,\theta}(f,x)$ convergence uniformly to f. Also, by using Theorem (2.5.) we have $\left\|B_{n,k,r,\theta}(f,x) - f(x)\right\|_{C^*[0,\infty)} \le 2\omega^*\left(f, \sqrt{2b_n + c_n}\right)$.■

Now, we will check the asymptotic behaviors of the sequence $B_{n,k,r,\theta}(f,x)$ by proving the Voronovskaya-type Theorem.

Theorem 3.2. *Let* $f, f', f'' \in C^*[0,\infty)$. *Then, we have*

$\left|n\left(B_{n,k,r,\theta}(f,x) - f(x)\right) + ax(1+x)f'(x) - \frac{x(x+1)}{2}f''(x)\right| \le |p_n(x)||f'(x)| + |q_n(x)||f''(x)| + 2\big(2q_n + x(x+1) + r_n(x)\big)\omega^*(f'', \frac{1}{\sqrt{n}})$, *where*

$$p_n(x) = n\,B_{n,k,r,\theta}((t-x),x) + ax(1+x),$$

$q_n(x) = \frac{1}{2}(n\, B_{n,k,r,\theta}((t-x)^2, x) - x(1+x)),$

$r_n(x) = n^2 \sqrt{B_{n,k,r,\theta}((e^{-x} - e^{-t})^4, x)}\ \sqrt{B_{n,k,r,\theta}((t-x)^4, x)}.$

Proof. By Taylor's expansion we have,

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + h(t,x)(t-x)^2$$

where $h(t,x) = \frac{f''(\eta) - f''(x)}{2}$, and η is a number lying between x and t.

Now. Applying $B_{n,k,r,\theta}$ to both sides of Taylor's expansion, we get

$$B_{n,k,r,\theta}(f,x) = B_{n,k,r,\theta}(f(x),x) + f'(x)B_{n,k,r,\theta}((t-x),x) + \frac{f''(x)}{2}B_{n,k,r,\theta}((t-x)^2,x) + B_{n,k,r,\theta}(h(t,x)(t-x)^2,x).$$

We multiply both sides with n and applying Lemma (2.3), we get

$$\left| n\left(B_{n,k,r,\theta}(f,x) - f(x)\right) + ax(x+1)f'(x) - \frac{x(x+1)}{2}f''(x)\right|$$

$$\leq |p_n(x)||f'(x)| + |q_n(x)||f''(x)| + \left|nB_{n,k,r,\theta}(h(t,x)(t-x)^2,x)\right| \qquad (8)$$

where

$p_n(x) = n\, B_{n,k,r,\theta}((t-x),x) + ax(1+x),$

$q_n(x) = \frac{1}{2}\left(n\, B_{n,k,r,\theta}((t-x)^2,x) - x(1+x)\right).$

By (6) and (7), one can have that if $n \to \infty$, $p_n(x) \to 0$ and $q_n(x) \to 0$, for all $x \in [0,\infty)$. Now, an evaluation of the term $\left|nB_{n,k,r,\theta}(h(t,x)(t-x)^2,x)\right|$ is given as follows, $|f(t) - f(x)| \leq \left(1 + \frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\right)\omega^*(f,\delta)$, $\delta > 0$, we get

$$|h(t,x)| \leq \left(1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2}\right)\omega^*(f'',\delta).$$

Since, $|h(t,x)| \leq \begin{cases} 2\omega^*(f'',\delta), & |e^{-x} - e^{-t}| \leq \delta \\ 2\left(\frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\right)\omega^*(f'',\delta), & |e^{-x} - e^{-t}| > \delta \end{cases}$,

then, $|h(t,x)| \leq 2\left(1 + \frac{\left(e^{-x}-e^{-t}\right)^2}{\delta^2}\right)\omega^*(f'',\delta).$

If we use it (8), we get

$$\left|nB_{n,k,r,\theta}(h(t,x)(t-x)^2,x)\right| \leq 2n\omega^*(f'',\delta)B_{n,k,r,\theta}((t-x)^2,x)$$

$$+\frac{2n}{\delta^2}\omega^*(f'',\delta)B_{n,k,r,\theta}((e^{-x}-e^{-t})^2(t-x)^2,x).$$

Apply Cauchy-Schwarz inequality, we get

$$nB_{n,k,r,\theta}(h(t,x)(t-x)^2,x) \leq 2n\omega^*(f'',\delta)B_{n,k,r,\theta}((t-x)^2,x)$$

$$+\frac{2n}{\delta^2}\omega^*(f'',\delta)\sqrt{B_{n,k,r,\theta}((e^{-x}-e^{-t})^4,x)}\sqrt{B_{n,k,r,\theta}((t-x)^4,x)}.$$

Choose $\delta = \frac{1}{\sqrt{n}}$ and

$$r_n(x) = \sqrt{n^2B_{n,k,r,\theta}((e^{-x}-e^{-t})^4,x)}\sqrt{n^2B_{n,k,r,\theta}((t-x)^4,x)},$$

Getting

$$\left|n\left(B_{n,k,r,\theta}(f(t),x)-f(x)\right)+ax(1+x)f'(x)-\frac{x(x+1)}{2}f''(x)\right|$$

$$\leq |p_n(x)||f'(x)|+|q_n(x)||f''(x)|+2\left(2q_n+x(x+1)+r_n(x)\right)w^*\left(f'',\frac{1}{\sqrt{n}}\right). \blacksquare$$

Corollary 3.3. *Let* $f, f', f'' \in C^*[0,\infty)$. *Then, the inequality*

$$\lim_{n\to\infty} n\left(B_{n,k,r,\theta}(f(t),x)-f(x)\right) = -ax(1+x)f'(x)+\frac{x(x+1)}{2}f''(x)$$

holds for any $x \in [0,\infty)$.

Proof. $\left|n\left(B_{n,k,r,\theta}(f,x)-f(x)\right)+ax(1+x)f'(x)-\frac{x(x+1)}{2}f''(x)\right| = 0$ as $n\to\infty$.

Hence, the Corollary holds. ■

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تقريب بواسطة عائلة من نمط-مجموع مؤثر باسكاكوف تحفظ دوال أسية

سارة عادل حسين و علي جاسم محمد

جامعة البصرة، كلية التربية للعلوم الصرفة، قسم الرياضيات

المستخلص

ندرس في هذا البحث تحسين لعائلة عامة من نمط-مجموع مؤثر باسكاكوف نشير اليها بواسطة $B_{n,k,r,\theta}(f(t),x)$.

في هذه ألدراسة نقوم بتعريف هذه المتتابعة ألتي تحفظ ألدوال 1 و e^{2ax}، عندما $a > 0$ يكون ثابتا.

حيث تناقش البناء، ألتقارب ألمنتظم، خطأ تقريبي، صيغة مشابهة لنمط-فرونوفسكي باستخدام مقياس الاستمرارية للدالة المقربة ألمدروسة في فضاء بناخ.