On Weakly π–Regular Rings

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Abstract

The main purpose of this paper is to study right(left)weakly π-regular ring. Also we give some properties of a weakly π-regular ring, and the connection between such rings and CS-ring, MP-ring and Quasi-Duo ring, and finally we gives the discrimination of this ring with the generalized right principally injective modules and simple singular right R-module which is GP-injective .

1.Introduction:

Throughout this paper, R represents an associative rings with identity and all right(left) R-module are unitary. J(R) denotes the Jacobson radical of a ring R.. A ring R is said to be right(left) weakly regular if $a \in aRaR$ ($a \in RaRa$) for every $a \in R$, R is weakly regular ring if it is both right and left weakly regular [1]. A ring R is said to be π-regular ring if for every element a in R there exists a positive integer n=n(a) depending on a, such that $a^n \in a^n Ra^n$ [٤]. A ring R is said to be reduced if R has no non-zero nilpotent element [5]. A ring R is said to be right(left) generalized semi regular ring if for all $a \in R$, and there exists $b \in R$, and there exists a positive integers n such that $a^n = a^n b$ (ba^n), and $r(a^n) = r(b)$ ($l(a^n) = l(b)$). For every $a \in R$,r (a) and l (a) will stand respectively for right and left annihilators of a. A right R-Module is called generalized right principally injective (briefly right GP- injective) if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$, and any right R-homomorphism of a^nR in to M extends to one of R in to M[6].

2. Weakly π–Regular Rings (Basic properties):

A ring R is said to be right(left) weakly π-regular ring if for every element a in R, there exists a positive integer n=n(a)depending on a, such that:

$a^n \in a^nRa^nR$ ($a^n \in Ra^nRa^n$). A ring R is said to be weakly π-regular ring if it is both right and left weakly π-regular[4].

Clearly every π–regular ring is weakly π–regular ring, but the converse is not true, if R be a commutative weakly π–regular ring, then R is π–regular ring, also if R is reduced then every right weakly π–regular ring is left weakly π–regular ring.

Lemma 2.1: [3]

The J(R) Jacobson radical of a right weakly π–regular rings are nil.

Lemma 2.2: Let R be a right weakly π–regular ring, then $R=Ra^nR$ (for all a^n is not a zero divisor) and n is a positive integer depending on a.

Proof:

Let a^n is not a zero divisor of the ring R, since R is a right weakly π–regular ring, then $a^nR=(a^nR)^2= a^nRa^nR$, hence $a^n(R- Ra^nR)=0$ and $(R- Ra^nR) \subseteq r(a^n)$.

Since $r(a^n)=L(a^n)=0$, (a^n is not a zero divisor). Therefore $R=Ra^nR$.

Proposition 2.3: Let R be a ring with a condition that for all non-zero elements a in R,there exists a positive integer n=n(a)depending on a, such that $r(a^n) \cap Ra^nR=0$. Then:

1- If for every $b,c \in R$, $(1- ba^nc) \in Ra^nR$, then every right weakly π–regular ring is a left weakly π–regular ring.
2- Every right weakly π–regular ring is a right generalized semi regular ring.

Proof(1):

Let $0 \neq a \in R$ then there exists a positive integer n=n(a)depending on a, such that $r(a^n) \cap Ra^nR=0$, since R is a right weakly π–regular ring, then $a^n \in a^nRa^nR$, let $b,c \in R$, then $a^n= a^nba^nc$, then $a^n(1- ba^nc)=0$, then $(1- ba^nc) \in r(a^n)$.

Since for every $b,c \in R$ $(1- ba^nc) \in Ra^nR$, then there exist $x \in Ra^nR$ such that:

$1- ba^nc=x \in Ra^nR$, then $1= ba^nc+x \in Ra^nR$, therefore $R = Ra^nR$.

Then $(1- ba^nc) \in r(a^n) \cap Ra^nR=0$, $(1- ba^nc) = 0$, hence $1= ba^nc$, implies $a^n = ba^nca^n$.

Therefore $a^n \in Ra^nRa^n$, then R is a left weakly π–regular rings.

Proof(2):

Let $0 \neq a \in R$ then there exists a positive integer n=n(a)depending on a, such that $r(a^n) \cap Ra^nR=0$, since R is a right weakly π–regular ring, then $a^n \in a^nRa^nR$, let $b,c \in R$, then $a^n = a^nba^nc$, let $d = ba^nc$, implies $a^n = a^nd$, we must show that $r(d) = r(a^n)$, let $x \in r(d)$, then dx=0, (multiply by a^n from left) then $(a^n d)x=0$.

Since ($a^n= a^nd$) then $a^nx = 0$, therefore $x \in r(a^n)$, then $r(d) \subseteq r(a^n)$.

Let $x \in r(a^n)$, then $a^n x=0$, since $d= ba^nc \in Ra^nR$, and $dx \in Ra^nR$, and $a^n= a^nd$, then $a^ndx =0$, implies $dx \in r(a^n)$, therefore $dx \in r(a^n) \cap Ra^nR = 0$, then dx =0, $x \in r(d)$,we get that $r(a^n) \subseteq r(d)$, therefore R is a right generalized semi regular ring.

Lemma 2.4: Let R be a right weakly π–regular ring, then Cent(R) is π– regular.

Proof:

Let $a \in$ Cent(R), since R is a right weakly π–regular ring, then for every element a in R, there exists a positive integer n=n(a)depending on a, such that:

$a^n \in a^nRa^nR$, let $b,c \in R$, then $a^n = a^nba^nc = a^nba^{n-1}ca$, and so we have that $a^n = a^nbca^n$.

Let d = bc, implies $a^n = a^nda^n$, therefore Cent(R) is π–regular.

Theorem 2.5:

Let I be a right weakly regular ideal and R/I be a right weakly π–regular ring, then R is a right weakly π–regular ring.

Proof:

Let R/I be a right weakly π–regular ring, then for all a in R there exists a positive integer n=n(a)depending on and there exists r, sbelongs to R such that:

$a^n + I = (a^n + I)(r + I)(a^n + I)(s + I)$ implies that $a^n + I = a^n r a^n s + I$, then $a^n - a^n r a^n s \in I$, since I is a right weakly regular ideal, then there exists t, w in I, such that:

$a^n - a^n r a^n s = (a^n - a^n r a^n s) t (a^n - a^n r a^n s) w$.

$a^n - a^n r a^n s = (a^n t - a^n r a^n s t)(a^n w - a^n r a^n sw)$.

$a^n - a^n r a^n s = a^n t a^n w - a^n t a^n r a^n sw - a^n r a^n s t a^n w + a^n r a^n s t a^n r a^n sw$.

$a^n = a^n r a^n s + a^n t a^n w - a^n t a^n r a^n sw - a^n r a^n s t a^n w + a^n r a^n s t a^n r a^n sw$.

$a^n = a^n (r a^n s + t a^n w - t a^n r a^n sw - r a^n s t a^n w + r a^n s t a^n r a^n sw)$

Since $(r a^n s + t a^n w - t a^n r a^n sw - r a^n s t a^n w + r a^n s t a^n r a^n sw) \in Ra^nR$.

Then $a^n = a^n (r a^n s + t a^n w - t a^n r a^n sw - r a^n s t a^n w + r a^n s t a^n r a^n sw) \in a^nRa^nR$, hence there exist g,h in R such that:

$ga^nh = r a^n s + t a^n w - t a^n r a^n sw - r a^n s t a^n w + r a^n s t a^n r a^n sw$.

Therefore, R is a right weakly π–regular ring.

3.The Connection Between Weakly π–Regular Rings and Other Rings:

In this section, we study the connection between Weakly π–regular rings and CS-ring, MP-ring, Quasi-Duo ring, the right R-Module which generalized right principally injective (briefly right GP- injective module), and simple singular right R-module which is GP-injective .

Recall that a ring R is said to be right (left) CS-ring if every non-zero right (left) ideal is essential in a direct summand [2], equivalently, every right (left) closed ideal is a direct summand, clearly every maximal right ideal is right closed.. An ideal I of a ring R is said to be right(left) pure ideal if for all $a \in I$, there exists $b \in I$ such that a=ab(ba), An ideal I is said to be right(left) Generalized pure ideal if for all $a \in I$, there exists $b \in I$, and there exists a positive integers n such that $a^n = a^n b(ba^n)$.A ring R is said to right(left) MP-ring if every maximal right(left) ideal is a left(right) pure ideal. A ring R is called right(left) Quasi-Duo ring if every maximal right(left) ideal of R is two sided ideal[3].

Theorem 3.1:

Let R be a right CS-ring with for all element a in R, L (a^n) is two sided ideal then R is a left weakly π–regular ring.

Proof:

Let R be a right CS-ring and to prove that R is a right weakly π–regular ring. Let $Ra^nR + L(a^n) = R$, if not then there a maximal right ideal M containing $Ra^nR + L(a^n)$, such that $Ra^nR + L(a^n) \subseteq M$, since R is a right CS-ring, then every maximal right ideal of R is a direct summand such that: $M \oplus K = R$, where K is a right ideal of R.

Hence $(Ra^nR + L(a^n) \cap K) \subseteq M \cap K = 0$, then $(Ra^nR + L(a^n)) \cap K = 0$, also $Ra^nR \cap K = 0$, $L(a^n) \cap K = 0$, then $Ka^n \in Ra^nR$ and $Ka^n \in K$, therefore $Ka^n \in Ra^nR \cap K = 0$, then $Ka^n = 0$, then $K \subseteq L(a^n)$, a contradiction. Thus $Ra^nR + L(a^n) = R$.

Let b,c be any two elements in R, and $d \in L(a^n)$ and $1 \in R$, then $ba^nc + d = 1$, then $ba^nca^n + da^n = a^n$. Therefore $ba^nca^n = a^n$, then $a^n \in Ra^nRa^n$, then R is a left weakly π–regular ring.

Theorem 3.2:

Let R be a right MP-ring with for all element a in R there exists a positive integer n such that $L(a^n) \subseteq r(a^n)$, then R is a right weakly π–regular ring.

Proof:

Let R be a right MP-ring and to prove that R is a right weakly π–regular ring. Let $Ra^nR + r(a^n) = R$, if not, then there exists a maximal right ideal M containing $Ra^nR + r(a^n)$ such that $Ra^nR + r(a^n) \subseteq M$, since R is a right MP-ring, then every maximal right ideal of R is a left pure ideal (since every left pure ideal is generalized left pure ideal). Then for all $a \in M$ there exists $b \in M$ and a positive integer n such that $a^n = ba^n$, then

$a^n - ba^n = 0$, implies $(1-b) a^n = 0$, then $(1-b) \in L(a^n) \subseteq r(a^n) \subseteq M$, hence $1 \in M$, a contradiction .

Hence $Ra^nR + r(a^n) = R$. Let b,c be any two elements in R, and $d \in r a^n$) and $1 \in R$, then $ba^nc + d = 1$, then $a^nba^nc + a^nd = a^n$, therefore $a^nba^nc = a^n$, then $a^n \in a^nRa^nR$, then R is a right weakly π–regular ring.

Corollary 3.3:

Let R be a ring with every maximal right ideal is a right generalized pure ideal then R is a right weakly π–regular ring.

Proof:

We must show that R is a right weakly π–regular ring, let $Ra^nR + r(a^n) = R$, if not then there a maximal right ideal M containing $Ra^nR + r(a^n)$ such that $Ra^nR + r(a^n) \subseteq M$, since every maximal right ideals is a right generalized pure ideal, then for all $a \in M$ there exists $b \in M$ and a positive integer n such that $a^n = a^n b$, then $a^n (1-b) = 0$.

Then $(1-b) \in r(a^n) \subseteq M$, then $1 \in M$, a contradiction. Therefore $Ra^nR + r(a^n) = R$, let b , c be any two elements in R, and $d \in r(a^n)$ and $1 \in R$, then $ba^nc + d = 1$.

Then $a^nba^nc + a^nd = a^n$, therefore $a^nba^nc = a^n$, then $a^n \in a^nRa^nR$, then R is a right weakly π–regular ring.

Lemma 3.4: Let R be a right weakly π–regular ring, then every two sided ideal is a right generalized pure ideal.

Proof:

Let R be a right weakly π–regular ring, , then $a^n \in a^nRa^nR$,there exists $b,c \in R$, such that $a^n = a^nba^nc$, Let I be a two sided ideal of R. Then for all $a \in I$, there exists a positive integers n such that $a^n \in I$(I is an ideal of R), then $ba^nc \in I$ (I is a two sided ideal of R).

Let $d = ba^nc \in I$, then $a^n = a^n d$, therefore I is a right generalized pure ideal.

Lemma 3.5: If R is quasi-duo ring and left MP-ring, then R is a right weakly π–regular ring.

Proof:

We must show that R is a right weakly π–regular ring, that is meaning $Ra^nR + r(a^n) = R$, if not then there a maximal ideal M containing $Ra^nR + r(a^n)$ such that $Ra^nR + r(a^n) \subseteq M$, since R is left MP-ring (every maximal left ideal is a right pure ideal)and every right pure ideals are right generalized pure ideals, then there exists a and b in M and there exists a positive integer n such that $a^n = a^n b$,

then a^n (1- b)=0 , then (1-b)$\in$r(a^n)$\subseteq$M, then 1$\in$M a contradiction. Hence Ra^nR+ r(a^n)=R. Let b , c be any two elements in R, and
d$\in$ r(a^n) and 1$\in$R, then ba^nc+ d=1.
Then a^nba^nc + a^nd=a^n, therefore a^nba^nc =a^n, then a^n $\in$ a^nRa^nR, then R is a right weakly π–regular ring.

Lemma 3.6: [7]
If R is left(right) quasi-duo ring and J(R)=0, then R is reduced ring.

Corollary 3.7: If R is quasi-duo ring and J(R)=0 and left MP-ring, then R is a weakly π–regular ring.

Proof:
By using (Lemma 3.5) R is a right weakly π–regular ring, by using (Lemma 3.6) R is reduced ring, then R is a weakly π–regular ring.

Theorem 3.8: If R is ring with every simple right R-module is GP-injective, then R is a right weakly π–regular ring.

Proof:
Let Ra^nR+ r(a^n)=R, if not then there a maximal right ideal M containing Ra^nR+ r(a^n) such that: Ra^nR+ r(a^n) $\subseteq$M.
We define f: a^nR$\longrightarrow$R/M by f(a^nr)=r+M, for every element r in R, f is a well defined function. Indeed if $a^n r_1$,$a^n r_2$ belong to a^nR, and $a^n r_1$=$a^n r_2$,then $a^n r_1$ -$a^n r_2$ =0, implies that a^n(r_1 - r_2)=0, then (r_1 - r_2) $\in$r(a^n) $\subseteq$M, hence (r_1 - r_2) $\in$M, then r_1 +M= r_2 +M.
Then f(a^n r_1) = f(a^n r_2).
Since R/M is simple module then R/M is GP-injective module, then there exists f: a^nR$\longrightarrow$R/M such that f(a^nr)=r+M = (c+M) a^nr, then r+M = ca^nr +M, let(r=1), then 1+M= ca^n +M, from this, we get 1- c$a^n$$\in$M. Since c$a^n$$\in$ Ra^nR$\subseteq$M, then 1$\in$M.
A contradiction. Hence Ra^nR+ r(a^n)=R, let b,c be any two elements in R, and d$\in$ r(a^n) and 1$\in$R, then ba^nc+ d=1.
Then a^nba^nc+ a^nd=a^n, therefore a^nba^nc =a^n, then $a^n$$\in$ a^nRa^nR, then R is a right weakly π–regular ring.

Lemma 3.9: Let R be a ring with for all elements a in R there exists a positive integer n such that L(a^n)$\subseteq$r(a^n), then Ra^nR+ r(a^n) is essential right ideal of R.

Proof:
Let a be any element of a ring R, and assume that Ra^nR+ r(a^n)$\cap$I=0, where I be a non zero right ideal of R, then Ra^nR$\cap$I=0, r(a^n)$\cap$I=0 , since I$a^n$$\subseteq$R$a^nR, Ia^n$$\subseteq$ I, then I$a^n$$\subseteq$R$a^nR\cap$I=0, then I$a^n$=0, then I$\subseteq$ L(a^n)$\subseteq$r(a^n), I$\subseteq$r(a^n), then I=0, a contradiction. Therefore, Ra^nR+ r(a^n) is essential right ideal of R.

Theorem 3.10: If R is a ring with every simple singular right R-module is GP-injective, and for all element a in R, L(a^n)$\subseteq$r(a^n), where n is a positive integers then R is a right weakly π–regular ring.

Proof:
We must show that R is a right weakly π–regular ring, that is meaning that Ra^nR+ r(a^n)=R, if not then there is a maximal right ideal M containing Ra^nR+ r(a^n) such that Ra^nR+ r(a^n) $\subseteq$M, then by (Lemma 3.9), Ra^nR+ r(a^n) is essential right ideal of R, then R/M is GP-injective module, then there exist a positive integers n such that any R-homomorphism of a^n R in to R/M extends to one of R in to R/M, we define f: a^nR$\longrightarrow$R/M by f(a^nr)=r+M, for every elements r in R, f is a well defined function because that if $a^n r_1$,$a^n r_2$ belongs to a^nR, and $a^n r_1$=$a^n r_2$, then $a^n r_1$ -$a^n r_2$ =0, implies a^n(r_1 - r_2)=0.
Then (r_1 - r_2)$\in$r(a^n)$\subseteq$M, hence (r_1 - r_2)$\in$M, then r_1 +M= r_2 +M, then f(a^n r_1)= f(a^n r_2), then there exists f: a^nR$\longrightarrow$R/M such that f(a^nr)=r+M = (c+M) a^nr, then r+M = ca^nr +M.
Let(r=1), then 1+M= ca^n +M, Thus 1- c$a^n$$\in$ M, and hence 1$\in$M , a contradiction.
Therefore Ra^nR+ r(a^n)=R , let b,c be any two elements in R, and d$\in$ r(a^n) and 1$\in$R, then ba^nc+ d=1.
Then a^nba^nc+ a^nd=a^n, therefore a^nba^nc =a^n, then $a^n$$\in$$a^nRa^n$R, then R is a right weakly π–regular ring.

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حول الحلقات الضعيفة المنتظمة من النمط π

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الملخص

الهدف الرئيسي من هذا البحث هو دراسة الحلقة المنتظمة الضعيفة اليمنى (اليسرى) من النمط π ، كذلك أعطينا بعضا من خواصها الأساسية وترابطها مع الحلقة من النمط CS، حلقة الـ MP وحلقة الـQuasi-Duo ، وأخيرا أعطينا التميز لهذه الحلقة مع المقاسات الغامرة من النمط GP والمقاسات البسيطة الشاذة(المنفردة) من النمط GP .