# On The Projective Plane Pg (2, 3)

Esam R. Kakayee

*Tech. Inst. Of Kirkuk Electronic Department* (Recieved / / , Accepted / / )

### abstract

In this work it is shown that the Steiner system S (2, 4, 13) (the projective plane PG(2, 3) of order three) can be constructed from the Steiner system S (5, 6, 12), by adding a new point to its twelve points. The third contraction of S (5, 6, 12) with respect to the same triads defined in (Kakayee, 1990)  $t_1$ ,  $t_2$ ,  $t_3$  and  $t_4$  respectively yields four Steiner

systems S (2, 3, 9) (the affine plane  $_{AG(2, 3)}$  of order three), denoted by  $A^{t_1}$ ,  $A^{t_2}$ ,  $A^{t_3}$  and  $A^{t_4}$ . In order to construct the same projective plane PG(2, 3) (without any isomorphic forms in case of using Singers theorem) considered in (Kakayee, 1990), the following procedure was used:-

- i- (9) Blocks formed from the union of all blocks of  $A^{t_1}$ ,  $A^{t_2}$ ,  $A^{t_3}$  and  $A^{t_4}$  that intersect in two points,
- ii- (4) blocks formed from the union of all blocks of  $A^{t_1}$ ,  $A^{t_2}$ ,  $A^{t_3}$  and  $A^{t_4}$  that intersect in three points each with a new point.

#### Introduction

A triple (t, k, v), where v is a set of points of size v, k is a collection of all  $\binom{v}{t} / \binom{k}{t}$  subsets of v of size k called

<u>blocks</u> and an incidence relation between the points and blocks such that each subset of distinct points of size t contained in a unique block, is called a <u>Steiner system</u> and denoted by S (t, k, v) (Jónsson, 1972).

Steiner systems S (2, 3, 9), S (2, 4, 13) and S (5, 6, 12) are existing and unique up to isomorphism (Linder and Rosa, 1980), (Witt, 1938) and (Barrau, 1908). The first and the second systems were constructed each from another geometrically (Hughes and Piper 1973). The third one was constructed geometrically by extending the first system three times (Lüneburg, 1968), conversely, the first system can be constructed from the third one by contraction (Gross, 1974). The aim of the present paper is to continue an investigation begun by (Kakayee, 1990), who shows that the system S(5,6,12) on twelve points can be construct geometrically by deleting a point of the system S(2,4,13) on thirteen points, to construct exactly the same system S(2,4,13) considered in the previous research fro the system S(5,6,12), that is a work converse to that done by (Kakayee, 1990).

#### **Definitions And Notations**

The definitions and notations given below are useful for the construction of the system.

A triple (t, k, v), where v is a set of points, k is a set of lines and t is an incidence relation between points and lines such that:

- 1- Each two distinct points determine one line,
- 2- each two distinct lines intersect in one point,
- 3- there are four points no three of them are collinear.

Is called a <u>projective plane</u> PG(2, n) of order n, where

n > 2 is a positive integer. A projective plan contains

 $n^2 + n + 1$  points and  $n^2 + n + 1$  lines, any line

contains n+1 points and each point lies on n+1 lines (Hughes and Piper 1973).

A triple (t, k, v), where v, k, t are as indicated above satisfying the following conditions:

1- Each two distinct points determine one line,

2- let *l* be any line and *p* be any point,  $p \notin l$  then there is one line *m* such that  $p \in m$  and l//m,

3- there are three non-collinear points.

Is called an <u>affine plane</u> AG(2, n) (Hughes and Piper 1973). Deleting any line of PG(2, n) and all its points gives an AG(2, n) of the same order, therefore an affine plane contains  $n^2$  points and  $n^2 + n$  lines, any line contains n points, each point lies on n+1 lines and there are n+1 parallel classes in the affine plane and each parallel class has n lines (Hughes and Piper 1973). Conversely, adding a new point to all the lines of each parallel class and a line incident with all these points gives a ptojective plane, this is known as Singers theorem (Linder and Rosa, 1980).

- PG(2,n) is  $S(2, n+1, n^2 + n+1)$  (Jónsson, 1972),
- AG(2,n) is  $S(2,n,n^2)$  (Jónsson, 1972),

- an i-th contraction of a Steiner system S(t,k,v) with respect to i-set of points,  $0 \le i \le t$  is the Steiner system S(t-i,k-i,v-i) derived by deleting the i-set of points and all the blocks that do not containing them, note that whenever a Steiner system S(t,k,v) exists then so does its (t-2)-nd contraction S(2,k-t+2,v-t+2)

(Gross, 1974) and (Linder and Rosa, 1980).Finally, the reader is referred to the original research (Kakayee, 1990), for different properties of the different Steiner systems.

### Construction Of S (2, 4, 13)

In this part, the Steiner system S (2, 4, 13) is constructed from four affine planes S (2, 3, 9) which can immediately be derived from the Steiner system S(5, 6, 12) by contraction. Considering the list of all blocks of S (5, 6, 12) as obtained in (Kakayee, 1990), and the triads  $t_1 = \{1, 2, 3\}, \quad t_2 = \{4, 5, 6\}, \quad t_3 = \{7, 8, 9\}$  and  $t_4 = \{10, 11, 12\}$  as defined in (Kakayee, 1990). On the third contraction of S(5, 6, 12) with respect to the triads each at a time, we will be left with four Steiner systems S(2, 3, 9), each on nine distinct points, each with twelve blocks and all together have twelve distinct points as below:

$\underline{A^{t_1}}$	$\underline{A^{t_2}}$	$\underline{A^{t_3}}$	$\underline{A^{t_4}}$
4 7 10	1 7 10	1 4 10	1 4 7
4 9 11	1 8 11	1 5 11	1 5 8
4 8 12	1 9 12	1 6 12	1 6 9
5 9 10	2 9 10	2 5 10	2 6 7
5 8 11	2 7 11	2 6 11	2 4 8
5 7 12	2 8 12	2 4 12	2 5 9
6 8 10	3 8 10	3 6 10	3 5 7
6 7 11	3 9 11	3 4 11	3 6 8
6 9 12	3 7 12	3 5 12	3 4 9
4 5 6	1 2 3	1 2 3	1 2 3
7 8 9	7 8 9	4 5 6	4 5 6
10 11 12	10 11 12	10 11 12	7 8 9

where,  $A^{t_1}$ ,  $A^{t_2}$ ,  $A^{t_3}$  and  $A^{t_4}$  refers to the contracted systems S(2, 3, 9).

### **Construction Of The Blocks Of S(2, 4, 13)**

In this part, the blocks of S(2, 4, 13) are constructed from the twelve points of the four affine planes  $A^{t_1}, A^{t_2}, A^{t_3}$  and  $A^{t_4}$  plus a new point (13) say, which is already deleted in (Kakayee, 1990), i.e. the thirteen points required for the construction of our Steiner system S(2, 4, 13)

The deduction of a projective plane [S(2, 4, 13)] from an affine plane [S(2, 3, 9)] of the same order follows the traditional treatment (Singers theorem), if it is applied on

### References

- Dr. Barau, 1908, On the combinatory Problem of Steiner, Lon. Akad. Wetensch. Amst. Proc. Sect. Sei 11. 352 – 360.
- B. H. Gross, 1974 Intersection Triangles and Block Intersection Numbers of Steiner Systems, Math. Z. 139. 87 – 104.
- 3. D. R. Hughes and F.C. Piper, 1973, Projective Planes, Springer Verlag. New Yourk, Heidelbery. Berlin.
- 4. W. Jónsson, 1972, On the Mathieu Groups  $M_{22}$ ,  $M_{23}$ ,  $M_{24}$  and the Uniqueness of the Associated Steiner Systems, Math. Z. 125. 193–214.

any of the affine planes  $A^{t_1}$ ,  $A^{t_2}$ ,  $A^{t_3}$  or  $A^{t_4}$  using the suitable triads  $t_1$ ,  $t_2$ ,  $t_3$ , or  $t_4$  each with the new point (13), four isomorphic Steiner systems S(2, 4, 13) will be obtained, but not necessary to be the same system from which we begun from in (Kakayee, 1990), so, to avoid the isomorphic forms the procedure given below which depends on the intersection relation among all the blocks of  $A^{t_1}$ ,  $A^{t_2}$ ,  $A^{t_3}$  and  $A^{t_4}$  is to be used. Obviously, the blocks of these affine planes intersect in 0, 1, 2 or 3 points, since the blocks of S(5, 6, 12) intersect in 0, 2, 3 or 4 points (Mendelson, 1970) and (Gross, 1974), and the procedure consists of the following two parts:

i- There are (9) blocks formed from the union of all the blocks of  $A^{t_1}, A^{t_2}, A^{t_3}$  and  $A^{t_4}$  that intersect exactly in two points, they are:

1	4	7	10	2	5	9	10	3	6	8	10
3	4	9	11	1	5	8	11	2	6	7	11
2	4	8	12	3	5	7	12	1	6	9	12

ii-There are (4) blocks formed from the union of all blocks of  $A^{t_1}, A^{t_2}, A^{t_3}$  and  $A^{t_4}$  that intersect exactly in three points (the triads) each of which with the new point (13), they are:

4	5	6	13
7	8	9	13
10	11	12	13
1	2	3	13

The number of the blocks in (i) and (ii) is the total number of the blocks (lines) of S(2, 4, 13)[PG(2, 3)].

### Conclusion

In the view of the present work and the previous research it is clear that the two remarkable Steiner systems S (2, 4, 13) and S (5, 5, 12) may be constructed each from another

- E. R. Kakayee, 1990, On the Steiner System S (5, 6, 12), M. S. C. Al- Mustansiriyah University.
- 6. H. Lüneburg, 1968, Über die Gruppen von Mathieu. Jornal of Algebra 10. 194 – 210.
- C. Linder and A. Rosa, 1980, Topics on Steiner System, North Holland.
- N. S. Mendelsohn, 1970, A theorem on Steiner Systems, Canadian J, Math. 22. 1010 – 1015.
- 9. E. Witt, 1938, Über Steinersche Systeme, Abh. Math. Sem-Hamburg.

## الملخص:

. (9) طوابير نحصل عليها من اتحاد كل الطوابير  $A^{t_4}$   $A^{t_3}$   $A^{t_4}$  التي تشترك في نقطتان.

(4) طوابير نحصل عليها من اتحاد كل الطوابير  $A^{t_4}$   $A^{t_2}$   $A^{t_2}$   $A^{t_2}$   $A^{t_3}$  التي تشترك في ثلاثة نقاط (المجاميع الثلاثية) مضافاً لكلٍ منها النقطة الجديدة.