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New operational matrices approach for optimal control based on modified Chebyshev polynomials

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Abstract

The purpose of this paper is to introduce interesting modified Chebyshev orthogonal polynomial. Then, their new operational matrices of derivative and integration or modified Chebyshev polynomials of the first kind are introduced with explicit formulas. A direct computational method for solving a special class of optimal control problem, named, the quadratic optimal control problem is proposed using the obtained operational matrices. More precisely, this method is based on a state parameterization scheme, which gives an accurate approximation of the exact solution by utilizing a small number of unknown coefficients with the aid of modified Chebyshev polynomials. In addition, the constraint is reduced to some algebraic equations and the original optimal control problem reduces to optimization technique, which can be solved easily, and the approximate value of the performance index is calculated. Moreover, special attention is presented to discuss the convergence analysis and an upper bound of the error for the presented approximate solution is derived. Finally, some important illustrative examples of obtained results are shown and proved that powerful method in a simple way to get an optimal control of the considered.

Introduction:

Direct method is one of the techniques for solving the optimal control problems by reducing the dynamic optimal control problem into on optimization problem (nonlinear programming problem [1-4]. They are working by either discretization or parameterization the state variables, the control variables or both the state and control variables [5-9]. The most logical way to solve optimal control problems is numerical solution because analytical solutions are not always available. The study of numerical methods had been presented by many researchers of mathematical sciences using different computational methods and efficient algorithms to solve optimal control problems [10-12]. Orthogonal functions and polynomials series and used in dealing with various numerical algorithm for solving optimal control problem. Examples are the utilize of Boubaker polynomials [13-16], the Chebyshev polynomials [17], shifted modified Chebyshev polynomials [18], variation problem [19] and continuous optimal control problem [20]. In this work, new operation matrix of derivative for modified Chebyshev polynomial is considered for solving quadratic optimal control problem.

The outline of this paper is organized as follow: Section two concerns the preliminaries of modified Chebyshev polynomials. Section three presents a new modified Chebyshev operational matrix of derivative as well as some other important properties. Section four discusses the convergence analysis of modified Chebyshev functions. Last section illustrates how the new introduced properties for modified Chebyshev can be utilized to convert the optimal control problem into nonlinear programming technique. The efficiency and accuracy of the proposed algorithm is shown by solving two examples.

Modified Chebyshev Polynomials:

Let $M_n(t)$ denote the n^{th} modified Chebyshev polynomials

$$M_n(t) = 2\,T_n\left(\frac{t}{2}\right), n \in N \tag{1}$$

where $T_n(t)$ is nth chebyshev polynomials of the first kind satisfying the identity

$$T_n(\cos\theta) = \cos(n\theta) \tag{2}$$

According to Eq. 1, we have

$M_0(x) = 2\,,\ M_1(t) = t$

and

$$M_{n+2}(t) = tM_{n+1}(t) - M_n(t) \quad , n \in N \tag{3}$$

From Eq.3 the following explicit formula of modified Chebyshev polynomials can be obtained $M_n(t) = \sum_{k=0}^{\frac{n}{2}} (1)^k \frac{n}{n-k} \binom{n-k}{k} t^{2k}$

The coefficients of power x for modified Chebyshev polynomials until degree 7 is given in table 1 which can be obtained from Eq. 4

Table 1: The coefficients of power t for modified Chebyshev polynomials

n/t	t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8
0	2								
1	0	1							
2	-2	0	1						
3	0	3-	0	1					
4	2	0	-4	0	1				
5	0	5	0	-5	0	1			
6	-2	0	9	0	-6	0	1		
7	0	7	0	14	0	-7	0	1	

The entries in table 1 can be calculated as

$$m_{ij} = \begin{cases} m_{i-1,j-1} - m_{i-2,j} & \text{for} \quad i-j \text{ even}, i > j \\ 0 & \text{for} \quad i-j \text{ odd}, i < j \end{cases}$$

Some New Properties of Modified Chebyshev Polynomials

Modified Chebyshev Operational Matrix of Derivative

The objective of this subsection is to state and prove new analytical formula expressing explicit the first derivative of the modified first kind Chebyshev polynomials in terms of their modified first kind Chebyshev polynomials themselves.

Theorem 1:

For all $n \geq 1$

$$\dot{M}_n(t) = n \begin{cases} \sum_{\substack{i=1 \\ \text{odd } i}}^{n-1} M_i(t) & \text{n ever} \\ \frac{1}{2} M_0(t) + \sum_{\substack{i=2 \\ \text{ever } i}}^{n-1} M_i(t) & \text{n odd} \end{cases}$$

where $M_0(t) = 0$

where n is the order of modified Chebyshev polynomials $M_n(t)$.

Proof:

Let us denote

$$S_n(t) = a_0 M_0(t) + a_1 M_1(t) + a_2 M_2(t) + \cdots + a_n M_n(t) = \sum_{i=0}^{n} a_i M_i(t) \tag{4}$$

Consider the recurrence relation given in Eq. 3

$$M_{n+1}(t) = tM_n(t) - M_{n-1}(t), n = 1,2,3 \ldots.$$

where $M_o = 0$ and $M_1 = t$

After differentiating the recurrence relation, one can obtain

$$\dot{M}_{n+1}(t) = t\dot{M}_n(t) + M_n(t) - \dot{M}_{n-1}(t)$$

This can be rewritten as:

$$\dot{M}_{n+1}(t) = M_1 \dot{M}_n(t) + M_n(t) - \dot{M}_{n-1}(t) \tag{5}$$

The polynomials $\dot{M}_n(t)$ and $\dot{M}_{n-1}(t)$ can be expressed as

$$\dot{M}_n(t) = \sum_{r=0}^{n-1} b_r M_r(t) \tag{6}$$

$$\dot{M}_{n-1}(t) = \sum_{r=0}^{n-2} c_r M_r(t) \tag{7}$$

Substituting Eqns. 6 and 7 into Eq. 5 yields

$$\dot{M}_{n+1}(t) = M_1 \sum_{r=0}^{n-1} b_r M_r(t) + M_n(t) - \sum_{r=0}^{n-2} c_r M_r(t)$$

On the other words

$$\dot{M}_{n+1}(t) = b_0 (M_1 M_0) + b_1 (M_1 M_1) + \cdots + b_{n-1} (M_n M_{n-1}) + M_n(t) - c_0 M_0 - c_1 M_1 - \cdots - c_{n-2} M_{n-2}$$

Using the product property $M_n(t) M_m(t) = M_{n+m}(t) + M_{n-m}(t)$

$\dot{M}_{n+1} = b_0(M_1 + M_1) + b_1 (M_2 + M_0) + \ldots\ldots + b_{n-1} (M_n + M_{n-1}) + M_n(t) - c_0 M_0(t) - c_1 M_1(t) - \ldots\ldots - c_{n-2} M_{n-2}(t)$

Collecting the coefficients of similar modified Chebyshev polynomials, one can get

$$\dot{M}_{n+1}(t) = (b_1 - c_0)M_0 + (2b_0 + b_2 - c_1)M_1 + (b_1 + b_3 - c_2)M_2 + \ldots + (b_{n-1} + b_{n+1} - c_{n-2})M_{n-2} + b_{n-2} + 1)\, M_n(t)$$

or

$$\dot{M}_{n-1}(t) = (b_1 - c_0)M_0 + \sum_{r=2}^{n-1}(b_{r-1} + b_{r+1} + c_r)\, M_r(t) + (b_{n-1} + 1)M_n(t) \qquad (8)$$

Note that $\dot{M}_0(t) = 0$ and $\dot{M}_1(t) = \frac{1}{2}M_0$

Applying Eq. 8 recursively, the following matrix can be obtained

$$D = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ \frac{1}{2} & 0 & 0 & \ldots & 0 \\ 0 & 2 & 0 & \ldots & 0 \\ \frac{3}{2} & 0 & 6 & \ldots & 0 \\ 0 & 4 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \\ \frac{n}{2} & 0 & n & \ldots & 0 \end{bmatrix} \qquad (9)$$

For odd n and the last row becomes [0 n 0 n … 0] for even n

Results in Eq. 9 can be written in the compact from

$\dot{M}_2 = e_{22}M_1$

$\dot{M}_3 = e_{31}M_0 + e_{33}M_2$

$\dot{M}_4 = e_{42}M_1 + e_{44}M_2$

$\dot{M}_5 = e_{51}M_0 + e_{53}M_2 + e_{55}M_4$

$\dot{M}_6 = e_{62}M_1 + e_{64}M_3 + e_{66}M_5$

where

$e_{ij} = d_{ij}$ in Eq. 9

From 8 and 9, one can get the following result

$$\dot{M}_n(t) = n\sum_{\substack{i=0 \\ \text{odd } i}}^{n-1} M_i \text{ for even n}$$

and

$$\dot{M}_n(t) = \frac{n}{2}M_0 + n\sum_{\substack{i=2 \\ \text{ever } i}}^{n-2} M_i, \text{ for even n}$$

This is the result we want to prove.

If a function x(t) can be approximated by a modified Chebyshev series of length n as follows

$$x(t) = \sum_{i=0}^{n} a_i M_i(t) \qquad (10)$$

Then the derivative of x(t) with respect to t is given by

$$\dot{x}(t) = \sum_{i=0}^{n-1} c_i M_i(t) \qquad (11)$$

where $c_{n-1} = n a_n$

$c_{n-2} = (n-1)a_{n-1}$
$c_{r-1} = c_{r+1} + r\,a_r \qquad \text{for} \quad r = (n-2), (n-1), \ldots, 2$
$c_0 = \dfrac{1}{2} + a_1$

Initial Values

$$M_n(0) = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ ever}, \frac{n}{4} = \text{integer} \\ -2 & n \text{ ever}, \frac{n}{4} \neq \text{integer} \end{cases}$$

and

$$M_n(1) = \begin{cases} (-1)^{n+1} & \text{otherwise} \\ (-1)^{n+2}.2 & \frac{n}{3} = \text{integer} \end{cases}$$

Integration Formula

For even $n, n \geq 2$

$$\int M_n(t)\,dt = \frac{1}{n+1} M_{n+1}(t) - \frac{1}{n-1} M_{n-1}(t)$$

For odd n

$$\int M_n(t)dt = \begin{cases} \frac{1}{n+1} M_{n+1}(t) - \frac{1}{n-1} M_{n-1}(t) - \frac{1}{2}\frac{n}{(n-1)} M_0 & \text{for } \frac{n+1}{2} \text{ even} \\ \frac{1}{n+1} M_{n+1}(t) - \frac{1}{n-1} M_{n-1}(t) + \frac{1}{2}\frac{n}{(n+1)} M_0 & \text{for } \frac{n+1}{2} \text{ odd} \end{cases}$$

where

$$\int M_0(t)\,dt = 2M_1(t) \quad , \int M_1(t)dt = \frac{1}{2}\left(M_2(t) + M_0\right)$$

Convergence Analysis and Error Estimate

Theorem 2:

$$|c_n| \leq \mu \sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \frac{n}{n-i} \binom{n-i}{i}$$

where μ is a constant that $|f(t)| \leq \mu$ (12)

Proof

The approximate function of $f(t)$ by using modified Chebyshev polynomials is as follows

$$f_n(t) = \sum_{i=0}^{n} c_i\, M_i(t) \tag{13}$$

$$\text{where } c_n = \langle f(t), M_n(t)\rangle = \int_{-1}^{1} f(t) M_n(t)\, dt \tag{14}$$

Since every continuous function on the closed interval is a bounded function, thus there is a constant μ such that $|f(t)| \leq \mu \ \forall t \in \{-1,1\}$ (15)

Then, one can conclude that

$$
\begin{aligned}
|c_n| &\le \int_{-1}^{1} |M_n(t)|\ dt \\
&\le \mu \int_{-1}^{1} \sum_{i=0}^{n/2} (-1)^i \frac{n}{n-i} \binom{n-i}{i} \left|t^{n-2i}\right| dt \\
&\le \mu \int_{-1}^{1} \sum_{i=0}^{n/2} (-1)^i \frac{n}{n-i} \binom{n-i}{i} dt \\
&\le \mu \int_{-1}^{1} \sum_{i=0}^{n/2} (-1)^i \frac{n}{n-i} \binom{n-i}{i} dt
\end{aligned}
$$

Therefore,

$$\int_{-1}^{1} |M_n(t)| dt \le \sum_{i=0}^{n/2} (-1)^i \frac{n}{n-i} \binom{n-i}{i}$$

$$|c_n| \le \mu \sum_{i=0}^{n/2} (-1)^i \frac{n}{n-i} \binom{n-i}{i}.$$

Assume that f be a continuous function and f_n be the approximate function of f by using modified Chebyshev polynomial then the error bound would be obtained as follows

$$\|E\|_2 = \|f - f_n\|_2 \le \left(\sum_{i=n+1}^{\infty} \sigma i^2\right)^{1/2} \quad \delta$$

where

$$\sigma\, i = \mu \sum_{i=0}^{n/2} (-1)^i \frac{n}{n-i} \binom{n-i}{i}.$$

New Technique for Solving Optimal Control Problem

Consider the linear quadratic optimal control problem

$$\int_{-1}^{1} (x^T Q\, x + u^T R u) dt \tag{16}$$

Subject to the linear system of state equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{17}$$

$$x(-1) = x_0 \tag{18}$$

where $x \in R^n$, $u \in R^m$, $m \le n$, the two real valued matrices A and B are n x n and n x m respectively while Q is n x n positive semi definite matrix, $x^T Q\, x \ge 0$ and R is positive definite matrix $u^T R u > 0$

The idea of the approximate methods using modified Chebyshev polynomials as basis function to approximate the state variables can be summarized as follows:

Approximate the state variables by a finite length polynomial series

$$x_j(t) = \sum_{i=0}^{N} a_{ij} M_i(t) \quad j = 1,2,\cdots,n$$

where a'_{ij} are the unknown parameters. The control variables $u_k(t)$, $k = 1,2,3,\dots.m$ are determined from the system state equation as a function of the unknown parameters of the state variables.

Note that, two cases can be distinguished when applying the state vector parameterization

Case 1: If the number of the states and the control variables are equal,

Case 2: If the number of the state variables is greater than the number of control variable.

The following two test examples will illustrate the two cases respectively.

First test example:

Minimize $\quad J = \frac{1}{2}\int_{-1}^{1} \left(x^2(t) + u^2(t)\right)dt \qquad (19)$

Subject to

$$\dot{x}(t) = \frac{1}{2}\, u(t) \qquad (20)$$

$$x(-1) = 1 \qquad (21)$$

This example contains state variable $x(t)$ and control variable $u(t)$. Here $x(t)$ is approximated by modified Chebyshev polynomial of unknown parameters, and then the control variable can be obtained using Eq. 20, yields

$$x(t) = \sum_{i=0}^{3} a_i\, M_i(t) = AM(t) \qquad (22)$$

$$u(t) = 2ADM(t) = a_1 M_0(t) + 4a_2 M_1(t) \qquad (23)$$

The initial condition in Eq. 21 is replaced by equation constraint as follows:

$$\sum_{i=0}^{3} a_i\, M_i(o) - x(0) = 0$$

Which represents the equality constraint. By substituting $x(t)$ and $u(t)$ into Eq. 19, an expression of J can be found.

$$J = \frac{1}{2}\int_{-1}^{1} (AMM^T A^T + 4ADMM^T D^T A^T)dt$$

Now, the quadratic optimal control problem is converted into parameters optimization problem, which is quadratic in the unknown parameters, and the new problem can be stated as:

$$\text{Min } J^* = \frac{1}{2}\, a^T H a$$

subject to

$$Fa - b = 0$$

The matrix H can be defined by finding Hessian of J^*

$$H = \frac{\partial^2 J^*}{\partial a_{ik}\, \partial a_{jk}}\ , i, j = 0,1,2, \cdots, n \text{ and } k = 1,2, \ldots\ldots, n$$

Finally, from the standard quadratic programming method $a^* = H^{-1}F^T(FH^{-1}F^T)^{-1}b$, the optimal value of the vector can be obtained as

$$a_0 = 0.4589,\ a_1 = 0.1758,\ a_2 = 0.0937,\ \text{ and } J = 0.7618.$$

The optimal values of J is shown in table. 2 versus the orders of modified Chebyshev series with $n = 2, 3, 4$. The graph of optimal value J using different orders of modified Chebyshev is illustrated in Fig. 1.

Table 2: The optimal values of J for Example 1

N	The optimal Value	Absolute error $\left\|J_{exact} - J_{app}\right\|$
2	0.7618	0.0002
3	0.7616	0.0000
4	0.7616	0.0000

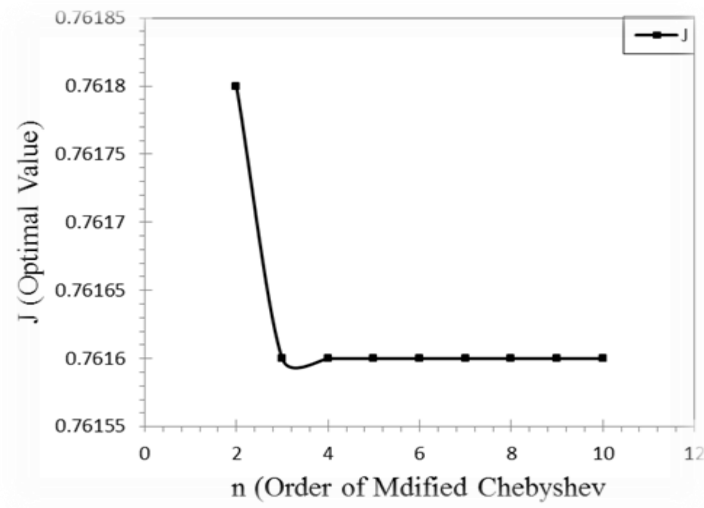

Fig. 1: The optimal values of J for Example 1 versus the orders of modified Chebyshev series.
Second test example:

Consider the following quadratic optimal control problem

$$\text{Minimize } J = \frac{1}{2}\int_{-1}^{1}(x_1^2(t) + x_2^2(t) + u^2(t))\,dt \tag{24}$$

subject to

$$\dot{x}_1(t) = 0.5x_2(t) \tag{25}$$
$$\dot{x}_2(t) = 0.5(-x_2(t) + u(t)) \tag{26}$$
$$x_1(-1) = 0\,,\ x_2(-1) = -1 \tag{27}$$

This example contains two state variables $x_1(t)$, $x_2(t)$ and one control variable $u(t)$. In this case, there is no need to approximate the two state variables; one of the two state variables is approximate which will enable us to find the other state variable. The quadratic optimal control problem in the second test example is reduced to a quadratic programming problem with fewer unknown parameters then the initial conditions in Eq. 27 are replaced by equation constraint. The state variable, which are not satisfied yet are added to the initial conditions to represent equality constraints. Here $x_1(t)$ is approximated by modified Chebyshev polynomials of unknown parameters, then $x_2(t)$ can be obtained from Eq. 25 and $u(t)$ can be obtained from Eq. 26. By substituting $x_1(t)$, $x_2(t)$ and $u(t)$ into Eq. 24, an expression of J can be found.

The optimal values of J is shown in Table 3 versus the orders of modified Chebyshev series with $n = 4, 5, 6$. The graph of optimal value J using different orders of modified Chebyshev is illustrated in Fig. 2.

Table 3: The optimal values of J for Example 2

N	The optimal Value	Absolute error $\lvert J_{exact} - J_{app} \rvert$
4	0.091738	-0.022377
5	0.075952	-0.006591
6	0.0696972	-0.000336

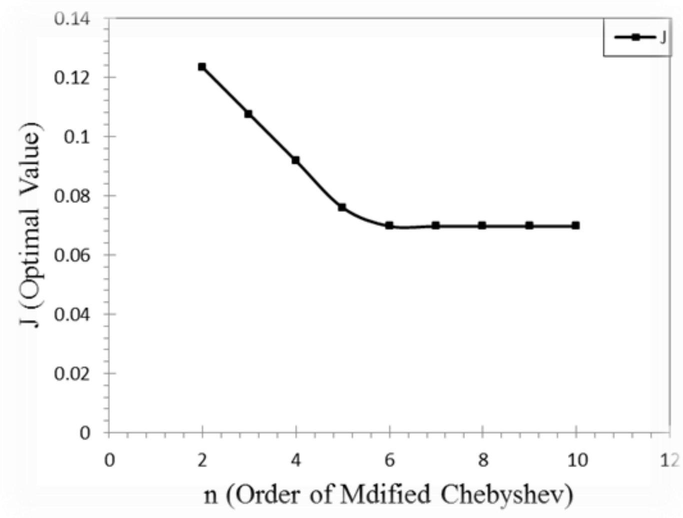

Fig. 2: The optimal values of J for Example 2 versus the orders of modified Chebyshev series

Conclusions

In this paper, we concerned with quadratic optimal control problems associated with finite time of minimizing a running cost or performance index subject to linear control dynamics. An approximate method for solving quadratic optimal control problem has been demonstrated. This is based on combining the obtained important operational matrices with state parameterization using modified Chebyshev polynomials to convert the original quadratic optimal control problem into a nonlinear programming problem. The effectiveness of the suggested method is demonstrated by solving some numerical examples. One of the advantages of the presented algorithm is using a computational technique with fast convergence.

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تقريب مصفوفات العمليات الجديدة للتحكم الأمثل المعتمدة على متعدد الحدود المطورة للشيبيشيف

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البحث مستل من رسالة ماجستير الباحث الاول

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الكلمات المفتاحية:

تحليل الاقتراب، متعددة حدود شيبيشف المعدلة، مصفوفة العمليات، مسألة السيطرة المثلى، تقنية الامثلية

الخلاصة:

الغرض من هذا البحث هو تقديم متعددات حدود شيبشيف المتعامدة المثيرة للاهتمام. بعد ذلك، تم تقديم مصفوفات العمليات الجديدة للمشتقات والتكاملات لمتعددة الحدود شيبيشيف من النوع الاول المعدلة مع صيغ صريحة. تم اقتراح طريقة عددية مباشرة لحل نوع خاص من مشكلة السيطرة المثلى تسمى مسألة السيطرة المثلى التربيعية باستخدام مصفوفة العمليات التي تم الحصول عليها. بتعبير أدق، تعتمد هذه الطريقة على مخطط معلمات الحالة، والذي يعطي تقريبا دقيقا للحل الدقيق من خلال استخدام عدد قليل من المعاملات غير المعروفة بمساعدة متعددة الحدود شيبيشف المعدلة. علاوة على ذلك، تم اختزال القيد إلى بعض المعادلات الجبرية وتقللت المشكلة الأصلية للسيطرة المثلى الى اسلوب الامثلية والتي يمكن حلها بسهولة ويتم حساب القيمة التقريبية لمؤشر الأداء. علاوة على ذلك، تم تقديم اهتمام خاص لمناقشة تحليل التقارب والحد الأعلى للخطأ للحل التقريبي المقدم. أخيراً، تم عرض بعض الأمثلة التوضيحية المهمة للنتائج التي تم الحصول عليها وإثبات أنها طريقة قوية بطريقة بسيطة للحصول على السيطرة المثلى المعتبرة.