Journal of Al-Qadisiyah for Computer Science and Mathematics Vol.16.(3) 2024,pp.Math 71–80

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

D-Index of Certain Ladder Graphs

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ARTICLE INFO

Article history:
Received: 26 /06/2024
Rrevised form: 03 /08/2024
Accepted : 03 /09/2024
Available online: 30 /09/2024

Keywords:

D-distance,

Wiener D-index,

Ladder graph, Semi-ladder graph

ABSTRACT

For any two distinct vertices u, v of a connected graph H, the D-distance $d^D(u,v) = min_s\{l(s) + \sum_{w\in V(s)} deg(w)\}$, in which the minimum is taken over all $u - v$ paths, and $l(s)$ is the length of the path s. The D-index of H is defined as $W^D(H) = \frac{1}{2}\sum_{u,v\in V(G),u\neq v} d^D(u,v)$. In this paper, we obtained a formula for D-index or Wiener D-index $W^D(L_n)$, where L_n is the ladder graph, $n \geq 3$. Also, we obtained the Wiener D-index and Wiener index of semi-Ladder graph L_n^*.

MSC..

https://doi.org/ 10.29304/jqcsm.2024.16.31668

1. Introduction

Finite undirected connected and simple graphs are the only considerations in this paper. We refer the reader to [6, 8]. Let H be a graph with vertex set $V(H)$ and edge set $E(H)$. The graph H is represented by ordered pair (V, H). For a vertex v, the degree of v is the number of vertices adjacent to v or is the number of edges incident to v, denoted by $deg_H(v)$ or simply $\deg(v)$. In graph theory one of the most appreciated concepts is distance, it has applications in isomorphism testing, graph operations, hamiltonicity problem, diameter and extermal problems on connectivity. Let u and v be two vertices of H. The standard distance $d(u,v)$ between any two arbitrary vertices u and v is the length of the shortest $u - v$ path in H. Various concepts of distances have been established by researchers as well as ordinary distance, such as width distance [9], Steiner distance [4], degree distance, etc. The concept of detour distance in graphs was instigated by [7] as follows: Let u and v be two distinct vertices in a graph H, then the detour distance $D(u,v)$ is defined as the length of the longest $u - v$ path in H. In [3] Ali and MohammedSaleh determined the detour polynomials and detour index of ladder graphs L_n. The authors in [1] obtained the detour index of some cog special graphs. The restricted detour distance between vertices u and v is the length of the longest $u - v$ path P such that $< V(P) >= P$ [11].

The concept of superior distance defined as: for any two vertices u and v in a graph H a $D_{u,v}$ −walk is a $u - v$ walk in H that contains every vertex of $D_{u,v}$ where $D_{u,v} = N[u] \cup N[v]$. The superior distance is the length of a shortest $D_{u,v}$ −walk is introduced by Kathiresan and Marimuthu [10]. All distances that have been put forward depend on the path length in a connected graph H. The concept of D-distance between two distinct vertices of a graph H was introduced in an earlier article; it was brought forward by considering the path length between vertices, as well as the degrees of all vertices that lie on this path. The D-distance and its properties were introduced and studied by Babu and Varma [5]. Rao and Varma [13] launched the concept of detour $D-$distance and related work. In [12], MohammedSaleh and Aziz, studied the detour D-index of various graphs, such as the French windmill, Kulli-wheel windmill, lollipop and general barbell graphs.

A topological index is a numerical parameter mathematically derived from the graph structure. It is a graph invariant, and has many applications in different fields. The D-index of graphs has various applications in different fields, particularly in network analysis, Chemistry, transportation and information systems. The index has been used in quantitative structure-activity relationship (QSAR) studies structure-boiling point modeling. The Wiener index of a graph H, represented by $W(H)$, is defined as the sum of distances between all pairs of vertices in a simple graph H. Ali and Aziz [2] found the relationship between Wiener index, and D-index for $r-$regular graphs of order n, they proved that

$$W^D(H) = (r+1)W(H) + r\binom{n}{2}.$$

The study of D-index in ladder graphs aims to investigate the relationships between the topological index and various graph properties, providing insight into the behaviors of ladder graphs in different applications such as computer network modeling, chemical and molecular structure analysis, and social network analysis. Molecules with higher D-indices might indicate the presence of alternative bonding paths, leading to more stable concepts.

In this article, we obtained a formula for Wiener D-index of a graph L_n, where L_n is the ladder graph, $n \geq 3$. Also, we obtained the Wiener D-index and Wiener index of semi-ladder graph L_n^* are also obtained.

Definition 1.1.[5] If u, v are vertices of a connected graph H, the D-length of a $u-v$ path s is defined as $l^D\,(s) = d(u,v) + \deg(u) + \deg(v) + \deg\sum(w)$ where the sum runs over all intermediate vertices w of s.

Definition 1.2.[5] The D-distance $d^D(u,v)$ between two vertices u,v of a connected graph H is defined as $d^D(u,v) = \min\{l^D(s)\}$ where the minimum is taken over all $u-v$ paths s in H. In otherwords, $d^D(u,v) = \min\{d(u,v) + \deg(u) + \deg(v) + \sum \deg(w)\}$ where the sum runs over all intermediate vertices w in s and minimum is taken over all $u-\ v$ paths in H.

1. D-index of Ladder L_n

By direct calculation we get $W^D(L_2) = 36$, $W^D(L_3) = 119$, $W^D(L_4) = 270$, $W^D(L_5) = 505$ and $W^D(L_6) = 840$.

To illustrate the method used in getting such results, we consider L_5. The D-distances between every pair of distinct vertices of L_5 are given in the following table, in which L_5 is shown in Fig. 2.1.

d^D	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	row sum
x_1	-	6	10	14	17	5	9	13	17	20	111
x_2		-	7	11	14	9	7	11	15	17	91
x_3			-	7	10	13	11	7	11	13	72
x_4				-	6	17	15	11	7	9	65
x_5					-	20	17	13	9	5	64
y_1						-	6	10	14	17	47
y_2							-	7	11	14	32
y_3								-	7	10	17

y_4									-	6	6
y_5										-	-

Therefore $W^D(L_5) = 111 + 91 + 72 + 65 + 64 + 47 + 32 + 17 + 6 = 505.$

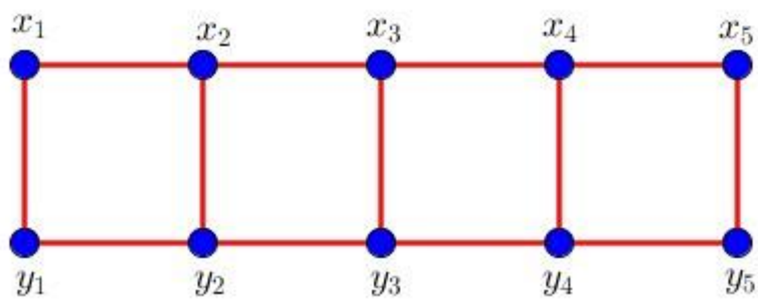
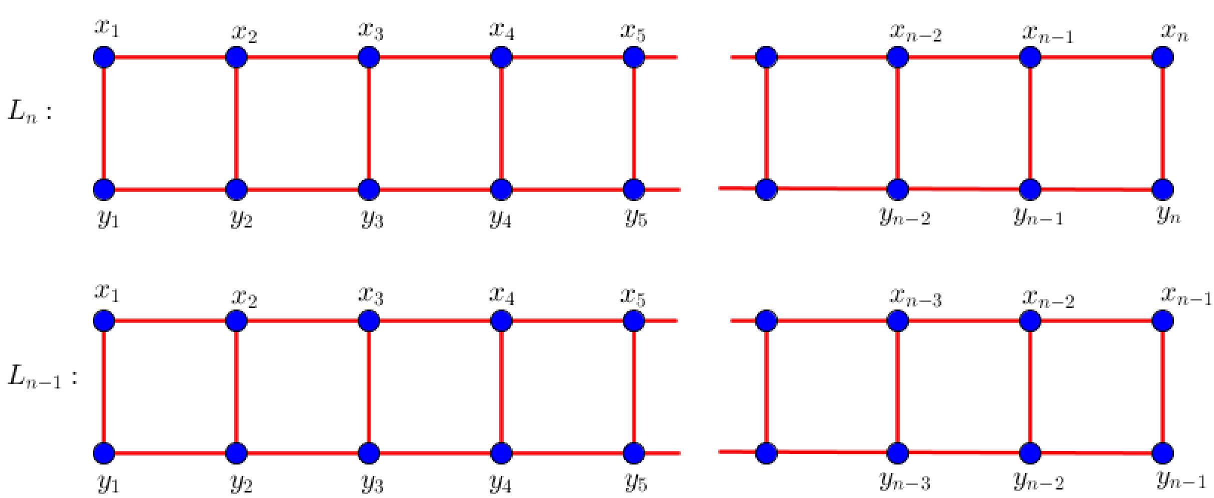


Fig. 2.1: L_5

In the following theorem, we obtain a recurrence formula for $W^D(L_n)$ in terms of $W^D(L_{n-1})$, $n \geq 5$.

Theorem 2.1. For $n \geq 5$

$$W^D(L_n) = W^D(L_{n-1}) + 8n^2 + 12n - 25.$$



Proof. To illustrate the steps in the proof, we draw L_{n-1} and L_n in Fig.2.2.

Fig. 2.2

For each pair (u, v) of L_n, we find $d^D_{L_n}(u, v)$ in terms of $d^D_{L_{n-1}}(u', v')$, for $u', v' \in V(L_{n-1})$.

(i) For $i, j \in \{3, 4, \dots, n\}$

$$d^D_{L_n}(x_i, x_j) = d^D_{L_{n-1}}(x_{i-1}, x_{j-1}),$$

$$d^D_{L_n}(y_i, y_j) = d^D_{L_{n-1}}(y_{i-1}, y_{j-1}),$$

$$d^D_{L_n}(x_i, y_j) = d^D_{L_{n-1}}(x_{i-1}, y_{j-1}).$$

(ii) For $j = 3, 4, \dots, n$

$$d^D_{L_n}(x_2, x_j) = 1 + d^D_{L_{n-1}}(x_1, x_{j-1}),$$

$$d^D_{L_n}(y_2, y_j) = 1 + d^D_{L_{n-1}}(y_1, y_{j-1}).$$

(iii) For $j = 2, 3, \dots, n-1$

$$d^D_{L_n}(x_2, y_j) = 2 + d^D_{L_{n-1}}(x_1, x_{j-1}),$$

$$d^D_{L_n}(y_2, x_j) = 2 + d^D_{L_{n-1}}(x_1, y_{j-1}), \text{for } j = 3, 4, \dots, n-1,$$

and $d^D_{L_n}(x_2, y_n) = 4n - 3$, $d^D_{L_{n-1}}(x_1, y_{n-1}) = 4n - 4$,

$$d^D_{L_n}(x_2, y_n) = 1 + d^D_{L_{n-1}}(x_1, y_{n-1}), \qquad d^D_{L_n}(y_2, x_n) = 1 + d^D_{L_{n-1}}(y_1, x_{n-1}).$$

(iv) For $j = 2, 3, \dots, n-1$, we consider x_1 and y_1 in L_n,

$$d^D_{L_n}(x_1, x_j) = (j-1) + 2 + 3(j-1) = 4j - 2,$$

$$d^D_{L_n}(x_1, x_n) = (n-1) + 2 + 3(n-2) + 2 = 4n - 3,$$

also,

$$d^D_{L_n}(y_1, y_j) = 4j - 2, \qquad d^D_{L_n}(y_1, y_n) = 4n - 3.$$

Moreover, for $j = 1, 2, \dots, n-1$,

$$d^D_{L_n}(x_1, y_j) = j + 2 + 2 + 3(j-1) = 4j + 1,$$

$$d^D_{L_n}(x_1, y_n) = n + 2 + 2 + 2 + 3(n-2) = 4n.$$

similarly, for $j = 2, 3, \dots, n-1$,

$$d^D_{L_n}(y_1, x_j) = j + 2 + 2 + 3(j-1) = 4j + 1,$$

$$d^D_{L_n}(y_1, x_n) = 4n.$$

For (i)-(iv), we get:

$$W^D(L_n) = W^D(L_{n-1}) + (n-2) + (n-2) + 2(n-2) + 2(n-3) + 1 + 1 + 2\sum_{j=2}^{n-1}(4j-2) + 2(4n-3)$$
$$+ \sum_{j=1}^{n-1}(4j+1) + 4n + \sum_{j=2}^{n-1}(4j+1) + 4n$$

$$= W^D(L_{n-1}) + (22n-18) + 8\sum_{j=2}^{n-1} j - 4(n-2) + 4\sum_{j=1}^{n-1} j + (n-1) + 4\sum_{j=2}^{n-1} j + (n-2)$$

$$= W^D(L_{n-1}) + (20n-13) + 8\left[\frac{1}{2}(n+1)(n-2)\right] + 4\left[\frac{1}{2}n(n-1)\right] + 4\left[\frac{1}{2}(n+1)(n-2)\right]$$

$$= W^D(L_{n-1}) + (20n-13) + 4(n^2-n-2) + 2(n^2-n) + 2(n^2-n-2)$$
$$= W^D(L_{n-1}) + 8n^2 + 12n - 25. \blacksquare$$

Remark: Let

$$R(n) = 8n^2 + 12n - 25,$$

then $R(3) = 83$, $R(4) = 151$, $R(5) = 235$, $R(6) = 335$,

$$W^D(L_3) = W^D(L_2) + 83 = 36 + 83 = 119,$$

$$W^D(L_4) = W^D(L_3) + 151 = 119 + 151 = 270,$$

$$W^D(L_5) = W^D(L_4) + 235 = 270 + 235 = 505,$$

$$W^D(L_6) = W^D(L_5) + 335 = 505 + 335 = 840.$$

Thus, Theorem 2.1 holds for $n \geq 3$.

Theorem 2.2. For $n \geq 3$,

$$W^D(L_n) = \frac{1}{3}(8n^3 + 30n^2 - 53n + 30).$$

Proof. Let us denote $R(n) = 8n^2 + 12n - 25$. Using the recurrence formula in Theorem 2.1, we get

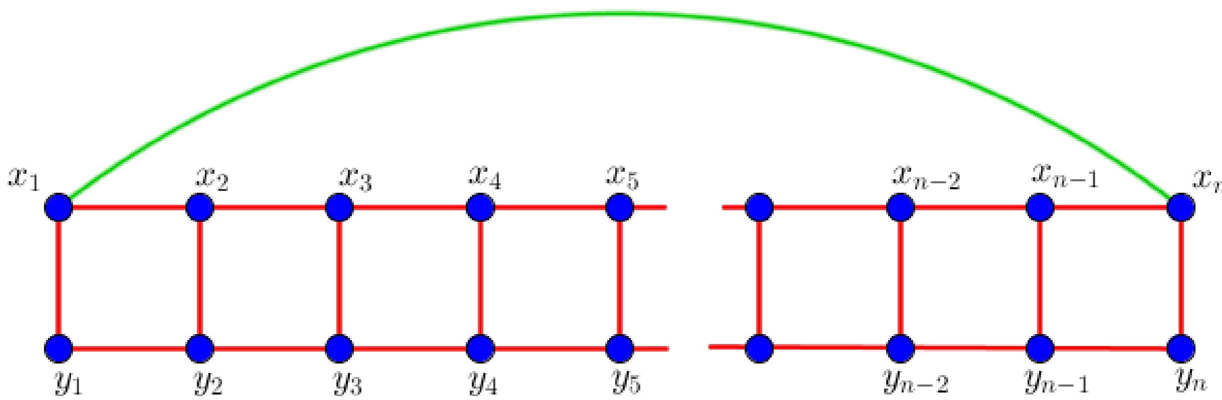
$$W^D(L_n) = W^D(L_{n-1}) + R(n) = W^D(L_{n-2}) + R(n-1) + R(n)$$
$$= W^D(L_{n-3}) + R(n-2) + R(n-1) + R(n) = \cdots$$
$$= W^D(L_3) + R(4) + R(5) + \cdots + R(n-1) + R(n) = W^D(L_3) + \sum_{i=4}^{n} R(i)$$
$$= W^D(L_3) + \sum_{i=4}^{n}(8i^2 + 12i - 25)$$

$$= 119 + 8\left(\sum_{i=1}^{n} i^2 - 14\right) + 12\left(\sum_{i=1}^{n} i - 6\right) - 25(n-3)$$
$$= 119 + 8\left[\frac{1}{6}n(n+1)(2n+1) - 14\right] + 12\left[\frac{1}{2}n(n+1) - 6\right] - 25n + 75$$
$$= 119 + \frac{4}{3}(2n^3 + 3n^2 + n) - 112 + 6n^2 + 6n - 72 - 25n + 75$$
$$= \frac{8}{3}n^3 + 10n^2 - \frac{53}{3}n + 10.$$

Hence the proof.■.

2. D-index of semi-ladder L_n^*

The semi-ladder graph L_n^* is defined as a ladder $L_n,\ n \geq 3$ with an edge joining vertex x_1 to vertex x_n as



shown in Fig. 3.1.

Fig. 3.1: L_n^*

It is clear that L_n^* contains exactly two vertices, namely y_1 and y_n, of degree 2, and other vertices are of degree 3.

For every pair $(u, v),\ u \neq v$ of vertices in L_n^*, a shortest $u - v$ paths contains y_1 and y_n or either y_1 (or y_n) or neither y_1 nor y_n. The following results gives us a relation between a shortest D −distance $u - v$ path and a shortest $u - v$ path.

Proposition 3.1. For every pair (u, v) of L_n^*, if s is a shortest D-distance $u - v$ path, then s is a shortest $u - v$ path.

Proof. Let s be the shortest D-distance $u - v$ path. If s is not shortest $u - v$ path, then there is a shortest $u - v$ path s' with $l(s') < l(s)$, that is $l(s) \geq 1 + l(s')$.

Thus

$$l^D(s') \leq l(s') + 3\big(1 + l(s')\big) = 4l(s') + 3,$$

$$l^D(s) \geq l(s) + 3(l(s) - 1) + 2 + 2.$$

Therefore

$$l^D(s) \geq 4l(s) + 1 \geq 4\big(1 + l(s')\big) + 1,$$

that is

$$l^D(s) \geq 4l(s') + 5 > l^D(s'),$$

a contradiction. Hence, the proof.■

Notice that the converse of Proposition 3.1 does not hold; that is, there may exist two shortest $u - v$ paths s_1 and s_2, but $l^D(s_1) \neq l^D(s_2)$; as for the pair of vertices (x_1, y_2) shown in Fig. 3.1.

Proposition 3.2. For each pair $(u, v),\ u \neq v$, of the vertices in L_n^*, let s be a shortest D-distance $u - v$ path, then

$$d^D(u, v) = \begin{cases} 4d(u, v) + 3, & \text{if } s \text{ contains neither } y_1 \text{ nor } y_n, \\ 4d(u, v) + 2, & \text{if } s \text{ contains } y_1 \text{ or } y_n \text{ (not both)}, \\ 4d(u, v) + 1, & \text{if } s \text{ contains } y_1 \text{ and } y_n\,. \end{cases}$$

Proof. By Proposition 3.1 and Definition 1.2, if s contains neither y_1 nor y_n then all the vertices of s are of degree 3, so

$$d^D(u,v) = l(s) + 3[l(s) + 1] = 4d(u,v) + 3.$$

If s contains either y_1 or y_n (not both), then all the vertices except one are of degree 3, thus

$$d^D(u,v) = l(s) + 3l(s) + 2 = 4d(u,v) + 2.$$

If s contains y_1 and y_n, then all the vertices of s except two vertices, are of degree 3, thus

$$d^D(u,v) = l(s) + 3[l(s) - 1] + 2 + 2 = 4d(u,v) + 1. \blacksquare$$

Theorem 3.3. For a semi-ladder graph L_n^*, $n \geq 3$, we have

$$W^D(L_n^*) = 4W(L_n^*) + \begin{cases} \frac{1}{2}(11n^2 - 12n + 4), & \text{for even } n, \\ \frac{1}{2}(11n^2 - 10n - 3), & \text{for odd } n. \end{cases}$$

Proof. (i) Let $n = 2r, r \geq 2$, and let $\mathcal{A}$ be the set of all distinct unorder pairs (u,v), $u \neq v$, $u,v \in V(L_n^*)$, then $|\mathcal{A}|$=$n(2n-1)$. Partition $\mathcal{A}$ into pairwise disjoint subsets $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$, where:

$\mathcal{A}_1 = \{(u,v) \in \mathcal{A}$: a shortest D-distance path $u - v$ contains neither y_1 nor $y_n\}$,

$\mathcal{A}_2 = \{(u,v) \in \mathcal{A}$: a shortest D-distance path $u - v$ contains either y_1 or y_n (not both)$\}$,

$\mathcal{A}_3 = \{(u,v) \in \mathcal{A}$: a shortest D-distance path $u - v$ contains y_1 and $y_n\}$.

To find $\mathcal{A}_1$, we partition it into $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$, where

$$\mathcal{B}_1 = \left\{(x_i,x_j): i \neq j,\ \ i,j \in \{1,2,\dots,n\}\right\},$$

$$\mathcal{B}_2 = \left[\bigcup_{i=2}^{r-1}\{(y_i,y_j): j = i+1, i+2, \dots, i+r\}\right] \cup \left[\bigcup_{i=r}^{n-2}\{(y_i,y_j): j = i+1, i+2, \dots, n-1\}\right],$$

and

$$\mathcal{B}_3 = \left[\bigcup_{i=2}^{r}\{(x_i,y_j): j = 2,3,\dots,r+i-1\}\right] \cup \left[\bigcup_{i=r+1}^{n-1}\{(x_i,y_j): j = n-1, n-2, \dots, i-r+1\}\right].$$

Counting the number of elements in $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$, we get:

$$|\mathcal{B}_1| = r(2r-1),$$

$$|\mathcal{B}_2| = r(r-2) + \sum_{k=1}^{r-1} k = r^2 - 2r + \frac{1}{2}r(r-1) = \frac{3}{2}r^2 - \frac{5}{2}r,$$

$$|\mathcal{B}_3| = \sum_{k=r}^{n-2} k + \sum_{k=n-2}^{r} k = 2\left[\frac{1}{2}(3r-2)(r-1)\right] = 3r^2 - 5r + 2.$$

Therefore,

$$\mathcal{A}_1 = (2r^2 - r) + \left(\frac{3}{2}r^2 - \frac{5}{2}r\right) + (3r^2 - 5r + 2) = \frac{13}{2}r^2 - \frac{17}{2}r + 2. \tag{3.1}$$

To find $\mathcal{A}_3$, we notice that for every pair of the form (x_i,y_j), (x_i,x_j), the shortest D-distance $x_i - y_j$ $(x_i - x_j)$ path does not contain y_1 and y_n. Therefore, we consider the pairs of the form (y_i,y_j), $i < j$, $i,j \in \{1,2,\dots,n\}$ in order to find $\mathcal{A}_3$.

For $1 \leq i < j \leq n$,

$$d^D(y_i, y_j) = \min\{l^D(s_1), l^D(s_2)\},$$

where s_1 is the path $(y_i, y_{i+1}, \dots, y_{j-1}, y_j)$ and s_2 is the path $(y_j, y_{j+1}, \dots, y_n, x_n, x_1, y_1, y_2, \dots, y_i)$. The length of s_1 is $j-i$, and the length of s_2 is $(n-j)+3+(i-1)=n+i+2-j$.

For $i>1$ and $j<n$,

$$l^D(s_1) = 4(j-i)+3, \quad l^D(s_2) = 4(n+i+2-j)+1.$$

If $(y_i, y_j) \in \mathcal{A}_3$, then $l^D(s_2) < l^D(s_1)$, that is $1+4(n+i+2-j) < 3+4(j-i)$. Thus $j-i > \frac{n}{2}+\frac{3}{4}$, that is $j-i>r$.

Therefore,

$$\mathcal{A}_3 = \bigcup_{i=1}^{r-1}\{(y_i, y_j): j = i+r+1, i+r+2, \dots, 2r\}.$$

Thus

$$|\mathcal{A}_3| = \sum_{i=1}^{r-1}(r-i) = \sum_{k=1}^{r-1} k = \frac{1}{2}r(r-1). \tag{3.2}$$

From (3.1) and (3.2), we have

$$|\mathcal{A}_2| = 2r(4r-1) - \left[\left(\frac{13}{2}r^2 - \frac{17}{2}r + 2\right) + \left(\frac{1}{2}r^2 - \frac{1}{2}r\right)\right] = r^2+7r-2. \tag{3.3}$$

By Proposition 3.2, we get

$$\begin{aligned} W^D(L_n^*) &= \sum_{\{u,v\}\subset\mathcal{A}} d^D(u,v) = \sum_{\{u,v\}\subset\mathcal{A}_1} d^D(u,v) + \sum_{\{u,v\}\subset\mathcal{A}_2} d^D(u,v) + \sum_{\{u,v\}\subset\mathcal{A}_3} d^D(u,v) \\ &= 4\sum_{\{u,v\}\subset\mathcal{A}_1} d(u,v) + 3|\mathcal{A}_1| + 4\sum_{\{u,v\}\subset\mathcal{A}_2} d(u,v) + 2|\mathcal{A}_2| + 4\sum_{\{u,v\}\subset\mathcal{A}_3} d^D(u,v) + |\mathcal{A}_3| \\ &= 4W(L_n^*) + 3\left(\frac{13}{2}r^2 - \frac{17}{2}r + 2\right) + 2(r^2+7r-2) + \frac{1}{2}r(r-1) \\ &= 4W(L_n^*) + 22r^2 - 12r + 2. \end{aligned}$$

Hence, the proof for even n is completed.

(ii) Let $n=2r+1$, $r\geq 2$, then by the steps similar to those used in the even n, we obtain

$$|\mathcal{B}_1| = \frac{1}{2}n(n-1) = r(2r+1).$$

$$|\mathcal{B}_2| = (r-2)(r+1) + \frac{1}{2}r(r+1) = \frac{3}{2}r^2 - \frac{1}{2}r - 2.$$

$$|\mathcal{B}_3| = 2(r-1) + 2\sum_{i=2}^{r}(i+r-1) = (2r-1) + 3r(r-1) = 3r^2 - r - 1.$$

Therefore

$$|\mathcal{A}_1| = r(2r+1) + \left(\frac{3}{2}r^2 - \frac{1}{2}r - 2\right) + (3r^2 - r - 1) = \frac{13}{2}r^2 - \frac{1}{2}r - 3. \tag{3.4}$$

For $\mathcal{A}_3$ in this case, we have:

$$\mathcal{A}_3 = \bigcup_{i=1}^{r-1}\{(y_i, y_j): j = i+r+2, i+r+3, \dots, 2r+1\}.$$

Thus

$$|\mathcal{A}_3| = \sum_{k=1}^{r-1} k = \frac{1}{2}r(r-1), \tag{3.5}$$

and

$$|\mathcal{A}_2| = n(2n-1) - \left[\frac{13}{2}r^2 - \frac{1}{2}r - 3 + \frac{1}{2}r^2 - \frac{1}{2}r\right] = r^2 + 7r + 4. \tag{3.6}$$

By Proposition 3.2, we get:

$$W^D(L_n^*) = 4W(L_n^*) + 3\left(\frac{13}{2}r^2 - \frac{1}{2}r - 3\right) + 2(r^2 + 7r + 4) + \left(\frac{1}{2}r^2 - \frac{1}{2}r\right) = 4W(L_n^*) + 22r^2 + 12r - 1$$
$$= 4W(L_n^*) + 22\left(\frac{n-1}{2}\right)^2 + 12\left(\frac{n-1}{2}\right) - 1 = 4W(L_n^*) + \frac{1}{2}(11n^2 - 10n - 3). \blacksquare$$

Remark: Theorem 3.3 holds for L_3^* because $W(L_3^*) = 22$ and $W^D(L_3^*) = 121$ obtained by calculation.

In order to obtain $W^D(L_n^*)$ in terms of n only, we shall find $W(L_n^*)$. We need the Weiner index of a cycle graph C_n given below [14]:

$$W(C_n) = \frac{n}{8}\begin{cases} n^2, & \text{if } n \text{ is even,} \\ n^2 - 1, & \text{if } n \text{ is odd.} \end{cases} \tag{3.7}$$

Theorem 3.4. For $n \geq 3$,

$$W(L_n^*) = \begin{cases} \frac{n}{4}(2n^2 + 5n - 2), & \text{for even,} \\ \frac{1}{4}(2n^3 + 5n^2 - 4n + 1), & \text{for odd } n. \end{cases}$$

Proof: (i) Let n be an even number, $n \geq 4$, and let V be the vertex set of L_n^*. It is clear that

$d(u,v) = d_{C_n}(u,v)$, for $u, v \in \{x_1, x_2, \dots, x_n\}$,

$d(u,v) = d_{C_{n+2}}(u,v)$, for $u, v \in \{y_1, y_2, \dots, y_n, x_n\}$,

where C_n is the cycle $(x_1, x_2, \dots, x_n, x_1)$ and C_{n+2} is the cycle $(y_1, y_2, \dots, y_n, x_n, x_1, y_1)$, and $d(u,v)$ is the distance in L_n^*.

Therefore,

$$W(L_n^*) = \sum_{\{u,v\}\subset V} d(u,v) = \sum_{\{u,v\}\subset V(C_n)} d(u,v) + \sum_{\{u,v\}\subset V(C_{n+2})} d(u,v) - 1 + \sum_{u\in X, v\in Y} d(u,v), \tag{3.8}$$

in which $X = \{x_2, x_3, \dots, x_{n-1}\}$ and $Y = \{y_1, y_2, \dots, y_n\}$.

Thus from (3.7) and the symmetry of the graph of L_n^* shown in Fig. 3.1, we obtain

$$W(L_n^*) = \frac{n^3}{8} + \frac{(n+2)^3}{8} - 1 + (n-2)\sum_{j=1}^{n} d(x_2, y_j)$$
$$= \frac{1}{8}(2n^3 + 6n^2 + 12n) + (n-2)\left[2 + \sum_{j=2}^{\frac{n}{2}+2} d(x_2, y_j) + \sum_{j=\frac{n}{2}+3}^{n} d(x_2, y_j)\right]. \tag{3.9}$$

It is clear that

$$d(x_2, y_j) = j - 1, \quad \text{for } 2 \leq j \leq \frac{n}{2} + 2,$$

and

$$d(x_2, y_j) = n + 3 - j, \quad \text{for } \frac{n}{2} + 3 \le j \le n.$$

Hence, from (3.8), we get

$$W(L_n^*) = \frac{n}{4}(n^2 + 3n + 6) + (n-2)\left[2 + \sum_{j=2}^{\frac{n}{2}+2}(j-1) + \sum_{j=\frac{n}{2}+3}^{n}(n+3-j)\right]$$
$$= \frac{n}{4}(n^2 + 3n + 6) + 2\,(n-2) + (n-2)\left[\sum_{k=1}^{\frac{n}{2}+1} k + \sum_{k'=\frac{n}{2}}^{3} k'\right],$$

in which $k = j - 1$ and $k' = n + 3 - j$. Thus

$$W(L_n^*) = \frac{n}{4}(n^2 + 3n + 14) - 4 + (n-2)\left[\frac{1}{2}\left(\frac{n}{2}+1\right)\left(\frac{n}{2}+2\right) + \frac{1}{2}\left(\frac{n}{2}+3\right)\left(\frac{n}{2}-2\right)\right]$$
$$= \frac{n}{4}(n^2 + 3n + 14) - 4 + \frac{1}{2}(n-2)\left(\frac{n^2}{2} + 2n - 4\right) = \frac{n}{4}(2n^2 + 5n - 2).$$

(ii) Now, let n be an odd number, $n \ge 3$. Thus, by (3.7) and the symmetry of the graph of L_n^*, we get:

$$W(L_n^*) = \frac{n}{8}(n^2 - 1) + \left(\frac{n+2}{8}\right)[(n+2)^2 - 1] - 1 + (n-2)\sum_{j=1}^{n} d(x_2, y_j) = \frac{n^3}{8} - \frac{n}{8}$$
$$+ \frac{1}{8}(n+2)(n^2 + 4n + 3) - 1 + 2(n-2) + (n-2)\left[\sum_{j=2}^{\frac{n+3}{2}}(j-1) + \sum_{j=\frac{n+5}{2}}^{n}(n-j+3)\right] =$$
$$= \frac{n^3}{4} + \frac{3}{4}n^2 + \frac{13}{4}n - \frac{17}{4} + (n-2)\left[\sum_{k=1}^{\frac{n+1}{2}} k + \sum_{k'=3}^{\frac{n+1}{2}} k'\right],$$

in which $k = j - 1$ and $k' = n - j + 3$. Therefor,

$$W(L_n^*) = \frac{1}{4}(n^3 + 3n^2 + 13n - 17) + (n-2)\left[\frac{1}{2}\left(\frac{n+1}{2}\right)\left(\frac{n+3}{2}\right) + \frac{1}{2}\left(\frac{n-3}{2}\right)\left(\frac{n+7}{2}\right)\right] =$$
$$= \frac{1}{4}(n^3 + 3n^2 + 13n - 17) + \frac{1}{4}(n-2)(n^2 + 4n - 9) = \frac{1}{4}(2n^3 + 5n^2 - 4n + 1). \blacksquare$$

The following result is derived from Theorems 3.3 and 3.4.

Corollary 3.4. For any semi-ladder L_n^*, $n \ge 3$ we have

$$W^D(L_n^*) = \begin{cases} 2n^3 + \frac{21}{2}n^2 - 8n + 2, & \text{for even } n, \\ 2n^3 + \frac{21}{2}n^2 - 9n - \frac{1}{2}, & \text{for odd } n. \blacksquare \end{cases}$$

Example

Wiener index	D-index
$W(L_3^*) = 22$	$W^D(L_3^*) = 121$

$W(L_4^*) = 50$	$W^D(L_4^*) = 266$
$W(L_5^*) = 89$	$W^D(L_5^*) = 467$
$W(L_6^*) = 150$	$W^D(L_6^*) = 764$
$W(L_7^*) = 226$	$W^D(L_7^*) = 1137$
$W(L_8^*) = 332.$	$W^D(L_8^*) = 1634.$

Acknowledgment: I am deeply grateful to Professor Dr. Ali Aziz Ali for his invaluable feedback on this research. I also thank the reviewers for their insightful comments.

Author Contributions: The author illustrated the proposed concept, formulated the theory, conducted the calculations, and handled all aspects related to this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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