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Some Characteristics Properties for Linear Operator on Class of Multivalent Analytic Functions Defined by Differential Subordination

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ABSTRACT

The purpose of this paper is to consider a linear operator and define a certain class $E_p(a, c, \lambda, \gamma: h)$ of analytic and multivalent functions in the open unit disk associated with differential subordination. Also, we discuss some geometric properties for this class.

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1. Introduction

Let $H = H(U)$ be the class of analytic function in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$. For n $\in$ N and $a \in \mathbb{C}$. Let $H[a, n]$ be the subclass of H and

$$H[a, n] = \{f \in H : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\} \quad (a \in \mathbb{C}).$$

Let $\mathrm{A_p}$ denote the subclass of H of function f of the form:

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$$f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n \quad (p\in \mathbb{N}=\{1,2,3,\dots\}), \qquad z\in U. \tag{1.1}$$

The Hadamard product (or convolution) $(f_1 * f_2)(z)$ of two functions

$$f_j(z)=z^p+\sum_{n=p+1}^{\infty} a_{n,j} z^n \in \mathcal{A}_p \ (j=1,2)$$

is given by

$$(f_1 * f_2)(z)=z^p+\sum_{n=p+1}^{\infty} a_{n,1}a_{n,2} z^n.$$

For two functions f and g, which are analytic in U, the function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function w analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$ such that $f(z)=g(w(z))$. In such a case we write $f\prec g$ or $f(z)\prec g(z)(z\in U)$. Furthermore, if g is univalent in U, then we have the following equivalent,

$$f\prec g \iff f(0)=g(0) \quad \text{and} \quad f(U)\subset g(U).$$

In the theory and widespread applications of fractional calculus (see, for example, [8,9]; see also the recent survey-cum-expository review article [19]), one of the most popular operators happens to be the Riemann-Liouville fractional integral operator of order $\alpha\in\mathbb{C}$ $(\mathrm{Re}(\alpha)>0)$ defined by

$$(\mathrm{I}^{\alpha}\mathrm{f})(\mathrm{x})=\frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{x}}(\mathrm{x}-\mathrm{t})^{\alpha-1}\mathrm{f}(\mathrm{t})d\mathrm{t} \qquad (\mathrm{x}>0;\mathrm{Re}(\alpha)>0). \tag{1.2}$$

In terms of the familiar (Euler's) Gamma function $\Gamma(\alpha)$. An interesting variant of the Riemann-Lioville operator I^{α},which is known as the Erdélyi-kober fractionl integral operator of order $\alpha\in\mathbb{C}$ (Re $(\alpha)>0$) defined by

$$\left(I_{\sigma,\eta}^{\alpha}f\right)(x)=\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_0^x t^{\sigma(\eta+1)-1}(x^{\sigma}-t^{\sigma})^{\alpha-1}f(t)dt \ \ (x>0;Re(\alpha)>0), \tag{1.3}$$

which corresponds essentially to (1.2) when $\sigma-1=\eta=0$, since

$$(I_{1,0}^{\alpha}f)(x)=x^{-\alpha}(I^{\alpha}f)(x) \qquad (x>0;Re(\alpha)>0).$$

Motivated essentially by the special case of the definition (1.3) when $x=\sigma=1, \eta=a-1$, and $\alpha=c-a$, here we consider a linear integral operator $\mathfrak{I}_p(a,c,\lambda)$ defined for a function $f\in\mathcal{A}_p$ by (see [6])

$$\mathfrak{I}_p(a,c,\lambda)f(z)=\frac{\Gamma(c+\lambda p)}{\Gamma(a+\lambda p)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}f\left(zt^{\lambda}\right)dt$$

$$(\lambda>0; a,c\in\mathbb{R}; c>a>-\lambda p; p\in\mathbb{N}).$$

When evaluated by means of the Eulerian Beta –function integral:

$$B(\alpha,\beta):=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt & (min\{Re(\alpha),Re(\beta)\}>0)\\ \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-),\end{cases}$$

we readily find that

$$\mathfrak{I}_p(a,c,\lambda)f(z)=\begin{cases} z^p+\dfrac{\Gamma(c+\lambda p)}{\Gamma(a+\lambda p)}\displaystyle\sum_{n=p+1}^{\infty}\dfrac{\Gamma(a+\lambda n)}{\Gamma(c+\lambda n)}a_n z^n & (c>a) \\ f(z) & (c=a), \end{cases} \tag{1.4}$$

where $\mathbb{Z}_0^-$ is the set of nonpositive integers. It is easy to deduce from (1.4) that

$$z\left(\mathfrak{I}_p(a,c,\lambda)f(z)\right)'=\left(\frac{a}{\lambda}+p\right)\mathfrak{I}_p(a+1,c,\lambda)f(z)-\frac{a}{\lambda}\mathfrak{I}_p(a,c,\lambda)f(z). \tag{1.5}$$

We also note that the linear operator $\mathfrak{I}_p(a,c,\lambda)$is a generalization of many other integral operators, which were considered in earlier works. For example, for $f\in\mathcal{A}_p$we have the following special:

Putting p=1, we obtain the operator $\tilde{I}(a,c,\lambda)$ studied by Raina and Sharma (see [16]).

Putting $a=\beta, c=\beta+1$ and $\lambda=1$,we obtain the operator $\mathfrak{I}_p^{\beta}(\beta>-p)$,which was studied by Saitoh et al. [20];

Putting $a=\beta, c=\alpha+\beta-\gamma+1$ and $\lambda=1$,we obtain the operator $\mathfrak{R}_{\beta,p}^{\alpha,\gamma}(\gamma>0;\alpha\geq\gamma-1;\beta>-p)$,which was studied by Aouf et al.[1];

Putting $a=\beta, c=\alpha+\beta$ and $\lambda=1$,we obtain operator $\mathcal{X}_{\beta,p}^{\alpha}(\alpha\geq 0;\beta>-p)$,which was studied by Liu and Owa[12];

Putting $p=1, a=\beta, c=\alpha+\beta$ and $\lambda=1$,we obtain operator $\mathfrak{R}_{\beta}^{\alpha}(\alpha\geq 0;\beta>-1)$,which was studied by Jung et al.[7];

Putting $p=1, a=\alpha-1, c=\beta-1$and $\lambda=1$,we obtain the operator $L(\alpha,\beta)$ $(\alpha,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-)$,which was studied by Carlson and Shaffer[3];

Putting $p=1, a=a-1, c=v$ and $\lambda=1$,we obtain the operator $I_{a,v}(a>0;v\geq -1)$, which was studied by Choi et al.[4];

Putting $p=1, a=\alpha, c=0$ and $\lambda=1$we obtain the operator $\mathfrak{D}^{\alpha}(\alpha>-1)$,which was studied by Ruscheweyh [17];

Putting $p=1, a=\alpha, c=m$ and $\lambda=1$,we obtain the operator $I_m(m\in\mathbb{N}_0:=\mathbb{N}\cup\{0\})$,which was studied by Noor[14];

Putting $p=1, a=\alpha, c=\beta+1$ and $\lambda=1$, we obtain the operator $\mathcal{J}_{\beta}$,which was studied by Bernardi [2];

Putting $p=1, a=1, c=2$ and $\lambda=1$,we obtain $\mathcal{J}$,which was studied by Libera [11].

Let H be the class of functions h with $h(0)=1$, which are analytic and convex univalent in U.

Definition 1. 1. A function $f\in\mathcal{A}_p$ is said to be in the class $E_p(a,c,\lambda,\gamma;h)$ if it satisfies the subordination condition:

$$(1-\gamma)z^{-p}\mathfrak{I}_p(a,c,\lambda)f(z)+\gamma z^{-p}\mathfrak{I}_p(a+1,c,\lambda)f(z)\prec h(z), \tag{1.6}$$

where $\gamma\in\mathbb{C}$ and $h\in H$.

A function $f\in\mathcal{A}$ is said to be in the class $S^*(\epsilon)$ if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\epsilon \qquad (z\in U).$$

for some ϵ $(\epsilon<1)$.
When $0\leq\epsilon<1$, $S^*(\epsilon)$ is the class of starlike functions of order ϵ in U.

A function $f\in\mathcal{A}$ is said to be prestarlike of order ϵ in U if

$$\frac{z}{(1-z)^{2(1-\epsilon)}}*f(z)\in S^*(\epsilon) \qquad (\epsilon<1).$$

We note this class by $\Re(\epsilon)$.
Clearly a function $f \in \mathcal{A}$ is in the class $\Re(0)$ if and only if f is convex univalent in U and $\Re\left(\frac{1}{2}\right) = S^*(\frac{1}{2})$.

Lemma 1. 1[13]. Let g be analytic in U and let h be analytic and convex univalent in U with $h(0) = g(0)$. If

$$g(z) + \frac{1}{\xi} z g'(z) \prec h(z), \tag{1.7}$$

where $\mathrm{Re}\,\xi \geq 0$ and $\xi \neq 0$, then

$$g(z) \prec \tilde{h}(z) = \xi z^{-\xi} \int_0^z t^{\xi-1} h(t)dt \prec h(z),$$

and $\tilde{h}(z)$ is the best dominant of (1.7) .

Lemma 1.2[18]. Let $\epsilon < 1$, $f \in S^*(\epsilon)$ and $g \in \Re(\epsilon)$. Then, for any analytic function F in U,

$$\frac{g * (fF)}{g * f}(U) \subset \overline{co}\big(F(U)\big),$$

where $\overline{co}\big(F(U)\big)$ denotes the closed convex hull of $F(U)$.

Some of the following properties studied for other classes in [5], [10], [15] and [21].

2. Main Results

Theorem 2.1. Let $0 \leq \gamma_1 < \gamma_2$. Then $E_p(a, c, \lambda, \gamma_2; h) \subset E_p(a, c, \lambda, \gamma_1; h)$.

Proof. Let $0 \leq \gamma_1 < \gamma_2$ and $f \in E_p(a, c, \lambda, \gamma_2; h)$.

Suppose that

$$g(z) = z^{-p} \mathfrak{I}_p(a, c, \lambda) f(z). \tag{2.1}$$

Then the function g is analytic in U with $g(0) = 1$.
Since $f \in E_p(a, c, \lambda, \gamma_2; h)$, then we have

$$(1 - \gamma_2) z^{-p} \mathfrak{I}_p(a, c, \lambda) f(z) + \gamma_2 z^{-p} \mathfrak{I}_p(a + 1, c, \lambda) f(z) \prec h(z). \tag{2.2}$$

From (2.1) and (2.2) we get

$$(1 - \gamma_2) z^{-p} \mathfrak{I}_p(a, c, \lambda) f(z) + \gamma_2 z^{-p} \mathfrak{I}_p(a + 1, c, \lambda) f(z) = g(z) + \frac{\lambda\gamma_2}{a + \lambda p} z g'(z) \quad \prec h(z). \tag{2.3}$$

By using Lemma 1.1 we have

$$g(z) \prec h(z)\,. \tag{2.4}$$

Nothing that $0 \leq \frac{\gamma_1}{\gamma_2} < 1$ and that h is convex univalent in U.
Hence

$$(1 - \gamma_1) z^{-p} \mathfrak{I}_p(a, c, \lambda) f(z) + \gamma_1 z^{-p} \mathfrak{I}_p(a + 1, c, \lambda) f(z)$$
$$= \frac{\gamma_1}{\gamma_2}\Big((1 - \gamma_2) z^{-p} \mathfrak{I}_p(a, c, \lambda) f(z) + \gamma_2 z^{-p} \mathfrak{I}_p(a + 1, c, \lambda)\Big) + \left(1 - \frac{\gamma_1}{\gamma_2}\right) g(z) \prec h(z)\,.$$

Thereforef $\in E_p(a, c, \lambda, \gamma_1; h)$, and we obtain the result.

Theorem 2.2. Let $f \in E_p(a, c, \lambda, \gamma; h)$, $g \in \mathcal{A}_p$ and

$$\mathrm{Re}\{z^{-p} g(z)\} > \frac{1}{2}. \tag{2.5}$$

Then

$$(f * g)(z) \in E_p(a, c, \lambda, \gamma; h).$$

Proof. Let $f \in E_p(a, c, \lambda, \gamma; h)$, and $g \in \mathcal{A}_p$. Then we have

$(1-\gamma)z^{-p}\mathfrak{I}_p(a,c,\mu)(f*g)(z)+\gamma z^{-p}\mathfrak{I}_p(a+1,c,\mu)(f*g)(z)$

$$=(1-\gamma)\left(z^{-p}g(z)\right)*\left(z^{-p}\mathfrak{I}_p(a,c,\lambda)f(z)+\gamma\left(z^{-p}g(z)\right)*\left(z^{-p}\mathfrak{I}_p(a+1,c,\lambda)f(z)\right)=\left(z^{-p}g(z)\right)*\phi(z),\qquad(2.6)$$

where

$$\phi(z)=(1-\gamma)z^{-p}\mathfrak{I}_p(a,c,\lambda)+\gamma z^{-p}\mathfrak{I}_p(a+1,c,\lambda)\prec h(z).\qquad(2.7)$$

From (2.5) note that the function $z^{-p}g(z)$ has the Herglotz representation

$$z^{-p}g(z)=\int_{|x|=1}\frac{d\mu(x)}{1-xz}\quad(z\in U),\qquad(2.8)$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x|=1$ and

$$\int_{|x|=1}d\mu(x)=1.$$

Since h is convex univalent in, it follows from (2.6) to (2.8) that

$$(1-\gamma)z^{-p}\mathfrak{I}_p(a,c,\lambda)(f*g)(z)+\gamma z^{-p}\mathfrak{I}_p(a+1,c,\lambda)(f*g)(z)=\int_{|x|=1}\phi(xz)d\mu(x)\prec h(z).$$

Therefore

$$(f*g)(z)\in E_p(a,c,\lambda,\gamma;h).$$

Corollary2.1. Let $f\in E_p(a,c,\lambda,\gamma;h)$, be defined as in (1.1) and let

$$\mathrm{Re}\left\{1+\sum_{n=p+1}^{\infty}\frac{\tau+p}{\tau+n}z^{n-p}\right\}>\frac{1}{2}.\qquad(2.9)$$

Then

$$r(z)=\frac{\tau+p}{z^{\tau}}\int_0^z t^{\tau-1}f(t)dt,\ (\tau>-p)$$

is also in the class $E_p(a,c,\lambda,\gamma;h)$.

Proof. Let $f\in E_p(a,c,\lambda,\gamma;h)$, be defined as in (1.1).

Then

$$r(z)=\frac{\tau+p}{z^{\tau}}\int_0^z t^{\tau-1}f(t)dt=z^p+\sum_{n=p+1}^{\infty}\frac{\tau+p}{\tau+n}a_nz^n$$

$$=\left(z^p+\sum_{n=p+1}^{\infty}a_nz^n\right)*\left(z^p+\sum_{n=p+1}^{\infty}\frac{\tau+p}{\tau+n}z^n\right)=(f*F)(z),\qquad(2.10)$$

where

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n\in E_p(a,c,\lambda,\gamma;h)$$

and

$$F(z)=z^p+\sum_{n=p+1}^{\infty}\frac{\tau+p}{\tau+n}z^n\in\mathcal{A}_p.$$

Note that

$$\mathrm{Re}\{z^{-p}F(z)\} = \mathrm{Re}\left\{1+\sum_{n=p+1}^{\infty}\frac{\tau+p}{\tau+n}z^{n-p}\right\} > \frac{1}{2}. \tag{2.11}$$

From (2.10) and (2.11) and by using Theorem 2.2, we get $r(z) \in E_P(a,c,\lambda,\gamma;h)$.

Theorem 2.3. Let $f \in E_p(a,c,\lambda,\gamma;h)$, $g \in \mathcal{A}_p$ and $z^{1-p}g(z) \in \Re(\epsilon)$, $(\epsilon < 1)$.
Then

$$(f * g)(z) \in E_p(a,c,\lambda,\gamma;h).$$

Proof. Let $f \in E_p(a,c,\lambda,\gamma;h)$, and $g \in \mathcal{A}_p$. Then, we have

$$(1-\gamma)z^{-p}\mathfrak{J}_p(a,c,\lambda) + \gamma z^{-p}(a+1,c,\lambda) \prec h(z). \tag{2.12}$$

Now

From (1.5), (2.12) is equivalent to

$$\frac{a+(1-\gamma)\lambda p}{a+\lambda p}z^{-p}\mathfrak{J}_p(a,c,\lambda) + \frac{\lambda\gamma}{a+\lambda p}z^{1-p}(\mathfrak{J}_p(a,c,\lambda)f(z))' \prec h(z). \tag{2.13}$$

Hence

$$\frac{a+(1-\gamma)\lambda p}{a+\lambda p}z^{-p}\mathfrak{J}_p(a,c,\lambda)(f*g)(z) + \frac{\lambda\gamma}{a+\lambda p}z^{1-p}\left(\mathfrak{J}_p(a,c,\lambda)(f*g)(z)\right)'$$

$$= \frac{a+(1-\gamma)\lambda p}{a+\lambda p}\left(z^{-p}g(z)\right) * \left(z^{-p}\mathfrak{J}_p(a,c,\lambda)f(z)\right) + \frac{\lambda\gamma}{a+\lambda p}\left(z^{-p}g(z)\right) \quad * \left(z^{1-p}\left(\mathfrak{J}_p(a,c,\lambda)f(z)\right)'\right)$$

$$= \frac{\left(z^{1-p}g(z)\right) * (z\psi(z))}{\left(z^{1-p}g(z)\right) * z} \quad (z \in U)\ , \tag{2.14}$$

where

$$\psi(z) = \frac{a+(1-\gamma)\lambda p}{a+\lambda p}z^{-p}\mathfrak{J}_p(a,c,\lambda) + \frac{\lambda\gamma}{a+\lambda p}z^{1-p}(\mathfrak{J}_p(a,c,\lambda)f(z))' \prec h(z). \tag{2.15}$$

Since h is convex univalent in U, $\psi(z) \prec h(z)$, $z^{1-p}g(z) \in \Re(\epsilon)$ and $z \in S^*(\epsilon)$, $(\epsilon < 1)$,

it follows from (2.14) and Lemma 1.2 , we get the result.

Theorem2.4. Let $\gamma > 0$, $\omega > 0$ and $f \in E_p(a,c,\lambda,\gamma;\omega h + 1 - \omega)$. If $\omega \le \omega_0$, where

$$\omega_0 = \frac{1}{2}\left(1 - \frac{a+\lambda p}{\lambda\gamma}\int_0^1 \frac{u^{\frac{a+\lambda p}{\lambda\gamma}-1}}{1+u}du\right)^{-1}, \tag{2.16}$$

then $f \in E_p(a,c,\lambda,0;h)$. The bound ω_0 is the sharp when $h(z) = \frac{1}{1-z}$.

Proof. Suppose that

$$g(z) = z^{-p}\mathfrak{J}_p(a,c,\lambda)f(z)\ . \tag{2.17}$$

Let $f \in E_p(a,c,\lambda,\gamma;\omega h + 1 - \omega)$ with $\gamma > 0$ and $\omega > 0$.

Then, we have

$$g(z) + \frac{\lambda\gamma}{a+\lambda p}zg'(z) = (1-\gamma)z^{-p}\mathfrak{J}_p(a,c,\lambda) + \gamma z^{-p}(a+1,c,\lambda) \prec \omega h(z) + 1 - \omega\ .$$

By using Lemma 1.1, we have

$$g(z) \prec \frac{\omega(a+\lambda p)}{\lambda\gamma} z^{-\frac{a+\lambda p}{\lambda\gamma}} \int_0^z t^{\frac{a+\lambda p}{\lambda\gamma}-1} h(t)dt + 1 - \omega = (h * \varphi)(z), \tag{2.18}$$

where

$$\varphi(z) = \frac{\omega(a+\lambda p)}{\lambda\gamma} z^{-\frac{(a+\lambda p)}{\lambda\gamma}} \int_0^z \frac{t^{\frac{a+\lambda p}{\lambda\gamma}-1}}{1-t} dt + 1 - \omega. \tag{2.19}$$

If $0 < \omega \le \omega_0$, where $\omega_0 < 1$ is given by (2.16), then it follows from (2.19) that

$$\begin{aligned} \mathrm{Re}(\varphi(z)) &= \frac{\omega(a+\lambda p)}{\lambda\gamma} \int_0^1 u^{\frac{a+\lambda p}{\lambda\gamma}-1} \mathrm{Re}\left(\frac{1}{1-uz}\right) du + 1 - \omega \\ &> \frac{\omega(a+\lambda p)}{\lambda\gamma} \int_0^1 \frac{u^{\frac{a+\lambda p}{\lambda\gamma}-1}}{1+u} du + 1 - \omega \ge \frac{1}{2}. \end{aligned}$$

Now, by using the Herglotz representation for(z) , from (2.17) and (2.18), we arrive at

$$z^{-p}\mathfrak{I}_p(a,c,\lambda) \prec (h * \varphi)(z) \prec h(z).$$

Since h is convex univalent in U, then $f \in E_p(a,c,\lambda,0;h)$.

For $h(z) = \frac{1}{1-z}$ and $f \in \mathcal{A}_p$ defined by

$$z^{-p}\mathfrak{I}_p(a,c,\lambda)f(z) = \frac{\omega(a+\lambda p)}{\lambda\gamma} z^{-\frac{(a+\lambda p)}{\lambda\gamma}} \int_0^z \frac{t^{\frac{a+\lambda p}{\lambda\gamma}-1}}{1-t} dt + 1 - \omega,$$

we have

$$(1-\gamma)z^{-p}\mathfrak{I}_p(a,c,\lambda) + \gamma z^{-p}(a+1,c,\lambda) = \omega h(z) + 1 - \omega.$$

Thus $f \in E_p(a,c,\lambda,\gamma;\omega h + 1 - \omega)$.

Also for $\omega > \omega_0$, we have

$$Re\{z^{-p}\mathfrak{I}_p(a,c,\lambda)f(z)\} \longrightarrow \frac{\omega(a+\lambda p)}{\lambda\gamma} \int_0^1 \frac{u^{\frac{a+\lambda p}{\lambda\gamma}-1}}{1+u} du + 1 - \omega < \frac{1}{2}, \qquad (z \to 1)$$

which implies that $f \notin E_p(a,c,\lambda,0;h)$.

Therefore the bound ω_0 cannot be increased when $h(z) = \frac{1}{1-z}$.

This completes the proof of the theorem.

Theorem 2.5. Let $f \in E_p\left(a+1,c,\lambda,\gamma;\frac{1+Az}{1+Bz}\right), a > -\lambda p, -1 \le B < A \le 1$. Then

$$z^{-p}\mathfrak{I}_p(a+1,c,\lambda)f(z) \prec \tilde{h}(z) = \frac{a+\lambda p+1}{\lambda\gamma} z^{-\frac{(a+\lambda p+1)}{\lambda\gamma}} \int_0^z t^{\frac{a+\lambda p+1}{\lambda\gamma}-1} \left(\frac{1+Az}{1+Bz}\right) dt$$

and $\tilde{h}$ is the best dominant.

Proof. Let $f \in E_p\left(a+1,c,\lambda,\gamma;\frac{1+Az}{1+Bz}\right)$. Then, we have

$$(1-\gamma)z^{-p}\mathfrak{I}_p(a+1,c,\lambda) + \gamma z^{-p}(a+2,c,\lambda) \prec \frac{1+Az}{1+Bz}. \tag{2.20}$$

Suppose that

$$g(z) = z^{-p}\mathfrak{I}_p(a+1,c,\lambda)f(z). \tag{2.21}$$

Then the function g is analytic in U with $g(0) = 1$.

From (1.5), (2.20) and (2.21), we get

$$(1-\gamma)z^{-p}\mathfrak{I}_p(a+1,c,\lambda) + \gamma z^{-p}(a+2,c,\lambda) = g(z) + \frac{\lambda\gamma}{a+\lambda p+1}zg'(z) \prec \frac{1+Az}{1+Bz}. \tag{2.22}$$

By Lemma 1.1, we obtain

$$g(z) \prec \tilde{h}(z) = \frac{a+\lambda p+1}{\lambda\gamma} z^{-\frac{(a+\lambda p+1)}{\lambda\gamma}} \int_0^z t^{\frac{a+\lambda p+1}{\lambda\gamma}-1}\left(\frac{1+Az}{1+Bz}\right)dt$$

and $\tilde{h}$ is the best dominant . Thus we have the result.

3.Conclusions

The study explored various inclusion relationships among subclasses of p-valent functions defined by a family of integral operators. It demonstrated that certain classes of multivalent analytic functions are closed under specific operations, confirming previous results and expanding the understanding of these function classes.

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