Approximation Using a Modified Type of Bernstein Operators

Mustafa K. Shehab, Amal K. Hassan *

Department of Mathematics, College of Science, University of Basrah, Basra, Iraq

*Corresponding author E-mail: **amal.khelil@uobasrah.edu.iq**

Doi: 10.29072/basjs.20220303

ARTICLE INFO

Keywords

Kantorovich operator, Voronovskaja theorem, Bernstein operators, Ordinary approximation.

ABSTRACT

A new generalization for the Bernstein-Kantorovich operator with a parameter is proposed in this study. First, we prove the Korovkin type approximation theorem, then we provide the Voronovskaja type theorem for our generalization, demonstrating that the order of approximation is improved, making the approximation by our operators better than the original Kantorovich, and finally, we provide some numerical data for two test functions to support the study.

Received 10 Aug 2022; Received in revised form 21 Nov 2022; Accepted 24 Dec 2022, Published 31 Dec 2022

1. Introduction

In his proof of the famous Weierstrass approximation theorem Sergej N. Bernstein introduced the following polynomials [1].

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right), \qquad (1)$$

Where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ $f \in C[0,1], x \in [0,1], n = 1,2,3, \dots$, After that Kantorovich in his works published in 1930 and 1931 defined and studied the following operator [2].

$$KB_n(f;x) = \eta \sum_{k=0}^{n} b_{n,k}(x) \int_{\frac{k}{\eta}}^{\frac{k+1}{\eta}} f(t)dt, \qquad (2)$$

Where $\eta = (n+1)$, Moreover, it's can be obtained by the classical Bernstein polynomials as follows.

$$B'_\eta(F;x) = KB_n(f;x)$$

Where $F(x) = \int_0^x f(t)dt$ and $x \in [0,1]$.

As for the sequence $KB_n(f;x)$ several modifications where presented. In 1983 J. Nagel study the Kantorovich operators in second order[3] and in 2013 N.I. Mahmudov, P. Sabancigil defined the following polynomial[4].

$$B_{n,q}(f;x) = \sum_{k=0}^{n} p_{n,k}(q;x) \int_0^1 f\left(\frac{[k] + q^k t}{[k+1]}\right) d_q t,$$

Also M. Mursaleen, F. Khan, A. Khan, A.Kiliçman us Erkuş-Srivastava multivariable polynomials to contract a new generalized Kantorovich-type operators[5] and in 2016 Ç. Atakut , İ. Büyükyazıcı introduced the following operators[6].

$$L_n^{\alpha_n,\beta_n}(f;x) = \frac{B_n}{A(1)B(\alpha_n x)} \sum_{k=0}^{\infty} p_k(\alpha_n x) \int_{\frac{k}{\beta_n}}^{\frac{k+1}{\beta_n}} f(t)dt,$$

In 2018 A.M. Acu, N. Manav, D.F. Sofonea defined and studied the following generalization.

$$K_{n,\lambda}(f;x) = (n+1)\sum_{k=0}^{n} b_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt,$$

Several researcher study other operators [8,9,10,11] .in this study our goal is improve the quality of approximation of (2) by introducing a modification based on a parameter $r \in N^o, f \in C_\rho[0,1]$ and $x \in [0,1]$ Defined as,

$$KB_{n,r}(f;x) = \eta\sum_{k=0}^{n} b_{n,k}(x)\int_{\frac{k}{\eta}}^{\frac{k+1}{\eta}} f(x+(t-x)^r)dt \qquad (3)$$

Where and $C_\rho[0,1] = \{f \in C[0,1], |f| \leq M|t|^\rho \; for\ some\ \rho > 0, M\ constant, t \in [0,1]\}$.

2. Preliminary Results

Defintion2.1.for $e \in \{0,1,2,3,\dots\}$ the e-th order moment for $KB_n(f;x)$ defined below [12].

$$S_{n,e}(x) = \eta\sum_{k=0}^{n} b_{n,k}(x)\int_{\frac{k}{\eta}}^{\frac{k+1}{\eta}} (t-x)^e dt,$$

Defintion2.2. if $e \in N^0$ then the e-th order moment for $KB_{n.r}(f;x)$ defined by.

$$W_{n,e,r}(x) = \eta\sum_{k=0}^{n} b_{n,k}(x)\int_{\frac{k}{\eta}}^{\frac{k+1}{\eta}} (t-x)^{re} dt,$$

Lemma 2.1 for $e \in N^0$ the operator $KB_n(f;x)$ we get the following [13].

1- $S_{n.0}(x) = 0,$

2- $S_{n.1}(x) = \frac{1-2x}{2(n+1)},$

3- $S_{n.2}(x) = \frac{nx-x-nx^2+x^2+\frac{1}{3}}{(n+1)^2},$

Lemma 2.2 for $x \in [0,1]$, $e \in N^0$and $x \in [0,1]$, then.

$$W_{n,e,r}(x) = S_{n,re}(x) \qquad (4)$$

Proof. By the definition of $W_{n,e,r}(x)$ and $S_{n,re}(x)$ (4) is holds.

Lemma 2.3

For $n \in \{1,2,3,\dots\}, re \geq 1$ we have.

$$S_{n,re+1}(x) = \frac{re+1}{re+2}\frac{(x-x^2)}{\eta}\left(S'_{n,re}(x) + arS_{n,re-1}(x)\right) - \frac{(re+1)x}{(re+2)\eta}S_{n,re}(x) + \frac{1}{re+2}H_{re+1}(x) \qquad (5)$$

Where $H_{re+1}(x) = \sum_{k=0}^{n} b_{n,k}(x)\left(\frac{k+1}{\eta} - x\right)^{re+1}$.

Moreover

$$W_{n,1,3}(x) = \frac{10nx - 30nx^2 + 20nx^3 - 4x + 3x^2 - x^3 + 1}{4(n+1)^3}, \qquad (6)$$

Proof. From Definition 2.1 we have:

$$S_{n,re}(x) = \eta \sum_{k=0}^{n} b_{n,k}(x) \int_{\frac{k}{\eta}}^{\frac{k+1}{\eta}} (t-x)^{re} dt$$

$$S_{n,re}(x) = \eta \sum_{k=0}^{n} b_{n,k}(x) \left[\frac{(t-x)^{re+1}}{re+1}\right]_{\frac{k}{\eta}}^{\frac{k+1}{\eta}}$$

$$S_{n,re}(x) = \eta \sum_{k=0}^{n} b_{n,k}(x) \left(\frac{\left(\frac{k+1}{\eta} - x\right)^{re+1}}{re+1} - \frac{\left(\frac{k}{\eta} - x\right)^{re+1}}{re+1}\right)$$

$$S_{n,re}(x) = \frac{\eta}{re+1}\left(\sum_{k=0}^{n} b_{n,k}(x)\left(\frac{k+1}{\eta} - x\right)^{re+1} - \sum_{k=0}^{n} b_{n,k}(x)\left(\frac{k}{\eta} - x\right)^{re+1}\right)$$

By the definition of $b_{n,k}(x)$ we get,

$$S'_{n,re}(x) = \frac{\eta}{re+1}\left(\frac{d}{dx}\left(\sum_{k=0}^{n} b_{n,k}(x)\left(\frac{k+1}{\eta}-x\right)^{re+1} - \sum_{k=0}^{n} b_{n,k}(x)\left(\frac{k}{\eta}-x\right)^{re+1}\right)\right)$$

$$S'_{n,re}(x) = \frac{\eta}{re+1}\left(\sum_{k=0}^{n} \frac{(k-nx)}{x(1-x)} b_{n,k}(x)\left(\frac{k+1}{\eta}-x\right)^{re+1} - (re+1)\sum_{k=0}^{n} b_{n,k}(x)\left(\frac{k+1}{\eta}-x\right)^{re} - \sum_{k=0}^{n} \frac{1}{x(1-x)}(k-nx) b_{n,k}(x)\left(\frac{k}{\eta}-x\right)^{re+1} + \sum_{k=0}^{n}(re+1) b_{n,k}(x)\left(\frac{k}{\eta}-x\right)^{re}\right)$$

$$S'_{n,re}(x) = \frac{\eta}{re+1}\left(\frac{\eta}{x(1-x)}\left(\sum_{k=0}^{n} b_{n,k}(x)\left(\frac{k+1}{\eta}-x\right)^{re+2} + \frac{(x-1)}{\eta}\sum_{k=0}^{n} b_{n,k}(x)\left(\frac{k+1}{\eta}-x\right)^{re+1}\right) - (re+1)\sum_{k=0}^{n} b_{n,k}(x)\left(\frac{k+1}{\eta}-x\right)^{re} - \frac{\eta}{x(1-x)}\left(\sum_{k=0}^{n} b_{n,k}(x)\left(\frac{k}{\eta}-x\right)^{re+2} + \frac{x}{\eta}\sum_{k=0}^{n} b_{n,k}(x)\left(\frac{k}{\eta}-x\right)^{re+1}\right) + (re+1)\sum_{k=0}^{n} b_{n,k}(x)\left(\frac{k}{\eta}-x\right)^{re}\right)$$

$$S'_{n,re}(x) = \frac{\eta}{x(1-x)}\frac{re+2}{re+1} S_{n,re+1}(x) + \frac{1}{(1-x)} S_{n,re}(x) - re S_{n,re-1}(x) - \frac{\eta}{(re+1)x(1-x)} H_{re+1}(x)$$

from the formula above (5) can be easily be obtained. Also, by using Lemmas 2.1 and 2.2 with (4) we get (6).

3.Main result

Theorem 3.1 (Korovkin type theorem)

For $f \in C_\rho[0,1], x \in [0,1]$, we have.

(1) $KB_{n,r}(1;x) = \alpha_n(x) + 1$

(2) $KB_{n,r}(t;x) = \beta_n(x) + x$

(3) $KB_{n,r}(t^2;x) = \gamma_n(x) + x^2$

Where $\alpha_n(x)$, $\beta_n(x)$ and $\gamma_n(x)$ goes to zero as $n \to \infty$.

Proof. By direct computation

$$KB_{n,r}(1;x) = \eta \sum_{k=0}^{n} b_{n,k}(x) \int_{\frac{k}{\eta}}^{\frac{k+1}{\eta}} dt \quad = 1$$

Next,

$$KB_{n,r}(t;x) = \eta \sum_{k=0}^{n} b_{n,k}(x) \int_{\frac{k}{\eta}}^{\frac{k+1}{\eta}} (x + (t-x)^r)dt$$

$$= \eta \left(x \sum_{k=0}^{n} b_{n,k}(x) \int_{\frac{k}{\eta}}^{\frac{k+1}{\eta}} dt + \sum_{k=0}^{n} b_{n,k}(x) \int_{\frac{k}{\eta}}^{\frac{k+1}{\eta}} (t-x)^r dt \right)$$

$$KB_{n,r}(t;x) = x + S_{n,r}(x)$$

Were,

$$\lim_{n\to\infty} S_{n,r}(x) = 0$$

By the same above method (3) is holds.

Theorem 3.2

Let $f \in C_\rho[0,1]$ and $\rho > 0$. If $f''(x)$ exists and continuous on $x \in (0,1)$,then;

$$\lim_{n\to\infty} n^2 \left(KB_{n,3}(f(t);x) - f(x)\right) = \left(\frac{10x - 30x^2 + 20x^3}{4}\right) f'(x). \qquad (7)$$

Proof. The Taylor's expunction of $f(t)$ about x is

$$f(t)=\sum_{i=0}^{2}\frac{f^{(i)}(x)}{i!}(t-x)^i+(t-x)^2\varepsilon(t,x),$$

Where $\varepsilon(t,x)\to 0$ as $t\to x$.

Then,

$$KB_{n,r}(f(t);x)=f(x)KB_{n,r}(1;x)+f^{(1)}(x)KB_{n,r}\big((t-x);x\big)+\frac{f^{(2)}(x)}{2!}KB_{n,r}((t-x)^2;x)$$
$$+KB_{n,r}((t-x)^2\varepsilon(t,x);x),$$

taking $r=3$

$$KB_{n,3}(f(t);x)$$
$$=f(x)KB_{n,3}(1;x)+f'(x)KB_{n,3}\big((t-x);x\big)+\frac{f''(x)}{2!}KB_{n,3}((t-x)^2;x)$$
$$+KB_{n,3}((t-x)^2\varepsilon(t,x);x),$$

By lemmas 2.1 and 2.2 and relation (5),we get.

$$KB_{n,3}(f(t);x)-f(x)$$
$$=f'(x)\left(\frac{10nx-30nx^2+20nx^3-4x+3x^2-x^3+1}{4(n+1)^3}\right)+\frac{f''(x)}{2!}S_{n,6}(x)$$
$$+KB_{n,3}((t-x)^2\varepsilon(t,x);x)$$

$$\lim_{n\to\infty}n^2\big(KB_{n,3}(f(t);x)-f(x)\big)$$
$$=\lim_{n\to\infty}f'(x)\left(\frac{10n^3x-30n^3x^2+20n^3x^3-4n^2x+3n^2x^2-n^2x^3+n^2}{4n^3+12n^2+12n+4}\right)$$
$$+\frac{f''(x)}{2!}\lim_{n\to\infty}n^2T_{n,6}(x)+\lim_{n\to\infty}n^2KB_{n,3}((t-x)^2\varepsilon(t,x);x)$$

$$\lim_{n\to\infty}n^2(K_n(f(t);x)-f(x))$$
$$=f'(x)\left(\frac{10x-30x^2+20x^3}{4}\right)+\lim_{n\to\infty}n^2KB_{n,3}((t-x)^2\varepsilon(t,x);x)$$

We take the terms $T=n^2KB_{n,3}((t-x)^2\varepsilon(t,x);x)$.then

$$|T| = n^2 K B_{n,3}(|(t-x)^2 \varepsilon(t,x)|; x)$$

$$|T| \leq n^2 \left(\eta \sum_{k=0}^{n} b_{n,k}(x) \int_{|t-x|<\delta} |(t-x)^6 \varepsilon(t,x)| dt + \eta \sum_{k=0}^{n} b_{n,k}(x) \int_{|t-x|\geq\delta} |(t-x)^6 \varepsilon(t,x)| dt \right)$$

Now, for $\epsilon > 0$ there is $\delta > 0$ such that if $|t-x| < \delta \to |\varepsilon(t,x)| < \epsilon$ and if $|t-x| \geq \delta \to |\varepsilon(t,x)(t-x)^6| \leq Z|t-x|^\rho$ then.

$$n^2|T| \leq \epsilon n^2 S_{n,6}(x) + n^2 M\eta \sum_{k=0}^{n} b_{n,k}(x) \int_{|t-x|\geq\delta} |t-x|^\rho dt$$

$$= \epsilon O(n^{-1}) + n^2 M\eta \sum_{k=0}^{n} b_{n,k}(x) \int_{|t-x|\geq\delta} |t-x|^\rho dt$$

Take

$$Q = n^2 Z\eta \sum_{k=0}^{n} b_{n,k}(x) \int_{|t-x|\geq\delta} |t-x|^\rho dt$$

By Schwarz inequality,

$$Q \leq n^2 \left(\eta \sum_{k=0}^{n} b_{n,k}(x) \int_{\frac{k}{C}}^{\frac{k+1}{C}} dt \right)^{\frac{1}{2}} \left(M^2 \eta \sum_{k=0}^{n} b_{n,k}(x) \int_{\frac{k}{C}}^{\frac{k+1}{C}} (t-x)^{2\rho} dt \right)^{\frac{1}{2}}$$

$$Q = Mn^2 \left(O(n^{-\rho})\right)^{\frac{1}{2}} \leq O(n^{-\mu}), \mu > 0$$

Then

$$\lim_{n\to\infty} n^2 K B_{n,3}((t-x)^2 \varepsilon(t,x); x) \to 0$$

And 3.1 is obtained.

4.Numerical example

in this part we gave two function to compare the approximation of this functions between $KB_n(f;x)$ and $KB_{n,3}(f;x)$.

Example 4.1

Let $f(x) = \sin 25x$ then in Fig.1, we have the convergence of $KB_n(\sin(25x);x)$ to $\sin 25x$ on the right and convergence of $KB_{n,3}(\sin(25x);x)$ to $\sin 25x$ on the left in different value of n on the left.

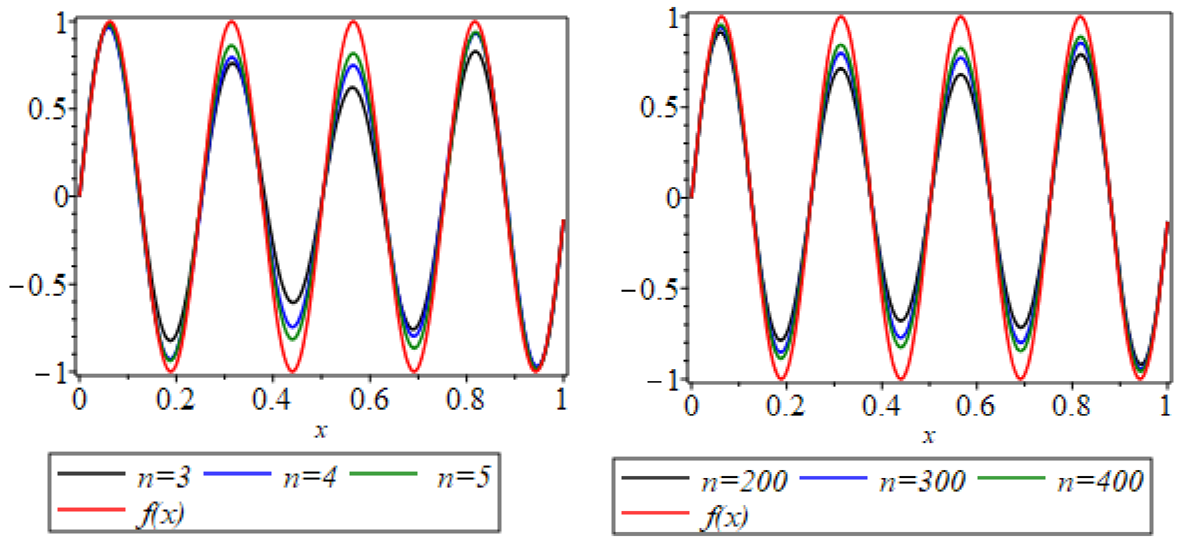

Figure1: The convergence of KB_n to the test function and the convergence of $KB_{n,3}$to the same function.

Example 4.2

Let $f(x) = 3\cos 3\pi x$ then in Fig.2, we have the convergence of $KB_n(3\cos 3\pi x\,;x)$ to $3\cos 3\pi x$ on the right and convergence of $KB_{n,3}(3\cos 3\pi x\,;x)$ to $3\cos 3\pi x$ on the left in different value of n on the left.

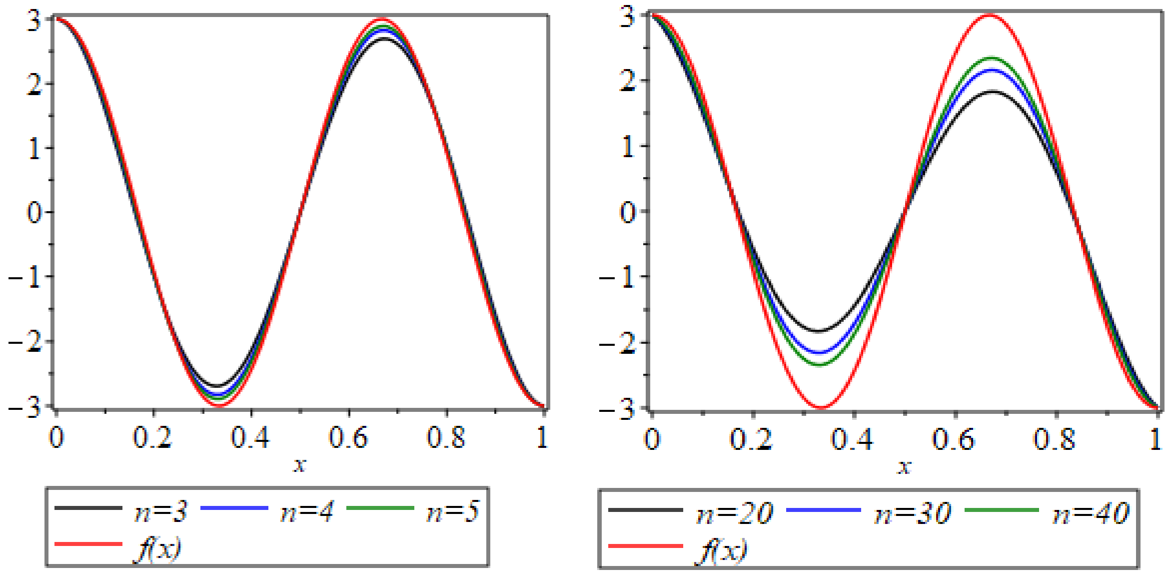

Figure 2: The convergence of KB_n to the test function and the convergence of $KB_{n,3}$to the same function.

5. Conclusion

In study interdicted a new modification of Bernstein-Kantorovich operators. some main theorems were discussed then we highlighted some numerical data to conclude that the level of accuracy of the new modification is improved then the original operator.

References

1. G.M. Phillips Interpolation and approximation by polynomials. first ed, Springer, New York, 2003.. https://doi.org/10.1007/b97417

2. V.S. Videnskii, Papers of LV Kantorovich on Bernstein polynomials, *Vestnik St. Vestn. St. Petersbg. Univ.: Math*, *46* (2013)85-88, https://doi.org/10.3103/S1063454113020088

3. J. Nagel, Kantorovich operators of second order, Monatsh. Math 95(1983)33-44, https://doi.org/10.1007/BF01301146

4. N.I. Mahmudov, P. Sabancigil, Approximation Theorems for q-Bernstein-Kantorovich Operators, *Filomat*, *27*(2013) 721-730, https://doi.org/10.2298/FIL1304721M

5. M. Mursaleen, F. Khan, A. Khan, A.Kiliçman, Some approximation results for generalized Kantorovich-type operators, *J Inequal Appl* , 585 (2013), https://doi.org/10.1186/1029-242X-2013-585

6. Ç. Atakut , İ. Büyükyazıcı, Approximation by Kantorovich-Szász Type Operators Based on Brenke Type Polynomials, Numer. Funct. Anal. Optim, 37(2016) 1488-1502, https://doi.org/10.1080/01630563.2016.1216447

7. A.M. Acu, N. Manav, D.F. Sofonea, Approximation properties of λ-Kantorovich operators, J. Inequalities Appl, 202(2018)1-12, https://doi.org/10.1186/s13660-018-1795-7

8. A.J. Mohammad, A.K. Hassan, Simultaneous Approximation by a New Sequence of Integral Type Based on Two Parameters, *Iraqi J. Sci.*,62(2021)1666-1674, https://doi.org/10.24996/ijs.2021.62.5.29

9. F. Ozsarac, V. Gupta, A. Aral, Approximation by Some Baskakov–Kantorovich Exponential-Type Operators, *Bull. Iran. Math. Soc*, *48*(2022)227-241, https://doi.org/10.1007/s41980-020-00513-3

10. A. Kajla, D. Miclăuş, Modified Bernstein–Durrmeyer Type Operators, *Mathematics*, *10*(2022) 1876, https://doi.org/10.3390/math10111876

11. A.J. Mohammad, A.K. Hassan, Approximation of New Sequence of Integral Type Operators with two Parameters, Adv. Math., *19*(2020)89–95, https://doi.org/10.24297/jam.v19i.8902

12. V.S. Videnskii, Papers of LV Kantorovich on Bernstein polynomials, *Vestn. St. Petersbg. Univ.: Math*, *46*(2013)85-88, https://doi.org/10.3103/S1063454113020088

13. A.M Acu, H. Gonska, Classical Kantorovich operators revisited, *arXiv preprint arXiv,1904(2019)11135*, https://doi.org/10.48550/arXiv.1904.11135

التقريب بواسطة مؤثر معدل من نوع برنستين-كونتروفج

مصطفى كامل شهاب، وامل خليل حسن

قسم الرياضيات، كلية العلوم، جامعة البصرة

المستخلص

في هذا الدراســة المقدمة ســوف نقدم تعميم جديد لمؤثر برنشــتاين-كانتروفج بالاعتماد على معامل أولا نثبت المبرهنة من نوع كوروفكن ثم نعطي المبرهنة من نوع فرونفســكيا التي تخص المؤثر الجديد عند الحالة الخاصــــة مثبتين ان رتبة التقارب قد تحسـنت مما يجعل التقريب بواسطة المؤثر الجديد أفضـل من المؤثر الأصـلي في النهاية بعد الرسـومات التقريبية سـوف تعطى لتدعيم هذا الدراسة.