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On Q-Injective, Duo Submodules of C_1-Module

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A B S T R A C T

This note investigates modules having quasi-injective and duo submodules. We introduce a new generalization of C_1-module. The main method that was adopted in this generalization is how to obtain a submodule $\mathcal{N}$ in $\mathcal{M}$ having the characteristic Quasi-injective. We investigate the relationship between pseudo-injective module and Quasi-injective property of C_1-module. Finally, we introduce a new relationship between Quasi-injective submodule and anti-hopfian module.

1. INTRODUCTION

All the modules in this paper have a unity. Many searchers studied Quasi-injective and injective modules in details. Here we study Quasi-injective of any submodule $\mathcal{N}$ of $\mathcal{M}$. In [1], An R-module P is a projective module if there exists an R-module Q such that $P \oplus Q$ is a free R-module; also more details about injective and projective module can find it in same reference. In [2], we can find the definition of a Quasi-injective module (briefly Q-injective). Also, in [3], the author said $\mathcal{M}$ is pseudo-injective module (p-injective module) if $\forall\, \mathcal{N} \leq \mathcal{M}$, each R-isomorphism $g: \mathcal{N} \to \mathcal{M}$ can be extended to an R-endomorphism of $\mathcal{M}$. In [4], A module $\mathcal{M}$ is called uniform if $\mathcal{N}_1$ and $\mathcal{N}_2$ are non-zero submodules of $\mathcal{M}$; $\mathcal{N}_1 \cap \mathcal{N}_2 \neq 0$ the intersection of any two non-zero submodules is nonzero, equivalently, $\mathcal{M}$ is uniform if $0 \neq \mathcal{N} \leq_{ess} \mathcal{M}$. In [5], $\mathcal{N} \leq \mathcal{M}$ is called stable if for each R-homomorphism $f: \mathcal{N} \to \mathcal{M}$ implies $f(\mathcal{N}) \subseteq \mathcal{N}$, and an R-module $\mathcal{M}$ is called fully stable in case every submodule of $\mathcal{M}$ is stable.

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In this article, we investigate some facts about any submodule $\mathcal{N}$ of C_1-module $\mathcal{M}$ like Q-injective and duo properties. Also we use other properties in order to satisfy the same goal such as hopfian, anti-hopfian and self-injective modules.

2.PSEUDO-INJECTIVE and QUASI-INJECTIVE SUBMODULES

In this section, we will study two important properties of submodule $\mathcal{N}$ of $\mathcal{M}$ namely Quasi-injective and P-injective. Via this submodule, we obtain a new characterization of C_1-module. Moreover; we should provide another property namely fully invariant of this submodule. Note that Q-injective itself injective.

Definition 2.1. [1]. An R-module $\mathcal{M}$ is called injective if for every monomorphism $h: \mathcal{M}_1 \to \mathcal{M}_2$ and homomorphism $f: \mathcal{M}_1 \to \mathcal{M}_3$ there exists a homomorphism $g: \mathcal{M}_2 \to \mathcal{M}_3$ such that $g \circ h = f$.

Definition 2.2. [2]. Let $\mathcal{M}$ be an R-module. Then $\mathcal{M}$ is said to be Q-injective if for each submodule $\mathcal{N}$ of $\mathcal{M}$ and R-homomorphism $f: \mathcal{N} \to \mathcal{M}$ can be extended to an R-endomorphism of $\mathcal{M}$.

Definition 2.3. [3]. An R-module $\mathcal{M}$ is called pseudo-injective, if for every submodule $\mathcal{N}$ of $\mathcal{M}$, each R-

isomorphism $g: \mathcal{N} \to \mathcal{M}$ can be extended to an R-endomorphism of $\mathcal{M}$.

Lemma 2.4. [6]. Let $\mathcal{M}$ be an R-module over P.I.D. If $\mathcal{M}$ is pseudo-injective module, so it is a Q-injective.

Now we need to find a submodule $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N}$ is a Q-injective with invariant property. From **[3]**, any pseudo-injective module over P.I.D is a Q-injective; this means if $\mathcal{M}$ is a module on P.I.D, so $\mathcal{N} \leq \mathcal{M}$ on P.I.D, but $\mathcal{M}$ is pseudo-injective; $\mathcal{N}$ is a pseudo-injective and hence $\mathcal{N}$ is a Q-injective.

Note that to understanding lemma **(2.4)**, we can see **[7]**.

The following theorem explain the relationship between pseudo-injective and C_1-module over P.I.D.

Theorem 2.5. Let a ring R be a P.I.D. If $\mathcal{M}$ is a pseudo-injective C_1-module over R, then any submodule $\mathcal{N} \leq \mathcal{M}$ is a Q-injective and f$(\mathcal{N}) \subseteq \mathcal{N}$; so $\mathcal{M}$ is Q-injective-duo-C_1-module.

Proof: Suppose that a module $\mathcal{M}$ is pseudo-injective. Let us take $\mathcal{N} \leq \mathcal{M}$. We have $\mathcal{M}$ any module on P.I.D So also $\mathcal{N} \leq \mathcal{M}$ on P.I.D. But $\mathcal{M}$ is pseudo-injective, then $\mathcal{N}$ is pseudo-injective over P.I.D. Hence $\mathcal{N}$ is Q-injective with f$(\mathcal{N}) \subseteq \mathcal{N}$ imply $\mathcal{N}$ is fully invariant (duo) submodule of $\mathcal{M}$. Thus $\mathcal{M}$ is Q-injective-duo-C_1-module.

Now we introduce another way to obtain any submodule $\mathcal{N}$ of C_1-module $\mathcal{M}$ and be Q-injective. This way depends on new domain namely Dedekind domain. (R is a Dedekind domain if it is integrally closed, Noetherian and if $0 \neq$ p is a maximal; p is prime ideal). So if R is a Dedekind domain, then it is a UFD if and only if R is P.I.D. See the next Lemma:

Lemma 2.6. [7]. Let $\mathcal{M}$ be an R-module over Dedekind domain. If $\mathcal{M}$ is pseudo-injective (P-injective), then $\mathcal{M}$ is a Q-injective and so $\mathcal{N} \leq \mathcal{M}$ is a Q-injective submodule.

Theorem 2.7. Let $\mathcal{M}$ be a Pseudo-injective- C_1-module over Dedekind domain. If $\mathcal{M}$ is stable, then $\mathcal{M}$ is Q-injective -duo-C_1-module.

Proof: Assume that a module $\mathcal{M}$ is Pseudo-injective and R is a Dedekind domain. From lemma **(2.6)**, $\mathcal{M}$ is a Q-injective. So $\mathcal{N} \leq \mathcal{M}$is also Q-injective. But $\mathcal{M}$ is stable, so $\mathcal{N}$ is a fully invariant. Therefore $\mathcal{N}$ is a duo submodule of $\mathcal{M}$.

Lemma 2.8. [7]. Let $\mathcal{M}$ be an R-module. If the following statements are true:

(1)- R is Multiplication ring;

(2)- $\mathcal{M}$ is P-injective;

(3)- $T(\mathcal{M}) = \mathcal{M}$;

then $\mathcal{M}$ is Q-injective and so $\mathcal{N}$ is also Q-injective.

Theorem 2.9. Let $\mathcal{M}$ be a module over a ring R. If:

(1)- R is multiplication ring;

(2)- $T(\mathcal{M}) = \mathcal{M}$;

(3)- $\mathcal{M}$ is stable;

(4)- $\mathcal{M}$ is D_1-module and Pseudo-injective;

then $\mathcal{M}$ is Q-injective-duo-C_1-module.

Proof: Assume that $T(\mathcal{M}) = \mathcal{M}$ and R is a multiplication ring. Then from **[8]**, $T(\mathcal{N}) = \mathcal{N}$ (any submodule of torsion module is torsion). Since $\mathcal{M}$ is P-injective, then $\mathcal{M}$ is a Q-injective and hence $\mathcal{N}$ is P-injective and $T(\mathcal{N}) = \mathcal{N} \ni \mathcal{N} \leq \mathcal{M}$. Hence $\mathcal{N}$ is a Q-injective. Since $\mathcal{M}$ is stable, then $\mathcal{N}$ is a fully invariant. But from condition (4), $\mathcal{M}$ is C_1- module. Then $\mathcal{M}$ is a Q-injective-duo-C_1-module.

Corollary 2.10. If $\mathcal{M}$ is C_1-pseudo-injective R-module, then $\mathcal{M}$ is Q-injective-duo-C_1-module, knowing that $f(\mathcal{N}) \subseteq \mathcal{N}$ *and* $T(\mathcal{M}) = \mathcal{M}$.

Recall that any R-module $\mathcal{M}$ is called nonsingular if, for all $m \in \mathcal{M}$ with $r(m) \leq_{ess} R$ implies that $m = 0$. Or $Z(\mathcal{M}) = \{x \in \mathcal{M};$ ∃ a right an ideal I of R such that $I \leq_{ess} R$ and $XI = 0\}$ $(Z(\mathcal{M}) = 0)$ **[9].**

Lemma 2.11. If $\mathcal{N} \leq_{\text{ess}} \mathcal{M}$ and $Z(\mathcal{M}) = 0$ in pseudo-injective module $\mathcal{M}$, then $\mathcal{N}$ is Q-injective.

Proof: Let $\mathcal{N} \leq_{\text{ess}} \mathcal{M}$ and $Z(\mathcal{M}) = 0$. Let $g: \mathcal{N} \to \mathcal{M}$ be an R-homomorphism. So $Ker(g) = 0$ or $Ker(g) = \mathcal{N}$. Suppose that $Ker(g) = \mathcal{N}$, so g can be extended to homomorphism $h: \mathcal{M} \to \mathcal{M}$. Now if $Ker(g) = 0$, so g is one to one and can be extended to R-homomorphism from $\mathcal{M} \to \mathcal{M}$ ($\mathcal{M}$ is Pseudo-injective). Hence $\mathcal{N}$ is Q-injective.

Corollary 2.12. Let $\mathcal{M}$ be a C_1-pseudo-injective R-module. If $f(\mathcal{N}) \subseteq \mathcal{N}$, $\mathcal{N} \leq_{ess} \mathcal{M}$ and $Z(\mathcal{M}) = 0$; then $\mathcal{M}$ is Q-injective-duo-C_1-module.

Now we present another way in order to obtain that any submodule $\mathcal{N} \leq \mathcal{M}$ is a Q-injective. But before that we need to present some important definitions that are closely related to the mentioned way. Firstly, a concept of Stable-Q-injective was explained in [6].

Let $\phi: \mathcal{N} \to M \ni \phi(\mathcal{N}) \subseteq \mathcal{N}$. Then $\mathcal{M}$ is called stable module. So if every $\mathcal{N} \leq \mathcal{M}$ is stable this means $\mathcal{M}$ is fully stable module (F-stable).

If $\mathcal{N} \leq \mathcal{M}$ is stable and can be extended R-homomorphism ($\mathcal{N} \to \mathcal{M}$) to an R-endomorphism ($\mathcal{M} \to \mathcal{M}$), then $\mathcal{M}$ is called stable-Q-injective R-module. Also, If R is an integral domain and $\mathcal{M}$ is an R-module, then an element $x \in \mathcal{M}$ is called torsion element if $\exists\, 0 \neq r \in R \ni rx = 0$. **[10]**. So we define:

$$T(\mathcal{M}) = \{\, x \in \mathcal{M};\ x \text{ is a torsion element}\}.$$

Note that:

1. If $T(\mathcal{M}) = \mathcal{M}$, then a module $\mathcal{M}$ is called torsion-module.
2. If $T(\mathcal{M}) = 0$, then a module $\mathcal{M}$ is called torsion-free-module.

Lemma 2.13. [6]. Let $\mathcal{M}$ be a stable-Q-injective R-module. If $\mathcal{M}$ is an injective R-module, then it is Q-injective.

Theorem 2.14. Let $\mathcal{M}$ be a C_1-module. If $\mathcal{M}$ is a F-stable and stable-Q-injective; then $\mathcal{M}$ is Q-injective-duo-C_1-module.

Proof: Let $\mathcal{N} \leq \mathcal{M}$ and let $\phi: \mathcal{N} \to \mathcal{M}$ be an R-homomorphism of $\mathcal{M}$. So $\mathcal{N}$ is a stable because $\mathcal{M}$ is a F-stable. But from stable-Q-injective of $\mathcal{M}$, there is an $\varphi: \mathcal{M} \to \mathcal{M} \ni \varphi$ extends ϕ. Hence $\mathcal{M}$ is a Q-injective. Thus $\mathcal{M}$ is Q-injective-duo-C_1-module.

Corollary 2.15. Let $\mathcal{M}$ be a C_1-module. If $\mathcal{N} \leq \mathcal{M}$; $\varphi(\mathcal{N}) \subseteq \mathcal{N} \ni \varphi: \mathcal{N} \to \mathcal{M}$ be a homomorphism and $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ is a stable-Q-injective, then $\mathcal{M}$ is Q-injective-duo-C_1-module.

Proof: By Theorem (2.14).

Remark 2.16. From definition of fully invariant submodule and definition of stable, we find the two meanings are same.

Recall that a ring R is called Quasi-Frobenius (QF-ring) if every projective module is injective; or every injective module is discrete. From **[11]**, every projective-module is injective and then every injective-module is Q-injective.

Corollary 2.17. Let $\mathcal{M}$ be a C_1-module over QF-ring. If $\mathcal{M}$ is a projective module and stable in R, then $\mathcal{M}$ is Q-injective-duo-C_1-module ($\mathcal{N}$ is Q-injective submodule).

Proof: Let R be a QF-ring. Since $\mathcal{M}$ is a projective R-module, then $\mathcal{M}$ is an injective module and hence Q-injective. Therefore any submodule $\mathcal{N}$ of $\mathcal{M}$ is Q-injective. Note that $\mathcal{M}$ is stable module; so for $\varphi: \mathcal{N} \to \mathcal{M}$ be a homo. we get $\varphi(\mathcal{N}) \subseteq \mathcal{N}$. Thus $\mathcal{M}$ is Q-injective-duo-C_1-module.

Recall that a module $\mathcal{M}$ is called D_1-module if for $\mathcal{N} < \mathcal{M}$, $\exists \mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ is a coessential sub of $\mathcal{N}$; or if $\mathcal{N}, K \leq \mathcal{M}$ and $H \leq \mathcal{N}$, then $\mathcal{M} = H \oplus K$ and $\mathcal{N} \cap H \leq \mathcal{M}$. So D_1-module is extending.

Proposition 2.18. Let $\mathcal{M}$ be an R-module over QF-ring R. If:

(1)- $\mathcal{M}$ is D_1-module;

(2)- $\mathcal{M}$ is stable module;

(3)- $\mathcal{M}$ is a free-module;

then $\mathcal{M}$ is Q-injective-duo-C_1-module.

Proof: From condition (1); $\mathcal{M}$ is C_1-module. From condition (2); $\exists$ an R-homomorphism $\varphi: \mathcal{N} \to \mathcal{M} \ni \varphi(\mathcal{N}) \subseteq \mathcal{N}$ ($\mathcal{N}$ is fully invariant). So $\mathcal{N}$ is a duo submodule. Condition (3); gives $\mathcal{M}$ is a free-module. So if we take F is a free-R-module on a set S. Suppose that $\mathcal{N}_1, \mathcal{N}_2$ two modules over the ring R. Let $\varphi: \mathcal{N}_1 \to \mathcal{N}_2$ is a homomorphism.

$$\forall x \in S;\ we\ choose\ a_x \in \mathcal{N}_1 \ni j(x) = a_x \ \ldots\ldots\ldots\ (1)$$

Also,

$$\forall x \in F, g(x) \in \mathcal{N}_2\ and\ \varphi: \mathcal{N}_1 \to \mathcal{N}_2\ is\ onto.$$

Then

$$\exists\, a_x \in \mathcal{N}_1 \ni \varphi(a_x) = g(x)\ \ldots\ldots\ldots\ (2)$$

Since F is a free-R-module on S, $\exists$ a unique homomorphism

$$h: F \to \mathcal{N}_1 \ni h \circ i = j \ \ldots\ldots\ldots\ (3)$$

To prove that $\varphi \circ h = g$. Let x $\in$ F. So

$$x = \sum r_k x_k;\ x_k \in S, r_k \in R;\ k = 1,2,\ldots\ldots,n$$

$$(because\ F\ is\ generated\ by\ s\ \ F = \langle s \rangle).$$

Now

$$(\varphi \circ h)(x) = (\varphi \circ h)(\sum r_k x_k)$$
$$= \varphi\,(h(\sum r_k x_k))$$
$$= \varphi(\sum r_k h(x_k)),$$

h is homomorphism.

$$= \varphi(\sum r_k (h(i(x_k)))\,.$$

Now

$$(\varphi \circ h) = \varphi(\sum r_k ((h \circ i)(x_k)))\,.$$
$$= \varphi(\sum r_k (j(x_k)))\,;\ (by\ (3)).$$
$$= \varphi(\sum r_k a_{x_k})\,;\ (by\ (1)).$$
$$= \sum r_k\ \varphi(a_{x_k})\,;\ \varphi\ \ homomorphism.$$
$$= \sum r_k g(x_k)\,;\ (by\ (2)).$$
$$= g(\sum r_k x_k).$$

g homomorphism. So $\varphi \circ h = g$. Thus $\mathcal{M}$ is a projective and hence $\mathcal{M}$ is injective ($\mathcal{M}$ is a Q-injective). Then $\mathcal{N} \leq \mathcal{M}$ is Q-injective. Thus $\mathcal{M}$ is a Q-injective-duo-C_1-module.

Lemma 2.19. For a ring R, we have R_R is a semi-simple if and only if R is a semisimple and so any module $\mathcal{M}$ over R is a semisimple module.

Proof: We need to prove the following,

1. R_R semisimple if and only if R is semisimple.
2. $\mathcal{M}$ is a semisimple module over R.

From [11], we can get the proof of (1).

Now we need to proof (2):

If R_R is a semisimple and if $\mathcal{M} = \mathcal{M}_R \ni m \in \mathcal{M}$, then R is a semisimple as an epimorphic image of R_R. So

$\mathcal{M} = \sum mR, m \in \mathcal{M}$ as a sum of semisimple module is again semisimple.

Lemma 2.20. Let a ring R be a semisimple, and $\mathcal{M}$ be an R-module. Then every submodule $\mathcal{N} \leq \mathcal{M}$ is Q-injective.

Proof: Since R is a semisimple ring, then every module $\mathcal{M}$ over R is a semisimple. So $\mathcal{N} \leq \mathcal{M}$ is a direct summand. Hence $\mathcal{M}$ is injective R-module. But every injective R-module is a Q-injective. Thus $\mathcal{N}$ is Q-injective.

Theorem 2.21. Let R be a semisimple ring and $\mathcal{M}$ is an R-module. If $\mathcal{M}$ is D_1-module and stable; then it is Q-injective-duo-C_1-module.

Proof: It is clear that from lemma (2.20), $\mathcal{N} \leq \mathcal{M}$ is Q-injective. But $\mathcal{M}$ is a stable, then $\exists\, f: \mathcal{N} \to \mathcal{M} \ni f(\mathcal{N}) \subseteq \mathcal{N}$. So $\mathcal{N}$ is a fully invariant and hence $\mathcal{M}$ is a duo ($\mathcal{N}$ is a duo submodule). We have $\mathcal{M}$ is D_1-module. So it is C_1-module. Thus $\mathcal{N}$ is Q-injective of $\mathcal{M}$.

Corollary 2.22. Let $\mathcal{M}$ be an R-module. If:

(1)- $\mathcal{M}$ is projective module;

(2)- $\mathcal{M}$ is a simple module;

(3)- $\mathcal{M}$ is Q-injective;

then $\mathcal{N}$ is Q-injective and duo submodule of C_1-module.

Proof: It is clear that projective module means C_1-module. Also, if a module $\mathcal{M}$ is simple, then $\mathcal{M}$ is duo-module. ($\mathcal{N} \leq \mathcal{M} \ni \mathcal{N}$ is fully invariant; $f(\mathcal{N}) \subseteq \mathcal{N}$ and $f: \mathcal{N} \to \mathcal{M}$ is an R-homomorphism). Now from condition (3), we have $\mathcal{M}$ is Quasi-projective. So $\mathcal{M}$ is a Q-injective and hence $\mathcal{N}$ is a Q-injective of C_1-module.

Recall that any ring R is called V-ring if every simple R-module is injective **[12]**.

Corollary 2.23. Let $\mathcal{M}$ be a D_1-R-module over V-ring. Then $\mathcal{M}$ is Q-injective-duo-C_1-module.

2.HOPFIAN, SELF-INJECTIVE MODULES AND Q-INJECTIVE SUBMODULE

From **[13]**, a module $\mathcal{M}$ is called self-p-injective if $\mathcal{M}$ satisfy the following condition; every homomorphism from a projection invariant submodule of $\mathcal{M}$ to $\mathcal{M}$ can be lifted to $\mathcal{M}$.

(#) Every self-injective is injective module.

Definition 3.1. Any R-module $\mathcal{M}$ is called indecomposable if $\mathcal{M}$ has no proper non trivial complement submodule $\mathcal{M}_1$ ($\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, so $\mathcal{M}_1 = 0$ or $\mathcal{M}_1 = \mathcal{M}$).

Example 3.2. Z is indecomposable Z − module, but Z is not simple Z − module (Z contains proper submodule 2Z).

Therefore, every simple module is indecomposable, but the converse is not true.

Theorem 3.3. Let $\mathcal{M}$ be an indecomposable self-P-injective R-module. Then any C_1- module is Q-injective-duo-C_1-module.

Proof: From definition of self-p-injective, there exists K submodule of $\mathcal{M}$ such that K is fully invariant. Assume that $\mathcal{M}$ is indecomposable module, so every submodule of $\mathcal{M}$ is projective invariant. Then $\mathcal{M}$ is Q-injective. Thus $\mathcal{M}$ is Q-injective-duo-C_1-module.

Recall that any module $\mathcal{M}$ is called Hopfian if every surjective f in $End(\mathcal{M})$ is isomorphism and a non simple module is called anti-Hopfian if proper submodule of $\mathcal{M}$ is a non-Hopfian kernel such that a submodule $\mathcal{N}$ of $\mathcal{M}$ is non-Hopfian kernel (for $\mathcal{M}$) if there exists an isomorohism $\mathcal{M}/\mathcal{N}$ to $\mathcal{M}$ **[14]**. Or an R-module $\mathcal{M}$ is anti-Hopfian if $\mathcal{M}$ is non simple and all nonzero factor modules of $\mathcal{M}$ are isomorphic to $\mathcal{M}$; that is for all $\mathcal{N} \leq \mathcal{M}, \mathcal{M}/\mathcal{N} \cong \mathcal{M}$ **[16]**.

Example 3.4. Any module of semisimple Artinian ring with finite length is Hopfian module.

Lemma 3.5. Let $\mathcal{M}$ be an R-module. If $\mathcal{M}$ is anti-Hopfian, then every submodule $\mathcal{N}$ of $\mathcal{M}$ is Q-injective **[15]**.

Theorem 3.6. Let $\mathcal{M}$ be C_1-R-module. If $\mathcal{M}$ has exactly one non-zero proper submodule and $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \ni \mathcal{M}_1, \mathcal{M}_2$ are simple modules, then $\mathcal{N} \leq \mathcal{M} \ni \mathcal{N}$ is a Q-injective of $\mathcal{M}$.

Proof: From **[14]**, $\mathcal{M}$ is anti-Hopfian module. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are simple modules, then $\mathcal{M}$ is a simple module and so it is a duo module ($\mathcal{N}$ is a duo submodule). From lemma (3.5), the proof is completed.

Corollary 3.7. Let R be a Dedekind domain, and $\mathcal{M}$ is C_1-module with $Rad(\mathcal{M}) \neq \mathcal{M}$. If $\mathcal{M} \cong R/I^2 \ni I$ is a non-zero ideal of R and $\mathcal{N}$ is duo submodule of $\mathcal{M}$, then $\mathcal{N} \leq \mathcal{M}$ is Q-injective in C_1-module.

Proof: From [14] and Lemma (3.5).

3. CONCLUSIONS

This paper investigated modules having a submodule are duo and Quasi-injective properties. Tow generalization of C_1-module have been studied. We proved that any module has pseudo-injective, $\mathcal{N}$ is essential in $\mathcal{M}$ and stable, this mean $\mathcal{M}$ is a Quasi-injective-duo-C_1-module where R is a Dedekind domain. Also same goal can obtained it if $\mathcal{M}$ is a projective and stable with $\mathcal{N}$ is an essential in $\mathcal{M}$.

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حول المقاسات الجزئية شبه الغامرة و الثنائية للمقاس من نوع C_1

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الخلاصة:

في هذا البحث تم تحقيق المقاس الذي يمتلك مقاسات جزئية شبه غامرة وثنائية. قدمنا تعميم جديد للمقاس من نوع C_1. الطريقة الرئيسية التي اعتمدت على كيفية الحصول على مقاس جزئي N في المقاس M له الخاصيتان السابقتان. تحققنا من العلاقة بين المقاس الجزئي شبه الغامر والمقاس الغامر الكاذب للمقاس الأصلي C_1. في نهاية البحث قدمنا العلاقة الجديدة بين المقاس الجزئي شبه الغامر والمقاس من نوع anti-hopfian.