Some properties on cyclic composition operators

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Abstract:

In this work, we study the cyclicity of the composition operator C_φ induced by an automorphism mapping φ on U = { z∈ ℂ : $|z|$ <1 },and give some conditions that are necessary and (or) sufficient for the operator $C_{\alpha_p} o C_{\alpha_q}$ to be cyclic, where α_p and α_q are the special automorphisim of the unit ball U .

Introduction:

Let H(U) be the set of all holomorphic functions on the unit ball U of the complex plane . If f belongs to H(U) then by Taylor theorem one can expand the function f about the origin as follows : $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n \quad ,(z\in U)$ If the sequence of the coefficients $\{\hat{f}(n)\}$ is a square - summable sequence, i.e. $\sum_{n=0}^{\infty}\left|\hat{f}(n)\right|^2<\infty$, then we say that the function f belongs to H^2 or $H^2(U)$. Therefore $H^2=\left\{f\in H(U):\sum_{n=0}^{\infty}\left|\hat{f}(n)\right|^2<\infty\right\}$. H^2 is called the Hardy space[1],[7].The Littlewood's principle theorem [7] shows that if φ is a self map of U (i.e. φ(U)⊆U) that belongs to H(U),then the composition operator C_φ defined by :

$C_\varphi f = f o\varphi$ (f holomorphic on U) takes the Hardy space H^2 into itself . Littlewood's principle also shows that C_φ is a bounded operator on H^2.

Recall that an operator T on a Hilbert space H is said to be cyclic if there is a vector x in H (called a cyclic vector for T) whose orbit , orb(T,x) = $\{T^n x : n=0,1,2,\ldots\}$ has dense linear span in H . The operator T is supercyclic if there is a vector x in H (called a supercyclic vector for T) such that the set $\{\lambda_n T^n x : \lambda_n\in ℂ, n=0,1,\ldots\}$ is dense in H . It may happen that orbit , orb(T,x) is dense in H , in this case T is called haypercyclic and x is a hypercyclic vector [2],[3],[4].

In this work we prove that if p∈U then C_{α_p} and $C^*_{\alpha_p}$ are not cyclic operators where $C^*_{\alpha_p}$ is the adjoint of C_{α_p} and $\alpha_p(z)=\frac{p-z}{1-\bar{p}z}$. We give some conditions that are necessary and (or) sufficient for the operator $C_{\alpha_p} o C_{\alpha_q}$ to be cyclic . Also we prove that $C_{\alpha_p} o C_{\alpha_q}$ is cyclic if and only if $C_{\alpha_q} o C_{\alpha_p}$ is cyclic .

Notations And Preliminary :

In this section we give some basic definitions and properties that we shall use in the next section .

Definition (2.1) [7]: A linear fractional transformation is a mapping φ defined on the complex plane ℂ by $\varphi(z)=\frac{az+b}{cz+d}$ where a,b,c,d are complex numbers .

Remarks (2.2) [6]:

1. A linear fractional transformation φ is constant function if and only if ad-bc = 0 .
2. every non constant linear fractional transformation is one – to – one and onto transformation.

We consider a linear fractional transformation $\varphi(z)=\frac{az+b}{cz+d}$ with ad-bc≠0 defined on the Riemann sphere $\hat{C}$ = ℂ$\bigcup\{\infty\}$ where $\varphi(\infty)=\frac{a}{c}$ and $\varphi(\frac{-d}{c})=\infty$.

Notation (2.3) [7]:

1. We denote the set of all linear fractional transformations is subject to the condition ad – bc ≠ 0 by LFT($\hat{C}$) .
2. For any $\varphi(z)=\frac{az+b}{cz+d}\in$ LFT($\hat{C}$) , we sometimes denote it by $\varphi_A(z)$ where A is the non – singular 2×2 complex matrix , $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Remarks (2.4) [7]:

1. it is easily prove that $\varphi_A o\varphi_B=\varphi_{AB}$ where o is the composition of maps.
2. It is easily to prove that (LFT($\hat{C}$),o) is a group where $\varphi_A^{-1}=\varphi_{A^{-1}}$ for all $\varphi_A\in$ LFT($\hat{C}$) .

Definition (2.5) [7]: A map φ∈ LFT($\hat{C}$) is called parabolic if it has a single fixed point in $\hat{C}$.

Remark (2.6) : If T $\in$ LFT($\hat{C}$) is not parabolic, then T has two fixed points $\alpha, \beta \in \hat{C}$. Let S $\in$ LFT($\hat{C}$) that takes α to 0 and β to ∞, then the map V=S o T o S^{-1} belongs to LFT($\hat{C}$) and fixes both 0 and ∞, so it must have the form V(z) = λz, λ is said to be the multiplier for T [7] .

In the following definition we classify the linear fractional transformation according to their multipliers .

Definition (2.7) [7] : Suppose that T $\in$ LFT($\hat{C}$) is neither parabolic nor the identity. Let λ ≠ 1 be the multiplier of T . Then T is called :

- Elliptic if $|\lambda| = 1$.
- Hyperbolic if λ > 0 . and
- Loxodromic if T is neither elliptic nor hyperbolic .

Our interest here is in LFT(U) the subgroup of LFT($\hat{C}$) consisting of self maps of the unite ball U (i.e. take U into itself) .

Proposition (2.8) [7] : If φ ∈ LFT(U) is parabolic then φ has its fixed point on ∂U , where ∂U is the boundary of U .

Theorem (2.9) [7] : Let φ be a linear fractional self map of U

1. If φ is hyperbolic , then it has fixed point in $\overline{U}$ with the other fixed point outside U, where $\overline{U}$ is the closed unit disc.
2. If φ is loxodromic or elliptic , then it has a fixed point in U and a fixed point outside $\overline{U}$.

Definition (2.10)[8]: Let φ be a holomorphic self map of U . If φ is one – to – one and onto U then φ is said to be a conformal automorphisim of U or just automorphisim .

Proposition (2.11)[6] : Let φ be a linear fractional self map of U then φ is elliptic if and only if φ is automorphisim with interior fixed point in U .

We end this section by the linear fractional transformation $\alpha_p(z) = \dfrac{p - z}{1 - \bar{p}z}$, p∈U which is called the special automorphisim of U . One can prove that α_p is automorphism and self inverse (i.e. $\alpha_p^{-1} = \alpha_p$),[8].

The following theorem shows that every conformal automorphism of U is linear fractional self map of U .

Theorem (2.12) [8] : If φ is a conformal automorphism of U then there exists p∈U and w∈ ∂U such that $\varphi(z) = w\alpha_p(z)$ for all z∈U .

Main Results:

In this section we show that if p∈U then C_{α_p} and $C^*_{\alpha_p}$ are not cyclic operators where $C^*_{\alpha_p}$ is the adjoint of C_{α_p} and we study the cyclicity of $C_{\alpha_p} o C_{\alpha_q}$ where α_p and α_q are the special automorphisim of the unit ball U . Also we give an example of two non cyclic composition operators C_φ and C_ψ which is its composition $C_\varphi C_\psi$ is hypercyclic .

We prove the following theorem .

Theorem(3.1) : If p ≠ 0 is interior point (p∈U) then $\alpha_p(z) = \dfrac{-z + p}{-\bar{p}z + 1}$ has interior fixed point $\dfrac{1 - \sqrt{1 - |p|^2}}{\bar{p}}$ and exterior fixed point $\dfrac{1 + \sqrt{1 - |p|^2}}{\bar{p}}$.

Proof : Put $\alpha_p(z) = z$ that is $\dfrac{-z + p}{-\bar{p}z + 1} = z$ therefore $\bar{p}z^2 - 2z + p = 0$. Hence α_p have two fixed points $z_1 = \dfrac{2 + \sqrt{4 - 4|p|^2}}{2\bar{p}} = \dfrac{1 + \sqrt{1 - |p|^2}}{\bar{p}}$ and $z_2 = \dfrac{2 - \sqrt{4 - 4|p|^2}}{2\bar{p}} = \dfrac{1 - \sqrt{1 - |p|^2}}{\bar{p}}$. Since p is interior point then $|\bar{p}| < 1 < \left|1 + \sqrt{1 - |p|^2}\right|$, therefore $|z_1| = \left|\dfrac{1 + \sqrt{1 - |p|^2}}{\bar{p}}\right| > 1$ .Thus z_1 is exterior fixed point .Now we prove $|z_2| = \left|\dfrac{1 - \sqrt{1 - |p|^2}}{\bar{p}}\right| < 1$.

Let r = $|p|$, then 0< r < 1 . Suppose that $|z_2| \geq 1$, so that $\left|1 - \sqrt{1 - r^2}\right| \geq r$. Since $1 \rangle \sqrt{1 - r^2}$ then $1 - \sqrt{1 - r^2} = \left|1 - \sqrt{1 - r^2}\right|$, therefore $1 - r - \sqrt{1 - r^2} \geq 0$.

This inequality implies that $-1 \leq r \leq 0$, this is contradict that 0<r<1 .Thus z_2 is interior fixed point .

Recall that if φ is a holomorphic self map of U that fixes a point q∈U then C_φ and C^*_φ are not supercyclic (hypercyclic)operators [6] .

From theorem(3.1) we can get easily the following corollary.

Corollary (3.2): If α_p is a special automorphisim of U (p∈U) then C_{α_p} and $C^*_{\alpha_p}$ are not supercyclic (hypercyclic)operators.

Bourdon and Shapiro in [1] prove the following result :

Theorem (3.3): If φ is elliptic then C_φ is cyclic if and only if the argument of λ is irrational multiple of π, where $\lambda = \varphi'(q), q$ is the interior fixed point of φ.

We prove the following theorem.

Theorem (3.4) : If p∈U then C_{α_p} and $C^*_{\alpha_p}$ are not cyclic operators .

Proof : Suppose p ≠ 0. Since α_p is automorphism [8] and has interior fixed point $\beta = \dfrac{1-\sqrt{1-|p|^2}}{\bar{p}}$ then α_p is elliptic(proposition(2.11)) .

Since

$$\lambda = \alpha'_p(\beta) = \frac{-1+|p|^2}{(-\bar{p}\beta+1)^2} = \frac{-1+|p|^2}{\left(-\bar{p}\left(\frac{1-\sqrt{1-|p|^2}}{\bar{p}}\right)+1\right)^2} = -1,$$

then from theorem (3.3), C_{α_p} is not cyclic . We see in [6, theorem(3.1.5)] that if φ is elliptic mapping then C_φ is cyclic if and only if C^*_φ is cyclic .Thus $C^*_{\alpha_p}$ is not cyclic operator . A trivial case when p = 0 then α_p(z) = - z and hence C_{α_p} is not cyclic. Since α_p(z) =-z then α_p is automorphism with fixed point z = 0 . Thus α_p is also elliptic , so that $C^*_{\alpha_p}$ is not cyclic operator [6, theorem (3.1.5)] .

Before we give the following theorem we need the following proposition .

Proposition (3.5) [7]: suppose that $w = e^{i\theta_o}$, where $-\pi < \theta_o \le \pi$ and p∈U then $w\alpha_p$ is

1. Elliptic if and only if $|p| < \cos\left(\frac{\theta_o}{2}\right)$
2. Parabolic if and only if $|p| = \cos\left(\frac{\theta_o}{2}\right)$
3. Hyperbolic if and only if $|p| > \cos\left(\frac{\theta_o}{2}\right)$

Theorem (3.6) : If φ is automorphism of U then φ is elliptic (parabolic ,hyperbolic) if and only if φ^{-1} is elliptic (parabolic ,hyperbolic).

Proof : Since φ is automorphism then from theorem (2.12) , $\varphi(z) = w\alpha_p(z)$ where $w = e^{i\theta_o} \in \partial U$ and p∈U. From the notation (2.3) we denote φ(z) by $\varphi_A(z)$ where $A = \begin{pmatrix} -w & wp \\ -\bar{p} & 1 \end{pmatrix}$, therefore $\varphi_A^{-1} = \varphi_{A^{-1}} = \frac{1}{w}\alpha_{wp} = e^{-i\theta_o}\alpha_{wp}$

If φ is elliptic then $|p| < \cos\left(\frac{\theta_o}{2}\right)$,(proposition (3.5)) , and hence $|wp| = |p| < \cos\left(\frac{\theta_o}{2}\right) = \cos\left(\frac{-\theta_o}{2}\right)$, that is φ^{-1} is elliptic (proposition (3.5)) . By the same way we can prove the theorem when φ is parabolic or hyperbolic.

Now we give the following corollary .

Corollary (3.7) :Let φ be a holomorphic self map of U . If φ is invertible then C_φ is cyclic (hypercyclic,supercyclic)if and only if C_φ^{-1} is cyclic (hypercyclic, supercyclic) .

Proof : Since φ and φ^{-1} are of the same type (theorem(3.6)),then C_φ is cyclic (hypercyclic,supercyclic)if and only if $C_\varphi^{-1} = C_{\varphi^{-1}}$ is cyclic (hypercyclic, supercyclic) .

Bourdon and Shapiro in [1] proves that if φ is a conformal automorphism of U with no fixed point in the interior of U ,then C_φ is hypercyclic operator .

Corollary (3.8) : Let $\varphi = w\alpha_p$ where $w = e^{i\theta}, -\pi < \theta \le \pi$ and $p \in U$. If $|p| \ge \cos\left(\frac{\theta}{2}\right)$ then C_φ and C_φ^{-1} are hypercyclic operators .

Proof : Since α_p is automorphism then $\varphi = w\alpha_p$ is automorphism . If φ has fixed point in the interior of U then φ is elliptic(proposition(2.11)) . This is contradict proposition (3.5), therefore φ is automorphism with no fixed point in the interior of U, so that C_φ is hypercyclic operator [1], and by using corollary(3.7) one can have C_φ^{-1} is also hypercyclic operator .

Now we study the cyclicity of $C_{\alpha_p} o C_{\alpha_q}$ where α_p and α_q are the special automorphisim mapping , a trivial case when p = q implies $C_{\alpha_p} o C_{\alpha_p}$ is not cyclic since $C_{\alpha_p} o C_{\alpha_p} = C_{\alpha_p o \alpha_p}$ = I ,where I is the identity operator.

We prove the following proposition .

Proposition (3.9) : If p_1, p_2 are interior points of U such that $p_1 \ne p_2$ then $\alpha_{p_1} o \alpha_{p_2} = w\alpha_p$ where $w = \frac{-1+p_1\bar{p}_2}{1-\bar{p}_1 p_2}$ with absolute value is equal to one , $p = \frac{p_2 - p_1}{1-p_1\bar{p}_2} \in U$.

Proof : The matrix of α_{p_1} and α_{p_2} are $\begin{pmatrix} -1 & p_1 \\ -\bar{p}_1 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & p_2 \\ -\bar{p}_2 & 1 \end{pmatrix}$ respectively, hence the matrix of $\alpha_{p_1} o \alpha_{p_2}$ is $\begin{pmatrix} -1 & p_1 \\ -\bar{p}_1 & 1 \end{pmatrix}\begin{pmatrix} -1 & p_2 \\ -\bar{p}_2 & 1 \end{pmatrix} = \begin{pmatrix} 1-p_1\bar{p}_2 & p_1-p_2 \\ \bar{p}_1-\bar{p}_2 & 1-\bar{p}_1 p_2 \end{pmatrix}$.Therefore

$$\alpha_{p_1} o \alpha_{p_2} = \frac{(1-p_1\bar{p}_2)z + p_1 - p_2}{(\bar{p}_1-\bar{p}_2)z + 1 - \bar{p}_1 p_2} = \frac{-1+p_1\bar{p}_2}{1-\bar{p}_1 p_2}\left(\frac{-z+\dfrac{p_2-p_1}{1-p_1\bar{p}_2}}{\dfrac{\bar{p}_1-\bar{p}_2}{1-\bar{p}_1 p_2}z+1}\right) = w\alpha_p(z)$$

,where $w = \dfrac{-1+p_1\bar{p}_2}{1-\bar{p}_1 p_2}$, $p = \dfrac{p_2-p_1}{1-p_1\bar{p}_2}$. It is clear that $|w| = 1$. To prove that p∈U, we see that $\left(1-|p_2|^2\right)\left(1-|p_1|^2\right) > 0$ since $p_1, p_2 \in U$, hence $1-|p_1|^2-|p_2|^2+|p_1|^2|p_2|^2 > 0$, therefore $|p_2|^2+|p_1|^2 < 1+|p_1|^2|p_2|^2$, this inequality implies that $|p|^2 = \left|\dfrac{p_2-p_1}{1-p_1\bar{p}_2}\right|^2 = \dfrac{(p_2-p_1)(\bar{p}_2-\bar{p}_1)}{(1-p_1\bar{p}_2)(1-\bar{p}_1 p_2)} < 1$. Thus p∈U.

Now we are ready to prove the following result .

Theorem (3.10) : Let p, q ∈ U , p ≠ q then $\alpha_p o \alpha_q$ is

1. elliptic if $|q-p| + \mathrm{Im}(p\bar{q}) < 0$
2. parabolic if $|q-p| + \mathrm{Im}(p\bar{q}) = 0$
3. hyperbolic if $|q-p| + \mathrm{Im}(p\bar{q}) > 0$

Proof : From proposition (3.9) , $\alpha_p o \alpha_q =$ wα_b where $w = \dfrac{-1+p\bar{q}}{1-\bar{p}q}$,

$b = \dfrac{q-p}{1-p\bar{q}}$ .Let r = $1-p\bar{q} = me^{i\theta}$, so that

$w = \dfrac{-r}{\bar{r}} = \dfrac{-me^{i\theta}}{me^{-i\theta}} = -e^{i2\theta} = e^{i(2\theta-\pi)}$

From proposition (3.5), $C_{\alpha_p} o C_{\alpha_q}$ is elliptic if $|b| < \cos\left(\frac{2\theta-\pi}{2}\right)$,that is $\dfrac{|q-p|}{|r|} < \cos\left(\theta-\frac{\pi}{2}\right) = \sin(\theta)$, so that $|q-p| < |r|\sin(\theta) = \mathrm{Im}(r) = \mathrm{Im}(1-p\bar{q}) = -\mathrm{Im}(p\bar{q})$. Thus $|q-p| + \mathrm{Im}(p\bar{q}) < 0$.

By the same way we can prove (2) and (3) .

Corollary (3.11) : Suppose that p, q are interior points of U , p ≠ q . If $|q-p| + \mathrm{Im}(p\bar{q}) \geq 0$ then $C_{\alpha_q} o C_{\alpha_p}$ is hypercyclic operator .

Proof : Since α_p and α_q are automorphisms then $\alpha_p o \alpha_q$ is automorphism .

From theorem (3.10) we have $\alpha_p o \alpha_q$ is either parabolic or hyperbolic , therefore $\alpha_p o \alpha_q$ does not have fixed point in the interior of U . Thus $C_{\alpha_q} o C_{\alpha_p} = C_{\alpha_p o \alpha_q}$ is hypercyclic [1, theorem (2.2)] .

In the following example we give two non cyclic composition operators C_{α_p} , C_{α_q} which is its composition $C_{\alpha_p} o C_{\alpha_q}$ is hypercyclic operator .

Example (3.12) : If p , q are two distinct real numbers in the interval (-1,1) then C_{α_q} and C_{α_p} are non cyclic composition operators (theorem (3.4)) , but $C_{\alpha_q} o C_{\alpha_p}$ is hypercyclic operator (corollary (3.11)) .

We recall that if H is a Hilbert space and T_1, T_2 are two operators on H , then T_1, T_2 are similar if there is an invertible operator S such that $T_2 = S^{-1}T_1S$ [7,p.93].

Remark(3.13) [5]:If T_1 and T_2 are similar operators then T_1 is cyclic (supercyclic , hypercyclic) if and only if T_2 is cyclic(supercyclic , hypercyclic) .

Now, we are in the position that we can give the following theorem .

Theorem (3.14) : Suppose that p, q are interior fixed points then $C_{\alpha_p} o C_{\alpha_q}$ is cyclic (supercyclic , hypercyclic) operator if and only if $C_{\alpha_q} o C_{\alpha_p}$ is cyclic (supercyclic , hypercyclic) operator.

Proof :We see that C_{α_q} is invertible operator where $C^{-1}_{\alpha_q} = C_{\alpha_q}$ .Since $C_{\alpha_q} o C_{\alpha_p} o C_{\alpha_q} o C_{\alpha_q} = C_{\alpha_q} o C_{\alpha_p}$ then $C_{\alpha_p} o C_{\alpha_q}$ and $C_{\alpha_q} o C_{\alpha_p}$ are similar operators , thus $C_{\alpha_p} o C_{\alpha_q}$ is cyclic (supercyclic , hypercyclic) operator if and only if $C_{\alpha_q} o C_{\alpha_p}$ is cyclic (supercyclic , hypercyclic) operator.

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عنوان البحث باللغة العربية

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المخلص:

درسنا في هذا العمل الصفة الدائرية للمؤثر التركيبي C_φ عندما φ دالة متقابلة (شاملة ومتباينة) على U، حيث $U = \{z \in \mathbb{C} : |z| < 1\}$. كما أعطينا بعض الشروط الكافية و (أو) الضرورية التي تجعل المؤثر التركيبي $C_{\alpha_p} o C_{\alpha_q}$ مؤثراً دائرياً ، حيث α_p & α_q هي الدوال الخاصة المعرفة على U.