An Inverse Scattering Problem

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Abstract:

In this paper we give an elementary method to study an inverse scattering problem for a pair of Hamiltonians (H(h), H_0(h)) on $L^2(IR^n)$, where $H_0(h) = - h^2 \Delta$, $H(h) = H_0(h) + V$, and V is a short-range potential. We show that, in dimension $n \geq 3$, the scattering operators $S(h); h \in (0, h_0]$ which are localized near a fixed energy $\lambda > 0$,determine the asymptotic of the potential V at infinity. This approach can be used to solve an inverse scattering problem for isotropic external metrics.

١. Introduction:

In this paper we study an inverse scattering problem for a pair (H(h), H_0(h)) on $L^2(IR^n)$, $n \geq 2$ where the free operator

$H_0(h) = -h^2\Delta$; $h \in (0, h_0]$ is the semi-classical parameter with h_0 small enough and

$$H(h) = H_0(h) + V, \qquad (1)$$

where $V \in C^\infty(IR^n)$ and satisfies $\forall \alpha \in IN^n$,

$$\left|\partial_x^\alpha V(x)\right| \leq C_\alpha < x >^{-\rho-|\alpha|}, \rho > 1. \qquad (a)$$

under the hypothesis (a) the wave operators:

$$W^\pm(H(h), H_0(h)) = s - \lim_{t \to \pm\infty} e^{itH(h)} e^{-itH_0(h)}, \qquad (2)$$

exist and complete, i.e. Ran

$W^\pm(H(h), H_0(h)) = H_{ac}(H)$ =subspace of absolute continuity of H(h) [8]. Let S(h) be the scattering operator defined by

$$S(h) = W^{+*}(H(h), H_0(h)) W^-(H(h), H_0(h)) \qquad (3)$$

In order to localize the scattering operator near a fixed energy $\lambda > 0$, we introduce a cut-off function $x \in C_0^\infty(0, +\infty), x = 1$ in a neighborhood of $\lambda > 0$.

The goal of this note is to obtain some information on the potential from S(h)x(H_0(h)), in the semi-classical limit $h \to 0$. We show that for $n \geq 3$. the operators S(h) x (H_0(h)), $h \in (0, h_0]$, determine the asymptotic of the potential at infinity, by studying the asymptotic of :

$$F(h) =< S(h) x(H_0(h)) \Phi_{h,w}, \Psi_{h,w} > \qquad (4)$$

where < , > is the usual scalar product in $L^2(IR^n)$, and $\Phi_{h,w}, \Psi_{h,w}$ are suitable test functions .

2. Semi-classical asymptotics for the localized cattering operator and application:

2.1 Definition of the test functions:

The dilation operator $U(h^\delta), \delta > 0$, on $L^2(IR^n)$ can be defined as follows [2]:

$$U(h^\delta)\Phi(x) = h^{\frac{n\delta}{2}} \Phi(h^\delta x). \qquad (5)$$

we also need an energy cut-off $x_0 \in C_0^\infty(IR^n)$ such that $x_0(\xi) = 1$ if $|\xi| \leq 1, x_0(\xi) = 0$ if $|\xi| \geq 2.$

For $w \in S^{n-1}$, we write $x \in IR^n$ as

$x = y + tw, y \in \Pi_w$ = orthogonal hyperplane to *w* and we consider:

$$X_w = \{x = y + tw \in IR^n : |y| \geq 1\}. \qquad (6)$$

Now, can define for $\Phi \in C_0^\infty(X_w)$ and suitable $\delta, \in > 0$,

$$\Phi_{h,w} = e^{\frac{i}{h}\sqrt{\lambda}x.w} U(h^\delta) x_0(h^\in D)\Phi, \qquad (7)$$

where $D = -i\nabla$, ($\Psi_{h,w}$ is defined in the same way with $\Psi \in C_0^\infty(X_w)$).

2.2 Semi-classical asymptotics for the scattering operator:

In this section we prove the following theorem:

Theorem 2.2.1:

$$< (S(h) - 1) x(H_0(h)) \Phi_{h,w}, \Psi_{h,w} > = \frac{1}{2i\sqrt{\lambda h}} < \int_{-\infty}^{+\infty} V(h^{-\delta}x + tw) dt \Phi, \Psi > + o(h^\mu),$$

(8)

where $\mu = \delta(\rho - 1) - 1 > 0.$

Remark:

Using (a), it is easy to see that the first term of the (R.H.S) is equal to $O(h^\mu)$ since Φ, Ψ have their support in X_w.

Proof:

Step 1 :

Let begin by an elementary lemma [8]

Lemma 2.2.1:

$\forall \in < 1 + \delta, \forall h \in (0, h_0]$, *we have* :

$$x(H_0(h)) \Phi_{h,w} = \Phi_{h,w} \qquad (9)$$

we easily obtain:

$$F[x(H_0(h))\Phi_{h,w}](\xi) = h^{-\frac{n\delta}{2}} x((h\xi)^2) x_0(h^{\in-\delta-1}(h\xi - \sqrt{\lambda}w)) F\Phi(h^{-\delta-1}(h\xi - \sqrt{\lambda}w)),$$

(10)

where *F* is the usual Fourier transform then, on *Supp* x_0 , we have $\left|h\xi - \sqrt{\lambda w}\right| \leq 2h^{1+\delta-\in}$.

so, for $\in < 1 + \delta$ and h_0 small enough, we have $x((h\xi)^2) = 1.$

Then by Lemma 2.2.1, we obtain

$$F(h) =< W^-(H(h), H_0(h))\Phi_{h,w}, W^+(H(h), H_0(h))\Psi_{h,w} >,$$

(11)

and calculation gives

$$F(h) =< \Omega^-(h, w) x_0, (h^\in D)\Phi, \Omega^+(h, w), x_0(h^\in D)\Psi >,$$

(12)

where

$$\Omega^{\pm}(h,w) = s - \lim_{t\to\pm\infty} \; e^{itH(h,w)} \quad e^{-itH_0(h,w)}$$
(13)

with

$$H_0(h,w) = (D + \sqrt{\lambda} \;\; h^{-(1+\delta)} w)^2, \qquad (14)$$

and

$$H(h,w) = H_0(h,w) + h^{-2(1+\delta)} V(h^{-\delta} x). \quad (15)$$

so by (12) we have to find the asymptotic of $\Omega^{\pm}(h,w)\mathrm{x}_0(h^{\in}D)\Phi.$ We follow the same strategy as in [6], [7] and we only treat the case (+).

Step 2:

we construct a modifier $J^+(h,w)$ in the form as in [6], [7]

$$J^+(h,w) = 1 + h^{v} d^+(h^{-\delta} x, w), \qquad (16)$$

where v a suitable parameter defined below, we denote :

$$T^+(h,w) = H(h,w)J^+(h,w) - J^+(h,w)H_0(h,w). \qquad (١٧)$$

A direct calculation shows that

$$\begin{aligned} T^+(h,w) = {} & h^{-2(1+\delta)}V(h^{-\delta}x) - 2i\sqrt{\lambda} \;\; h^{-1-2\delta+v} w.\nabla d^+(h^{-\delta}x.w) \\ & + h^{v-2(1+\delta)}V(h^{-\delta}x)d^+(h^{-\delta}x,w) - h^{-2\delta+v}\Delta d^+(h^{-\delta}x,w) \\ & - 2h^{-\delta+v}\nabla d^+(h^{-\delta}x,w).\nabla. \end{aligned} \qquad (18)$$

thus, we choose $v = -1$ and we solve the transport equation

$$w.\nabla d^+(x.w) = \frac{1}{2i\sqrt{\lambda}} V(x). \qquad (19)$$

The solution of (19) is given by

$$d^+(x.w) = \frac{i}{2\sqrt{\lambda}} \int_0^{+\infty} V(x+tw)dt. \qquad (20)$$

we obtain

$$\left|\partial_x^{\alpha} d^+(h^{-\delta}x,w)\right| \le C_{\alpha} h^{\delta(\rho-1+|\alpha|)} \quad < x >^{1-\rho-|\alpha|}, \forall (x,w) \in \Gamma^+, \qquad (21)$$

where

$$\Gamma^+ = \{(x,\xi): |x| \ge R, |\xi| \ge a, x.\xi \ge -\sigma |x|\,|\xi|\}, \sigma \in (-1,1).$$

From this, we deduce the following lemma [6].

Lemma 2.2.2:

$$\Omega^+(h,w)\mathrm{x}_0(h^{\in}D)\Phi = \lim_{t\to+\infty} e^{itH(h,w)} J^+(h,w) e^{-itH_0(h,w)} \mathrm{x}_0(h^{\in}D)\Phi.$$
(22)

Step 3:

Now, we can formulate [8]

Lemma 2.2.3:

For $\delta > \dfrac{1}{\rho-1}$ *and* $\in < 1+\delta, \rho \ne 1,$ *we have*

$$\left\|(\Omega^+(h,w) - J^+(h,w))\mathrm{x}_0(h^{\in}D)\Phi\right\| = o(h^{\delta(\rho-1)-1}). \qquad (23)$$

First, we write

$$(\Omega^+(h,w) - J^+(h,w))\mathrm{x}_0(h^{\in}D)\Phi = i\int_0^{+\infty} e^{itH(h,w)} T^+(h,w) e^{-itH_0(h,w)} \mathrm{x}_0(h^{\in}D)\Phi \;\, dt.$$
(24)

In order to introduce a cut-off, which localizes far from the origin, let V_w be a neighborhood of w in S^{n-1}. We define

$$O^+ = \{x + tw'; x \in Supp\Phi, t \ge 0, w' \in V_w\}. \qquad (25)$$

If V_w is rather small, it is clear that $O^+ \subset IR^n \setminus B$ where B is the unit ball. Let $X^+ \in C^{\infty}(IR^n \setminus B)$ be a cut-off function such that $\mathrm{x}^+ = 1$ in a conical neighborhood of O^+. Now, we have the following estimation which is obtained by using a standard non stationary phase argument

$$\forall \in < \delta + 1, \forall N \ge 1, (\mathrm{x}^+ - 1)e^{-itH_0(h,w)}\mathrm{x}_0(h^{\in}D)\Phi = O(< t >^{-N} \;\, h^N),$$
(26)

In the sense of the L^2-norm [7].

So, we deduce

$$\left\|(\Omega^+(h,w) - J^+(h,w))\mathrm{x}_0(h^{\in}D)\Phi\right\| \le \int_0^{+\infty} \left\|\mathrm{x}^+ T^+(h,w)\, e^{-itH_0(h,w)} \mathrm{x}_0(h^{\in}D)\Phi\right\| dt,$$
(27)

Modulo $O(h^{\infty})$. Since $Supp \;\, \mathrm{x}^+ \subset IR^n \setminus B,$ we obtain using (21) and (27) as in [7]

$$\left\|(\Omega^+(h,w) - J^+(h,w))\mathrm{x}_0(h^{\in}D)\Phi\right\| = O(h^{\delta(\rho-1)-1})[O(h^{1+\delta}) + O(h^{\delta(\rho-1)-1})],$$
(28)

and the last term is equal to $o(h^{\delta(\rho-1)-1})$ if $\delta > \dfrac{1}{\rho-1}.$; $\rho \ne 1$

Step 4:

Using the following estimation on $L^2(IR^n): \forall N \ge 1,$

$$(\mathrm{x}_0(h^{\in}D) - 1)\Phi = O(h^N), \mathrm{n} \ge 1 \qquad (29)$$

and Lemma 2.2.3, we obtain that

$$\Omega^+(h,w)\mathrm{x}_0(h^{\in}D)\Phi = (1 + \frac{i}{2\sqrt{\lambda h}} d^+(h^{-\delta}x,w) + o(h^{\delta(\rho-1)-1}))\Phi. \;\square$$
(30)

2.3 An application to an inverse scattering problem in semi-classical asymptotics :

We use equation (8) in the particular case where V is an asymptotic homogeneous function. Let V_j , j = 1, 2 be two potentials satisfying when $|x| \to +\infty,$

$$V_j(x) = |x|^{-\rho} \;\; fi(\frac{x}{|x|}) + o(|x|^{-\rho}), \rho > 1, \qquad (31)$$

where $fi \in C^{\infty}(S^{n-1}), S^{n-1}$ being the unit sphere of IR^n.

we denote $S_j(h)$ the scattering operator associated with the pair ($H_0(h)+V_j$, $H_0(h)$). We have the following result

Corollary 2.3.1:

For $n \ge 3,$ *assume that* $\forall h \in (0,h_0], S_1(h)\mathrm{x}(H_0(h)) = S_2(h)\mathrm{x}(H_0(h)).$

then $f_1 = f_2$.

Proof:

We have for $\delta > \dfrac{1}{\rho-1}$ and $\in < 1+\delta,$ by Theorem 2.2.1

$$< (S_j(h) - 1)\mathrm{x}(H_0(h))\Phi_{h,w}, \Psi_{h,w} > = \frac{h^{\mu}}{2i\sqrt{\lambda}} < \int_{-\infty}^{+\infty} V_{0,j}(x+tw)dt\Phi, \Psi > + o(h^{\mu}).$$
(32)

where $V_{0,j} = |x|^{-\rho} \quad f_j\left(\dfrac{x}{|x|}\right).$

So, if

$S_1(h)\mathrm{x}(H_0(h)) = S_2(h)\mathrm{x}(H_0(h)), \forall h \in (0,h_0],$ we deduce

$$\forall w \in S^{n-1}, \forall x \in X_w, \int_{-\infty}^{+\infty} V_0(x+tw)dt = 0,$$
(33)

where $V_0 = V_{0,1}$-$V_{0,2}$. So, using the support theorem for the Radon transform [3], [6] we obtain $V_0(x) = 0, |x| \ge 1.$ $\square$

Remark:

In a non semi-classical context (h = 1), let us mention the reference [5] where the inverse scattering problem at a fixed energy is treated. They showed that if $V_1, V_2 \in S_{cl}^{-2}(IR^n), n \ge 3$ and if the associated matrices at some non-zero fixed energy are equal up to smooth terms then $V_1 - V_2 \in S^{-\infty}(IR^n).$

3. An inverse scattering problem in the case of isotropic external metrics:

3.1 Notations:

We show that the previous approach can be used to solve an inverse scattering problem for perturbation of order 2 of the free Laplacian $H_0 = -\Delta$.

Let us consider the following Hamiltonian on $L^2(IR^n), n \geq 2$:

$$H = \sum_{i,j} D_i g^{ij}(x) D_{ij}, \qquad (34)$$

where $G(x) = (g^{ij}(x))$ is a C^{∞}- definite positive metric satisfying

$$\forall \alpha \in IN^n, \left|\partial_x^\alpha (G(x) - Id)\right| \leq C_\alpha < x >^{-\rho-|\alpha|}, \rho > 1. \qquad (b)$$

so, we can define the wave operators [1]

$$W^{\pm} = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}, \qquad (35)$$

and the scattering operator $S = W^{+*}W^{-}$.

In order to study the asymptotic at high energies of the scattering operator, we consider the following test function

$$\Phi_{\lambda,w} = e^{i\sqrt{\lambda}x.w} U\left(\lambda^{-\frac{\delta}{2}}\right) x_0 \left(\lambda^{-\frac{\delta}{2}} D\right)\Phi, \qquad (36)$$

where

$\Phi \in C_0^\infty(X_w), (\Psi_{\lambda,w}$ is defined in the same way with $\Psi \in C_0^\infty(X_w))$.

In [2] Enss and Weder also used such test functions with $\delta = 0, \in = 0$. The uniqueness of inverse potential scattering problems is given in [6] and [7]. We have the following result where (,) is the scalar product in IR^n and $H(x) = G(x)-Id$:

Theorem 3.1.1:

For $\delta > \frac{1}{\rho - 1}$ *and* $\in < 1 + \delta, \rho \neq 1$ *we have, when* $\lambda \to +\infty$,

$$< (S-1)\Phi_{\lambda,w}, \Psi_{\lambda,w} > = \frac{\sqrt{\lambda}}{2i} < \int_{-\infty}^{+\infty} (H(\lambda^{-\frac{\delta}{2}} x + tw)w, w) dt \Phi, \Psi > + o(\lambda^{-\frac{\mu}{2}}).$$

(37)

where $\mu = \delta(\rho - 1) - 1 > 0$.

Proof:

As in section 2, we define $F(\lambda) = < W^- \Phi_{\lambda,w}, W^+ \Psi_{\lambda,w} >$. We see that:

$$F(\lambda) = < \Omega^-(h,w) x_0 (h^{\in} D)\Phi, \Omega^+(h,w) x_0(h^{\in} D)\Psi >, \qquad (38)$$

where $\Omega^{\pm}(h,w)$ is defined by (29) with $h = \lambda^{-\frac{1}{2}}$ and

$$H(h,w) = \sum_{i,j} (D_i + h^{-1(1+\delta)} w_i) g^{ij} (h^{-\delta} x)(D_j + h^{-(1+\delta)} w_j). \qquad (39)$$

so, everything done section 2 also works in this situation.

3.2 An application to an inverse scattering problem of isotropic external metrics:

We consider isotropic homogeneous metrics $G_j, j = 1, 2$ satisfying (b) and when $|x| \to +\infty$

$$G_j(x) - Id = |x|^{-\rho} \quad f_j\left(\frac{x}{|x|}\right) Id + o(|x|^{-\rho}), \rho > 1 \qquad (40)$$

where $f_j \in C^\infty(S^{n-1})$. Let S_j be the associated scattering operator. As in section 3, we have:

Corollary 3.2.1:

In dimension n ≥ 3 we have

$$S_1 = S_2 \Rightarrow f_1 = f_2 \qquad (41)$$

Proof:

We have for $\delta > \frac{1}{\rho - 1}$ and $\in < 1 + \delta$, by Theorem 1

$$< (S_j(h) - 1) x(H_0(h))\Phi_{h,w}, \Psi_{h,w} > = \frac{h^\mu}{2i\sqrt{\lambda}} < \int_{-\infty}^{+\infty} G_{0,j}(x + tw) dt \Phi, \Psi > + o(h^\mu).$$

(42)

where $G_{0,j} = |x|^{-\rho} \quad f_j\left(\frac{x}{|x|}\right)$.

So, if

$S_1(h) x(H_0(h)) = S_2(h) x(H_0(h)), \forall h \in (0, h_0]$, we deduce $\forall w \in S^{n-1}, \forall x \in X_w, \int_{-\infty}^{+\infty} G_0(x + tw) dt = 0,$

(43).

So, using the support theorem for the Radon transform [3], [6] we obtain

$G_0(x) = 0, |x| \geq 1.$□

4. Discussion:

The present work deals with the inverse scattering problem of a pair of Hamiltonians (H(h), H_0(h)) on $L^2(IR^n)$. Section 2 discussed the approximately semi classical technique for the localized scattering operators through finding the wave operator, in addition to formulate the scattering operator equation in dimension $n \geq 3$ which centered near bounded energy $\lambda > 0$. From the scattering operator equation we determine the approximated potential at infinity, in addition to some applications and this is clear from corollary 2.3.1. Section 3 discussed the inverse scattering operators for isotropic external metrics, in addition to finding the wave operators which lead us to finding the scattering operator in dimension $n \geq 3$, also we mention some applications on this case which is clear from corollary 3.2.1.

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معضلة الاستطارة العكسية

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الملخص:

في هذا البحث تم إعطاء الطريقة الأساسية لدراسة مسألة الاستطارة العكسية لزوج من المؤثرات الهاملتونية $(H(h), H_0(h))$ على $L^2(IR^n)$ ، إذ أن :

$$H(h) = H_0(h) + V, \quad H_0 (h) = - h^2 \Delta$$

وتمثل V الجهد القصير المدى ، فقد تم إثبات أن مؤثرات الاستطارة $S(h)$ لقيم $h \in (0, h_0]$ في البعد $n \geq 3$ ، قد تمركزت قرب طاقة محددة قيمتها $\lambda > 0$ والتي تحدد الجهد التقريبي عند اللانهاية، إن هذه الطريقة يمكن استخدامها لحل مسألة الاستطارة العكسية لأي معيار متساوي الخواص .