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A New Class of Higher Derivatives for Harmonic Univalent Functions Established using a Generalized Fractional Integral Operator

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ABSTRACT: A new class of higher derivatives for harmonic univalent functions defined by a generalized fractional integral operator inside an open unit disk $\mathbb{E} = \{\xi \in \mathbb{C}: |\xi| < 1\}$ is the aim of this paper. The method for proving these theorems, which are used to derive new findings on this topic, is based on Lemma 1.1, which is stated in this study. Some of its geometric characteristics such as the Hadmared product, convex combination, and distortion bounds, were obtained using the recent finding of coefficient bounds for the new class. The originality of attempting to create a new class of harmonic functions, this work expands on the body of information currently accessible on the convolution of univalent analytic function and fractional integral operator for generate new class of harmonic functions.

Keywords: Harmonic function, Univalent function, Fractional operator, Higher derivatives

1. INTRODUCTION

Let $\mathcal{A}$ represent the class of analytic functions $\mathcal{L}$, normalized by $\mathcal{L}(0) = \mathcal{L}'(0) - 1 = 0$ in the open unit disk $\mathbb{E} = \{\xi \in \mathbb{C}: |\xi| < 1\}$. As a result, functions in the class $\mathcal{A}$ have the form $\mathcal{L}(\xi) = \xi + \sum_{w=2}^{\infty} a_w \xi^w$. Let S be a subclass of $\mathcal{A}$ which possesses univalent functions in $\mathbb{E}$. Dalia and et al. [1] introduced the generalized fractional integral operator in complex domain, which defined as

$$I_{\alpha,\beta,\gamma}^{\mu,\tau}\mathcal{L}(\xi) = \xi + \sum_{w=2}^{\infty}\left(\frac{\Gamma(\alpha+\beta+\gamma+2)\tau^{n-1}\mu(n)}{\mu(1)\Gamma(\alpha n+\beta+\gamma+2)}\right) a_w \xi^w$$

$$(\alpha,\beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, \mu = \{\mu(0),\mu(1),\dots,\mu(n)\}, \mu(j) \in \mathbb{C}\ \forall\, j = 0,\dots,n, \tau \neq 0, m \in \mathrm{Z})$$

Applying this operator to complex valued harmonic functions $\mathcal{L} = \chi + \bar{\psi}$, some of their geometric characteristic is investigated. Here, χ is referred as the analytic portion and ψ as the co-analytic portion of $\mathcal{L}$. Consider T_H, the class of complex valued harmonic mapping $\mathcal{L} = h + \bar{g}$, which is defined in $\mathbb{E}$ and normalized by $\chi(0) = \psi(0) = \chi'(0) - 1 = 0$. these mappings are represented by the power series as follows:

$$\mathcal{L}(\xi) = \xi + \sum_{w=2}^{\infty} a_w \xi^w + \sum_{w=1}^{\infty} b_w \bar{\xi}^w \tag{1.1}$$

The fact that $|\chi'(\xi)| > |\psi'(\xi)|$ for every $\xi \in \mathbb{E}$ is a necessary and sufficient condition for $\mathcal{L} \in T_H$ to be locally univalent and sense-preserving in $\mathbb{E}$(see [2]). In 1984, Clunie and Sheil-Small defined analogue classes and investigated harmonic mappings as a generalization of analytic functions (see [3]). As a generalization of the class S, they defined the class S_H of univalent and sense-preserving harmonic mappings and investigated the geometric characteristics of S_H including its subclasses. Extensive studies have been conducted in this field since the publication of Clunie and Sheil-

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Small's paper. Using several operators, authors have defined a lot subclasses for S_H; , see ([4,5,6,7,8,9,10]) and references within.

This study defines a new class $T^{\lambda}(\mu,\tau,q,\alpha,\beta,\gamma)$ of T_H that consists of harmonic functions $\mathcal{L}=\chi+\bar{\psi}$ of the forms (1.1) that satisfy $q\geq 1$ and $0\leq\lambda<1$,

$$\boldsymbol{Re}\left(\frac{1+\lambda\xi^{q}\left(I^{\mu,\tau}_{\alpha,\beta,\gamma}\mathcal{L}(\xi)\right)^{q+1}+(\mathbf{1}-\boldsymbol{\lambda})\xi^{q+1}\left(I^{\mu,\tau}_{\alpha,\beta,\gamma}\mathcal{L}(\xi)\right)^{q+2}}{\mathbf{1}+\xi^{q}\left(I^{\mu,\tau}_{\alpha,\beta,\gamma}\mathcal{L}(\xi)\right)^{q+1}}\right)\geq\boldsymbol{\lambda} \quad (1.2)$$

we therefore have

$$I^{\mu,\tau}_{\alpha,\beta,\gamma}\mathcal{L}(\xi)=I^{\mu,\tau}_{\alpha,\beta,\gamma}\chi(\xi)+\overline{I^{\mu,\tau}_{\alpha,\beta,\gamma}\psi(\xi)}$$

Let $\bar{T}_H$ represent the subclass of T_H that includes functions of the kind $\mathcal{L}_m=\chi+\bar{\psi}_m$, where

$$\chi(\xi)=\xi-\sum_{w=2}^{\infty}|a_w|\xi^w,\ \psi_m(\xi)=-\sum_{w=1}^{\infty}|b_w|\bar{\xi}^w \quad (1.3)$$

,

and consider the subclass of $\bar{T}_H$ whose functions meet condition (1.2) to be $\overline{T^{\lambda}}(\mu,\tau,q,\alpha,\beta,\gamma)$. For harmonic functions $\mathcal{L}=\chi+\bar{\psi}$ provided by (1.1), a sufficient condition in the next sections is derived that they must be in $T^{\lambda}(\mu,\tau,q,\alpha,\beta,\gamma)$. It is then demonstrated that this condition is also required for the functions in the class $\overline{T^{\lambda}}(\mu,\tau,q,\alpha,\beta,\gamma)$. For the functions in the class $\overline{T^{\lambda}}(\mu,\tau,q,\alpha,\beta,\gamma)$, Hadmared product, convex combination, and distortion bounds are studied. The following lemma was used as the way to establish our finding:

Lemma(1.1)[11]: Let $\lambda\geq 0$, then $Re(V)>\alpha$ if and only if $|V-(1+\lambda)|\leq|V+(1-\lambda)|$, where V be any complex number.

2. COEFFICIENT BOUNDS

A sufficient coefficient condition for the function in $T^{\lambda}(\mu,\tau,q,\alpha,\beta,\gamma)$ is where we start

Theorem(2.1): For $\alpha,\beta\in\mathbb{C}, Re(\alpha)>0, Re(\beta)>0, \mu(j)\in\mathbb{C}\ \forall\, j=0,\dots,n, \tau\neq 0, m\in\mathbb{Z}, q\geq 1$ and $0\leq\lambda<1$, let $\mathcal{L}(\xi)=\chi(\xi)+\overline{\psi(\xi)}$ be a function where $\chi(\xi)$ and $\psi(\xi)$ were define as in equation (1.1) and satisfy

$$\sum_{w=2}^{\infty}w!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^{w}(\Gamma,\tau,\mu)a_w|\xi|^{w-q}$$
$$+\sum_{w=1}^{\infty}w!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^{w}(\Gamma,\tau,\mu)b_w|\xi|^{w-q}\leq(1-\lambda) \quad (2.1)$$

where, $D^{w}(\Gamma,\tau,\mu)=\left(\frac{\Gamma(\alpha+\beta+\gamma+2)\tau^{w-1}\mu(w)}{\mu(1)\Gamma(\alpha w+\beta+\gamma+2)}\right)$, then $\mathcal{L}(\xi)$ is harmonic, univalent and sense-preserving in $\mathbb{E}$.

Proof: it was started to prove that $\mathcal{L}(\xi)$ is harmonic and univalent in $\mathbb{E}$, let $\xi_1,\xi_2\in\mathbb{E}$ for $|\xi_1|\leq|\xi_2|<1$. If it is given that $\xi_1\neq\xi_2$, therefore

$$\left|\frac{\mathcal{L}(\xi_1)-\mathcal{L}(\xi_2)}{\chi(\xi_1)-\chi(\xi_2)}\right|\geq 1-\left|\frac{\psi(\xi_1)-\psi(\xi_2)}{\chi(\xi_1)-\chi(\xi_2)}\right|=1-\left|\frac{\sum_{w=1}^{\infty}|b_w|(\xi_1^w-\xi_2^w)}{(\xi_1-\xi_2)-\sum_{w=2}^{\infty}|a_w|(\xi_1^w-\xi_2^w)}\right|\geq 1-\left|\frac{\sum_{w=1}^{\infty}w|b_w|}{1-\sum_{w=2}^{\infty}w|a_w|}\right|\geq$$

$$1-\left|\frac{\sum_{w=1}^{\infty}\frac{w!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^{w}(\Gamma,\tau,\mu)}{(1-\lambda)}|b_w|}{1-\sum_{w=2}^{\infty}\frac{w!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^{w}(\Gamma,\tau,\mu)}{(1-\lambda)}|a_w|}\right|\geq 0$$

$\mathcal{L}$ is a univalent function in $\mathbb{E}$ as a result.

Take note of how $\mathcal{L}$ is sense-preserving in $\mathbb{E}$. This is due to

$$|\chi'(\xi)|\geq 1-\sum_{w=2}^{\infty}w\,|a_w||\xi|^{w-1}>1-\sum_{w=2}^{\infty}w\,|a_w|\geq 1-\sum_{w=2}^{\infty}\frac{w!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^{w}(\Gamma,\tau,\mu)}{(1-\lambda)}|a_w|$$

$$\geq\sum_{w=1}^{\infty}\frac{w!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^{w}(\Gamma,\tau,\mu)}{(1-\lambda)}|b_w|\geq\sum_{w=1}^{\infty}w|b_w|\geq\sum_{w=1}^{\infty}w|b_w||z|^{w-1}\geq|\psi'(\xi)|.$$

It demonstrates that if equation (2.1) holds by using the condition of equation (1.2), then

$$Re\left(\frac{1+\xi^{q}\left(I_{\alpha,\beta,\gamma}^{\mu,\tau}\mathcal{L}(\xi)\right)^{q+1}+\xi^{q+1}\left(I_{\alpha,\beta,\gamma}^{\mu,\tau}\mathcal{L}(\xi)\right)^{q+2}}{1+\xi^{q}\left(I_{\alpha,\beta,\gamma}^{\mu,\tau}\mathcal{L}(\xi)\right)^{q+1}}\right)\geq\lambda=Re\left(\frac{A(\xi)}{B(\xi)}\right)\geq\lambda$$

Lemma$(\mathbf{1.1})$ suffices to demonstrate that

$|A(\xi)-(1+\lambda)B(\xi)|-|A(\xi)+(1-\lambda)B(\xi)|\leq 0$

It constructs for $A(\xi)$ and $B(\xi)$ in

$$|A(\xi)-(1+\lambda)B(\xi)|=\left|1+\xi^{\boldsymbol{q}}\left(I_{\alpha,\beta,\gamma}^{\mu,\tau}\mathcal{L}(\xi)\right)^{q+1}+\xi^{\boldsymbol{q+1}}\left(I_{\alpha,\beta,\gamma}^{\mu,\tau}\mathcal{L}(\xi)\right)^{q+2}-(1+\lambda)\left(1+\xi^{q}\left(I_{\alpha,\beta,\gamma}^{\mu,\tau}\mathcal{L}(\xi)\right)^{q+1}\right)\right|$$

$$\begin{aligned}=\Bigg|-\lambda+\sum_{w=2}^{\infty}w!\left(\frac{1}{(w-q-2)!}-\frac{\lambda}{(w-q-1)!}\right)D^{w}(\Gamma,\tau,\mu)a_{w}\xi^{w-q}\\
+\sum_{w=1}^{\infty}w!\left(\frac{1}{(w-q-2)!}-\frac{\lambda}{(w-q-1)!}\right)D^{w}(\Gamma,\tau,\mu)b_{w}\,\bar{\xi}^{w-q}\Bigg|\\
\leq\lambda+\sum_{w=2}^{\infty}w!\left(\frac{1}{(w-q-2)!}-\frac{\lambda}{(w-q-1)!}\right)D^{w}(\Gamma,\tau,\mu)a_{w}|\xi|^{w-q}\\
+\sum_{w=1}^{\infty}w!\left(\frac{1}{(w-q-2)!}-\frac{\lambda}{(w-q-1)!}\right)D^{w}(\Gamma,\tau,\mu)b_{w}|\xi|^{w-q}\end{aligned}\tag{2.2}$$

Now, $A(\xi)$ and $B(\xi)$ are compensate for in $|A(\xi)+(1-\alpha)B(\xi)|$, then

$$\begin{aligned}|A(\xi)+(1-b)B(\xi)|\\
=\Bigg|1+\sum_{w=2}^{\infty}w!\left(\frac{1}{(w-q-1)!}+\frac{1}{(w-q-2)!}\right)D^{w}(\Gamma,\tau,\mu)a_{w}\xi^{w-q}\\
+\sum_{w=1}^{\infty}w!\left(\frac{1}{(w-q-1)!}+\frac{1}{(w-q-2)!}\right)D^{w}(\Gamma,\tau,\mu)b_{w}\,\bar{\xi}^{w-q}\\
+(1-\lambda)\left(1+\sum_{w=2}^{\infty}\frac{w!}{(w-q-1)!}D^{w}(\Gamma,\tau,\mu)a_{w}\xi^{w-q}+\sum_{w=1}^{\infty}\frac{w!}{(w-q-1)!}D^{w}(\Gamma,\tau,\mu)b_{w}\,\bar{\xi}^{w-q}\right)\Bigg|\\
=\Bigg|2-\lambda+\sum_{w=2}^{\infty}w!\left(\frac{2-\lambda}{(w-q-1)!}+\frac{1}{(w-q-2)!}\right)D^{w}(\Gamma,\tau,\mu)a_{w}\xi^{w-q}\\
+\sum_{w=1}^{\infty}w!\left(\frac{2-\lambda}{(w-q-1)!}+\frac{1}{(w-q-2)!}\right)D^{w}(\Gamma,\tau,\mu)b_{w}\,\bar{\xi}^{w-q}\Bigg|\\
\geq(2-\lambda)-\sum_{w=2}^{\infty}w!\left(\frac{2-\lambda}{(w-q-1)!}+\frac{1}{(w-q-2)!}\right)D^{w}(\Gamma,\tau,\mu)a_{w}|\xi|^{w-q}\\
-\sum_{w=1}^{\infty}w!\left(\frac{2-\lambda}{(w-q-1)!}+\frac{1}{(w-q-2)!}\right)D^{w}(\Gamma,\tau,\mu)b_{w}\left|\bar{\xi}\right|^{w-q}\end{aligned}\tag{2.3}$$

Then compensation is made for equation (2.2) and (2.3), resulting in

$$\begin{aligned}|A(\xi)-(1+b)B(\xi)|-|A(\xi)+(1-b)B(\xi)|\\
=\lambda+\sum_{w=2}^{\infty}w!\left(\frac{1}{(w-q-2)!}-\frac{\lambda}{(w-q-1)!}\right)D^{w}(\Gamma,\tau,\mu)a_{w}|\xi|^{w-q}\\
+\sum_{w=1}^{\infty}w!\left(\frac{1}{(w-q-2)!}-\frac{\lambda}{(w-q-1)!}\right)D^{w}(\Gamma,\tau,\mu)b_{w}|\xi|^{w-q}-(2-\lambda)\\
+\sum_{w=2}^{\infty}w!\left(\frac{2-\lambda}{(w-q-1)!}+\frac{1}{(w-q-2)!}\right)D^{w}(\Gamma,\tau,\mu)a_{w}|\xi|^{w-q}\\
+\sum_{w=1}^{\infty}w!\left(\frac{2-\lambda}{(w-q-1)!}+\frac{1}{(w-q-2)!}\right)D^{w}(\Gamma,\tau,\mu)b_{w}\left|\bar{\xi}\right|^{w-q}\end{aligned}$$

$$= -2(1-\lambda) + 2\sum_{w=2}^{\infty} w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right) D^w(\Gamma,\tau,\mu) a_w + 2\sum_{w=1}^{\infty} w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right) D^w(\Gamma,\tau,\mu) b_w \le 0$$

Then

$$\sum_{w=2}^{\infty} w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right) D^w(\Gamma,\tau,\mu) a_w + \sum_{w=1}^{\infty} w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right) D^w(\Gamma,\tau,\mu) b_w \le (1-\lambda)$$

is obtained and completes the proof of **Theorem(2.1)**.

The function is harmonic univalent

$$\mathcal{L}(\xi) = \xi + \sum_{w=2}^{\infty} \frac{(1-\lambda)}{w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right) D^w(\Gamma,\tau,\mu)} X_w \xi^w + \sum_{w=1}^{\infty} \frac{(1-\lambda)}{w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right) D^w(\Gamma,\tau,\mu)} \overline{Y_w \xi^w} \quad (2.4)$$

In the case when $\sum_{w=2}^{\infty}|X_w| + \sum_{w=1}^{\infty}|Y_w| = 1$, the coefficient bound defined by (2.1) should be true

Because $\sum_{w=2}^{\infty} \frac{w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right) D^w(\Gamma,\tau,\mu)}{(1-\lambda)} |a_w| + \sum_{w=1}^{\infty} \frac{w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right) D^w(\Gamma,\tau,\mu)}{(1-\lambda)} |b_w| \le 1,$

$$\sum_{w=2}^{\infty} \frac{(1-\lambda)}{w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right) D^w(\Gamma,\tau,\mu)} \times \frac{w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right) D^w(\Gamma,\tau,\mu)}{(1-\lambda)} |X_w| + \sum_{w=1}^{\infty} \frac{(1-\lambda)}{w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right) D^w(\Gamma,\tau,\mu)} \times \frac{w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right) D^w(\Gamma,\tau,\mu)}{(1-\lambda)} |Y_w| = \sum_{w=2}^{\infty}|X_w| + \sum_{w=1}^{\infty}|Y_w| = 1. \blacksquare$$

Here, it must be demonstrated that the function $\mathcal{L}_m = \chi + \overline{\psi_m}$ must also satisfy the condition of (2.1), where χ and ψ_m are define by (1.3).

Theorem(2.2): Let $\mathcal{L}_m = \chi + \overline{\psi_m}$ is provided by (1.3), $\mathcal{L}_m \in \overline{T^\lambda}(\mu,\tau,q,\alpha,\beta,\gamma)$ if and only if the coefficient in condition (2.1) is true.

Proof: The "only if "part of the theorem is to be proven since $\overline{T^\lambda}(\mu,\tau,q,\alpha,\beta,\gamma) \subset T^\lambda(\mu,\tau,q,\alpha,\beta,\gamma)$.

Consequently,

$$Re\left(\frac{1 + \xi^q\left(I_{\alpha,\beta,\gamma}^{\mu,\tau}\mathcal{L}(\xi)\right)^{q+1} + \xi^{q+1}\left(I_{\alpha,\beta,\gamma}^{\mu,\tau}\mathcal{L}(\xi)\right)^{q+2}}{1 + \xi^q\left(I_{\alpha,\beta,\gamma}^{\mu,\tau}\mathcal{L}(\xi)\right)^{q+1}}\right) \ge \lambda$$

is obtained by(1.2) or equally

$$Re\left(\frac{\begin{array}{c}1 - \sum_{w=2}^{\infty} w!\left(\frac{1}{(w-q-1)!} + \frac{1}{(w-q-2)!}\right) D^w(\Gamma,\tau,\mu) a_w \xi^{w-q} - \\ \sum_{w=1}^{\infty} w!\left(\frac{1}{(w-q-1)!} + \frac{1}{(w-q-2)!}\right) D^w(\Gamma,\tau,\mu) b_w \bar{\xi}^{w-q} \\ -\lambda\left(1 - \sum_{w=2}^{\infty}\frac{w!}{(w-q-1)!} D^w(\Gamma,\tau,\mu) a_w \xi^{w-q} - \sum_{w=1}^{\infty}\frac{w!}{(w-q-1)!} D^w(\Gamma,\tau,\mu) b_w \bar{\xi}^{w-q}\right)\end{array}}{1 - \sum_{w=2}^{\infty}\frac{w!}{(w-q-1)!} D^w(\Gamma,\tau,\mu) a_w \xi^{w-q} - \sum_{w=1}^{\infty}\frac{w!}{(w-q-1)!} D^w(\Gamma,\tau,\mu) b_w \bar{\xi}^{w-q}}\right) \ge 0$$

$$Re\left(\frac{\begin{array}{c}(1-\lambda) - \sum_{w=2}^{\infty} w!\left(\frac{(1-\lambda)}{(w-q-1)!} + \frac{1}{(w-q-2)!}\right) D^w(\Gamma,\tau,\mu) a_w \xi^{w-q} - \\ \sum_{w=1}^{\infty} w!\left(\frac{(1-\lambda)}{(w-q-1)!} + \frac{1}{(w-q-2)!}\right) D^w(\Gamma,\tau,\mu) b_w \bar{\xi}^{w-q}\end{array}}{1 - \sum_{w=2}^{\infty}\frac{w!}{(w-q-1)!} D^w(\Gamma,\tau,\mu) a_w \xi^{w-q} - \sum_{w=1}^{\infty}\frac{w!}{(w-q-1)!} D^w(\Gamma,\tau,\mu) b_w \bar{\xi}^{w-q}}\right) \ge 0 \quad (2.5)$$

The previously mentioned requirement (2.5) must be met for every values of $\xi, |\xi| = r < 1$.when selecting the values of ξon the positive real axis, since $0 \le \xi = r < 1$, must be obtained

$$\left[\frac{\begin{array}{c}(1-\lambda)-\sum_{w=2}^{\infty} w!\left(\frac{(1-\lambda)}{(w-q-1)!}+\frac{1}{(w-q-2)!}\right) D^{w}(\Gamma,\tau,\mu) a_{w} r^{w-q} - \\ \sum_{w=1}^{\infty} w!\left(\frac{(1-\lambda)}{(w-q-1)!}+\frac{1}{(w-q-2)!}\right) D^{w}(\Gamma,\tau,\mu) b_{w} r^{w-q}\end{array}}{1-\sum_{w=2}^{\infty} \frac{w!}{(w-q-1)!} D^{w}(\Gamma,\tau,\mu) a_{w} r^{w-q}-\sum_{w=1}^{\infty} \frac{w!}{(w-q-1)!} D^{w}(\Gamma,\tau,\mu) b_{w} r^{w-q}}\right] \geq 0 \tag{2.6}$$

It is observed that when r goes to 1; the numerator (2.6) is negative if condition (2.1) is not satisfied. The proof is complete since contradicts the need for $\mathcal{L}_m \in \overline{T^{\lambda}}(\mu,\tau,q,\alpha,\beta,\gamma)$. ∎

3. CONVOLUTION (HADAMARD PRODUCT)

The class $\overline{T^{\lambda}}(\mu,\tau,q,\alpha,\beta,\gamma)$ is proved to be closed under convolution in this section

For harmonic functions

$\mathcal{L}_m(\xi)=\xi-\sum_{w=2}^{\infty}|a_w|\xi^w-\sum_{w=1}^{\infty}|b_w|\bar{\xi}^w$ & $\mathcal{J}_m(\xi)=\xi-\sum_{w=2}^{\infty}|A_w|\xi^w-\sum_{w=1}^{\infty}|B_w|\bar{\xi}^w$

The convolution of $\mathcal{L}_m(\xi)$ and $\mathcal{J}_m(\xi)$ is given by

$$(\mathcal{L}_m*\mathcal{J}_m)(z)=\mathcal{L}_m(\xi)*\mathcal{J}_m(\xi)=\xi-\sum_{w=2}^{\infty}|a_wA_w|\xi^w-\sum_{w=1}^{\infty}|b_wB_w|\bar{\xi}^w \tag{3.1}$$

Theorem(3.1): Let $\mathcal{L}_m(\xi)\in\overline{T^{\lambda'}}(\mu,\tau,q,\alpha,\beta,\gamma)$ and $\mathcal{J}_m(\xi)\in\overline{T^{\lambda''}}(\mu,\tau,q,\alpha,\beta,\gamma)$,then $\mathcal{L}_m*\mathcal{J}_m\in\overline{T^{\lambda''}}(\mu,\tau,q,\alpha,\beta,\gamma)\subset\overline{T^{\lambda'}}(\mu,\tau,q,\alpha,\beta,\gamma)$, for $0\leq\lambda'\leq\lambda''<1$.

Proof: It is intended to demonstrate that the coefficient $\mathcal{L}_m*\mathcal{J}_m$ meets the necessary requirement stated in **Theorem(2.2)**. For $\mathcal{J}_m(\xi)\in\overline{T^{\lambda''}}(\mu,\tau,q,\alpha,\beta,\gamma)$, it is noted that $|A_w|\leq 1$ and $|B_w|\leq 1$. Now, for the convolution function $\mathcal{L}_m*\mathcal{J}_m$, it is obtained that

$$\sum_{w=2}^{\infty}\frac{w!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^w(\Gamma,\tau,\mu)}{(1-\lambda'')}|a_w||A_w|+\sum_{w=1}^{\infty}\frac{w!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^w(\Gamma,\tau,\mu)}{(1-\lambda'')}|b_w||B_w|\leq$$

$$\sum_{w=2}^{\infty}\frac{w!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^w(\Gamma,\tau,\mu)}{(1-\lambda')}|a_w|+\sum_{w=1}^{\infty}\frac{w!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^w(\Gamma,\tau,\mu)}{(1-\lambda')}|b_w|\leq 1.$$

given that $0\leq\lambda'\leq\lambda''<1$ and $\mathcal{L}_m(\xi)\in\overline{T^{\lambda'}}(\mu,\tau,q,\alpha,\beta,\gamma)$. Consequently, $\mathcal{L}_m*\mathcal{J}_m\in\overline{T^{\lambda''}}(\mu,\tau,q,\alpha,\beta,\gamma)\subset\overline{T^{\lambda'}}(\mu,\tau,q,\alpha,\beta,\gamma)$. ∎

4. CONVEX COMBINATION

In this section, $\overline{T^{\lambda}}(\mu,\tau,q,\alpha,\beta,\gamma)$ is proved to be closed under convex combination of its member. Let the function $\mathcal{L}_{m_i}(\xi)$ be defined, for $i=1,2,\ldots$ by

$$\mathcal{L}_{m_i}(\xi)=\xi-\sum_{w=2}^{\infty}|a_{w_i}|\xi^w-\sum_{w=1}^{\infty}|b_{w_i}|\bar{\xi}^w \tag{4.1}$$

Theorem(4.1): For each $i=1,2,\ldots$, let the function $\mathcal{L}_{m_i}(\xi)$, defined by (4.1) belong to the class $\overline{T^{\lambda}}(\mu,\tau,q,\alpha,\beta,\gamma)$, then the function $t_i(z)$ which is defined by $t_i(\xi)=\sum_{i=1}^{\infty}C_i\,\mathcal{L}_{m_i}(\xi)$, $0\leq C_i\leq 1$, is also in the class $\overline{T^{\lambda}}(\mu,\tau,q,\alpha,\beta,\gamma)$, where $\sum_{i=1}^{\infty}C_i=1$.

Proof: The definition of $t_i(z)$ allows us to write

$$t_i(z)=z-\sum_{w=2}^{\infty}\left(\sum_{i=1}^{\infty}C_i\left|a_{w,i}\right|\right)z^w-\sum_{w=1}^{\infty}\left(\sum_{i=1}^{\infty}C_i\left|b_{w,i}\right|\right)\bar{z}^w.$$

Furthermore, for each $i=1,2,\ldots$, since $\mathcal{L}_{m_i}(\xi)$ are in $\overline{T^{\lambda}}(\mu,\tau,q,\alpha,\beta,\gamma)$, therefore

$$\sum_{w=2}^{\infty}w!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^w(\Gamma,\tau,\mu)\left(\sum_{i=1}^{\infty}C_i\left|a_{w,i}\right|\right)+\sum_{w=1}^{\infty}w!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^w(\Gamma,\tau,\mu)\left(\sum_{i=1}^{\infty}C_i\left|b_{w,i}\right|\right)=$$

$$\sum_{i=1}^{\infty}C_i\left(\sum_{w=2}^{\infty}w!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^w(\Gamma,\tau,\mu)\left|a_{w,i}\right|+\sum_{w=1}^{\infty}!\left(\frac{1}{(w-q-2)!}+\frac{(1-\lambda)}{(w-q-1)!}\right)D^w(\Gamma,\tau,\mu)\left|b_{w,i}\right|\right)\leq$$

$$\sum_{i=1}^{\infty}C_i(1-\lambda)\leq(1-\lambda)$$

This completes the theorem (4.1) proof. ∎

5. DISTORTION BOUNDS

The distortion bounds for functions in $\overline{T^{\lambda}}(\mu,\tau,q,\alpha,\beta,\gamma)$ are given by the following theorem, which also provides a covering result for this class.

Theorem(5.1): If $\mathcal{L}_m = \chi + \overline{\psi_m} \in \overline{T^{\lambda}}(\mu,\tau,q,\alpha,\beta,\gamma)$, given by (1.3) and $|\xi| = r < 1$, then

$$(1-|b_1|)r - \frac{1}{\mathbb{T}}\left((1-\lambda) - \left[\prod_{\substack{\Lambda=0\\ \Lambda\neq 1}}^{q+1}(1-\Lambda) + \left((1-\lambda)\prod_{\substack{\Lambda=0\\ \Lambda\neq 1}}^{q}(1-\Lambda)\right)\right]D^1(\Gamma,\tau,\mu)|b_1|\right)r^2 \le |\mathcal{L}_m(\xi)|$$

$$\le \ (1-|b_1|)r + \frac{1}{\mathbb{T}}\left((1-\lambda) - \left[\prod_{\substack{\Lambda=0\\ \Lambda\neq 1}}^{q+1}(1-\Lambda) + \left((1-\lambda)\prod_{\substack{\Lambda=0\\ \Lambda\neq 1}}^{q}(1-\Lambda)\right)\right]D^1(\Gamma,\tau,\mu)|b_1|\right)r^2 \qquad \mathbf{(5.1)}$$

Where $\mathbb{T} = \left[\prod_{\substack{\Lambda=0\\ w\neq\Lambda}}^{q+1}(w-\Lambda) + \left((1-\lambda)\prod_{\substack{\Lambda=0\\ w\neq\Lambda}}^{q}(w-\Lambda)\right)\right]D^2(\Gamma,\tau,\mu)$

Proof: The left-hand side of inequality (6.1) will be proved first. Let $\mathcal{L}_m \in \overline{T^{\lambda}}(\mu,\tau,q,\alpha,\beta,\gamma)$; then
$|\mathcal{L}_m(\xi)| = \left|\xi - \sum_{w=2}^{\infty}|a_w|\xi^w - \sum_{w=1}^{\infty}|b_w|\bar{\xi}^w\right|$

$\ge (1-|b_1|)|\xi| - \sum_{w=2}^{\infty}(|a_w| + |b_w|)r^w$
$\ge (1-|b_1|)|\xi| - \sum_{w=2}^{\infty}(|a_w| + |b_w|)r^2$

$$= (1-|b_1|)r - \frac{1-\lambda}{\left[\prod_{\substack{\Lambda=0\\ w\neq\Lambda}}^{q+1}(w-\Lambda) + \left((1-\lambda)\prod_{\substack{\Lambda=0\\ w\neq\Lambda}}^{q}(w-\Lambda)\right)\right]D^2(\Gamma,\tau,\mu)}\sum_{w=2}^{\infty}\frac{\left[\prod_{\substack{\Lambda=0\\ w\neq\Lambda}}^{q+1}(w-\Lambda) + \left((1-\lambda)\prod_{\substack{\Lambda=0\\ w\neq\Lambda}}^{q}(w-\Lambda)\right)\right]D^2(\Gamma,\tau,\mu)}{1-\lambda}(|a_w| + |b_w|)r^2 \ge (1-|b_1|)r -$$

$$\frac{1-\lambda}{\left[\prod_{\substack{\Lambda=0\\ w\neq\Lambda}}^{q+1}(w-\Lambda) + \left((1-\lambda)\prod_{\substack{\Lambda=0\\ w\neq\Lambda}}^{q}(w-\Lambda)\right)\right]D^2(\Gamma,\tau,\mu)}\sum_{w=2}^{\infty}\left(\frac{w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right)D^w(\Gamma,\tau,\mu)}{1-\lambda}|a_w| + \frac{w!\left(\frac{1}{(w-q-2)!} + \frac{(1-\lambda)}{(w-q-1)!}\right)D^w(\Gamma,\tau,\mu)}{1-\lambda}|b_w|\right)r^2$$

$$\ge (1-|b_1|)r - \frac{1-\lambda}{\left[\prod_{\substack{\Lambda=0\\ w\neq\Lambda}}^{q+1}(w-\Lambda) + \left((1-\lambda)\prod_{\substack{\Lambda=0\\ w\neq\Lambda}}^{q}(w-\Lambda)\right)\right]D^2(\Gamma,\tau,\mu)}\left(1 - \frac{\left[\prod_{\substack{\Lambda=0\\ \Lambda\neq 1}}^{q+1}(1-\Lambda) + \left((1-\lambda)\prod_{\substack{\Lambda=0\\ \Lambda\neq 1}}^{q}(1-\Lambda)\right)\right]D^1(\Gamma,\tau,\mu)}{1-\lambda}|b_1|\right)r^2$$

$$\geq (1-|b_1|)r - \frac{1}{\begin{bmatrix} \prod_{\substack{\Lambda=0\\ w\neq\Lambda}}^{q+1}(w-\Lambda)+ \\ \left((1-\lambda)\prod_{\substack{\Lambda=0\\ w\neq\Lambda}}^{q}(w-\Lambda)\right)\end{bmatrix} D^2(\Gamma,\tau,\mu)} \left((1-\lambda) - \begin{bmatrix} \prod_{\substack{\Lambda=0\\ \Lambda\neq1}}^{q+1}(1-\Lambda)+ \\ \left((1-\lambda)\prod_{\substack{\Lambda=0\\ \Lambda\neq1}}^{q}(1-\Lambda)\right)\end{bmatrix} D^1(\Gamma,\tau,\mu)|b_1|\right) r^2$$

The right-hand inequality in (5.1) can be readily demonstrated using similar argument. ■

6. CONCLUSION

The generalized fractional integral operator on an open unit disk is connected to a new class of harmonic univalent analytic functions with higher derivatives was created in order to compute coefficient bounds and uses the results to determine the Hadmared product, convex combination, and distortion bound theorems. In the future, it might be able to get different conclusions for the same class by using another strategy rather than Lemma 1.1 or by expanding this new class to the multivalent analytic or meromorphic functions by employing different kind of operators to arrive at new results.

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