Wasit Journal for Pure Science

Journal Homepage: https://wjps.uowasit.edu.iq/index.php/wjps/index
e-ISSN: 2790-5241 p-ISSN: 2790-5233

On Graph T-topology

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DOI: https://doi.org/10.31185/wjps.567
Received 10 October 2024; Accepted 24 November 2024; Available online 30 December 2024

ABSTRACT: In this work, a new type of topology called T-topology is generated based on graph $G = (V(G), X(G))$, and specific types of graphs are discussed for this purpose. The way to generate this type of topology was the open neighborhood $N_\omega = \{\acute{\omega} \in (V(G): \omega\acute{\omega} \in X(G)\}$, which is a subbase of topology.

Keywords: t-set, graph topology, T-topological space.

1. INTRODUCTION

Topology has been related to several branches of mathematics including graph theory. The method of relate was through constructing a topological space based on a specific graph. Thus, it is a method for generating a topology from the elements of the graph. The matter may be the opposite, i.e. inducing a graph by a topological space. Among the first to present a method for generating a topological space by a graph were Diesto and Gervacio [1], where they used a topological rule based on the closed neighborhood of the set of vertices of the graph. Al-khafaji generated topology on graphs and topology on subgraph [2], after that KA Abdo introduced the concept of compatible and incompatible edge topologies [3]. On the other hand, the concepts of T-closed set and T-open set was introduced by Saad [4]. In this paper only finite undirected graphs are dealt with. Important definitions and preliminaries of topological space can be viewed through the source [5].

Definition 1.1[6]: A graph is a pair (V, X), where V is the set of vertices and X is the set of edges.

Definition 1.1[6]: A graph G in which contains a list of vertices ω_1,ω_2,…,ω_p of G such that for $1 \le i \le p-1$,there is an edge $\omega_i\omega_{i+1}$ in G is called path graph and denoted by P_n with n vertices.

Definition 1.2[6]: A graph G in which form by joining the two end-vertices of a path graph is called cycle graph and denoted by $C_n$with n vertices.

Definition 1.3[6]: A star graph is a graph in which $n-1$ vertices have degree 1 and a single vertex have degree $n-1$ and denoted by $S_n$with n vertices.

Definition 1.4[6]: A graph in which form by a cycle graph C_{n-1} with a star graph S_n by joining an edge from a center vertex of a star graph S_n to each vertex of $C_{n-1}$is called wheel graph and denoted by W_n with n vertices.

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Definition 1.5[6]: A complete graph is a graph G which have each pair of distinct vertices are adjacent, and denoted by K_n with n vertics.

Definition1 1.6[3]: Given a graph $G = (V(G), X(G))$ with $\omega \in V(G)$, the neighborhood set $N_\omega = \{\acute{\omega} \in V(G): \omega\acute{\omega} \in X(G)\}$.

Definition 1.7 [3]: Given a graph $G = (V(G), X(G))$, then a topology T_A which has a subbase open neighborhood $S_N = \{N_\omega: \omega \in V(G)\}$ is called graph adjacency topology.

Remark: A pair $(V(G), T_A)$ is called a graph adjacency topological space, denoted by GATS.

2. GRAPH T-TOPOLOGY (GATTS)

Definition 2.1: Let $(V(G), T_A)$ be GATS and H be a vertex form subgraph of G. Thus:

1- The closure of $V(H)$, defined by: $Cl(V(H)) = V(H) \cup \{\omega \in V(G): N_\omega \cap V(H) \neq \emptyset\}$.
2- The interior of $V(H)$, defined by: $Int(V(H)) = \{\omega \in V(G): N_\omega \subseteq V(H)\}$.

Definition 2.2: Let $(V(G), T_A)$ be GATS and H be a vertex induced subgraph of G. Then $V(H)$ is called t-set if $Int(V(H)) = Int(Cl(V(H)))$.

Example 2.3: Since, $S_N = \{\{\omega_2, \omega_3, \omega_5\}, \{\omega_1, \omega_3\}, \{\omega_1, \omega_2, \omega_4\}, \{\omega_3, \omega_5\}, \{\omega_1, \omega_4\}\}$, then a basis $\beta = \{\emptyset, \{\omega_2, \omega_3, \omega_5\}, \{\omega_1, \omega_3\}, \{\omega_1, \omega_2, \omega_4\}, \{\omega_3, \omega_5\}, \{\omega_1, \omega_4\}, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$
And $T_A = \{\emptyset, V(G), \{\omega_2, \omega_3, \omega_5\}, \{\omega_1, \omega_3\}, \{\omega_1, \omega_2, \omega_4\}, \{\omega_3, \omega_5\}, \{\omega_1, \omega_4\}, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}$ $, \{\omega_1, \omega_2, \omega_3, \omega_5\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_1, \omega_3, \omega_5\}, \{\omega_1, \omega_3, \omega_4\}, \{\omega_1, \omega_2, \omega_3\}, \{\omega_1, \omega_3, \omega_4, \omega_5\}$ $\{\{\omega_2, \omega_3\}\}, \{\omega_2, \omega_3, \omega_4\}, \{\{\omega_2, \omega_3, \omega_4, \omega_5\}\}, \{\omega_1, \omega_2, \omega_5\}\}$.
Take $V(H)=\{\omega_2, \omega_4, \omega_5\}$, then $Cl(V(H)) = V(H)$
So, $Int(V(H)) = Int\left(Cl(V(H))\right) = \{\omega_2\}$
Thus, $V(H)$ is t-set. See FIGURE .1

Remarks 2.4: Let $(V(G), T_A)$ be GATS with a vertex H which form subgraph of G. Then:

i- If $V(H)$ is a closed set, then it is t-set.
ii- $V(H)$ is called regular open set if $V(H) = Int(Cl(V(H)))$.
iii- If $V(H)$ is regular open set, then it is t-set.
iv- If $V(H)$ is a clopen set, then it is t-set.

Definition 2.5: A topological space $(V(G), T_A)$ is called GT_1-space if for any two different vertices from $V(G)$, there are two open sets in $T_A$each of which contains one but not the other.

Definition 2.6: A topological space $(V(G), T_A)$ is called definitely disconnected if every open set in it is clopen.

Definition 2.7: Let $(V(G), T_A)$ be GATS and H be a vertex form subgraph of G. $\omega \in V(G)$ is said to be T-cluster vertex of H if $U \cap H \neq \emptyset$, for any t-set U contain v.
The set of all T-cluster vertexes of H (denoted by $TCl(H)$) is called T-closure of H.

Definition 2.8: Let $(V(G), T_A)$ be GATS with a vertex H which form subgraph of G, then H is called T-closed set if $H = TCl(H)$.The complement of T-closed set is called T-open, a family of T-open sets denoted by $TO(V)$.

Definition 2.9: Let $(V(G), T_A)$ be GATS. Then a family $TO(V)$ forms a topology on $V(G)$ denoted by TT_A and the pair $(V(G), TT_A)$ is called a graph adjacency topological space, dented by GATTS.

Proposition 2.10: Let $(V(G), T_A)$ be GATS, then:

(i) Every discrete GATS is GATTS.
(ii) Every GT_1-space is GATTS.
(iii) Every definitely disconnected GATS is GATTS.

Proof: (i) and (iii) are obvious.

(ii) Let $(V(G), T_A)$ be GATS such that it is GT_1-space and $V(H) \subseteq V(G)$ is open set. Thus, For any $\omega \in V(H)$, $\{\omega\}$ is a closed set and so, $\{\omega\}$ is a T-open, but $V(H) = \bigcup_{v \in H}\{\omega\}$, hence $V(H)$ is T-open.

Proposition 2.11: Every complete graph $K_{n\geq 3}$ is GATTS.
Proof: Take n=3, $V(G) = \{\omega_1, \omega_2, \omega_3\}$, so $S_N = \{\{\omega_2, \omega_3\}, \{\omega_1, \omega_3\}, \{\omega_1, \omega_2\}\}$ and $\beta = \{\{\omega_2, \omega_3\}, \{\omega_1, \omega_3\}, \{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_1\}, \{\omega_2\}\}$.
Thus, T_A is a discrete topology. Hence, K_3 is GATTS by Proposition 2.10(i). See FIGURE .3
Take $n \geq 3$, $V(G) = \{\omega_1, \omega_2, \omega_3, \omega_4, \dots, \omega_n\}$, then $S_N = \{V(G)\backslash\{\omega_1\}, V(G)\backslash\{\omega_2\}, V(G)\backslash\{\omega_3\}, \dots, V(G)\backslash\{\omega_n\}\}$ and so, $\beta = \{V(G)\backslash\{\omega_1\}, V(G)\backslash\{\omega_2\}, V(G)\backslash\{\omega_3\}, \dots, V(G)\backslash\{\omega_n\}\} \cup \{V(G)\backslash\{\omega_1, \omega_2\}, V(G)\backslash\{\omega_1, \omega_3\}, \dots, V(G)\backslash\{\omega_1, \omega_n\}\} \cup V(G)\backslash\{\omega_2, \omega_3\}, \dots, V(G)\backslash\{\omega_2, \omega_n\}\}$.
For any $\omega_i, \omega_j \in V(G)$, there exist $\{V(G)\backslash\{\omega_i, \omega_k\}, V(G)\backslash\{\omega_j, \omega_k\} \in \beta$, for some $\omega_k \in V(G)$ such that $\omega_i \in V(G)\backslash\{\omega_j, \omega_k\}$ and $\omega_j \notin V(G)\backslash\{\omega_j, \omega_k\}$, also $\omega_j \in V(G)\backslash\{\omega_i, \omega_k\}$ and $\omega_i \notin V(G)\backslash\{\omega_i, \omega_k\}$.
Therefore, T_A is GT_1-space, hence $K_{n\geq 4}$ is GATTS from Proposition 2.10(ii). See FIGURE .2

Example 2.12: The complete graph K_4 is GATTS.
Since $S_N = \{\{\omega_3, \omega_2, \omega_4\}, \{\omega_4, \omega_1, \omega_3\}, \{\omega_1, \omega_2, \omega_4\}, \{\omega_2, \omega_3, \omega_1\}\}$, then a basis $\beta = \{\{\omega_3, \omega_2, \omega_4\}, \{\omega_4, \omega_1, \omega_3\}, \{\omega_1, \omega_2, \omega_4\}, \{\omega_2, \omega_3, \omega_1\}, \{\omega_3, \omega_4\}, \{\omega_2, \omega_4\}, \{\omega_2, \omega_3\}, \{\omega_1, \omega_4\}, \{\omega_1, \omega_3\}, \{\omega_1, \omega_2\}\}$
Note that the topological space generated by basis β will be GT_1-space, from Proposition 2.10(ii).
K_4 is GATTS.

Remark 2.13: The complete graph K_3 is cycle graph.

Proposition 2.14: Every cycle graph C_n is GATTS.
Proof: If n=3 or n=4, then C_n is GATTS, from Example 2.12 and Example 2.14.
If n ≥ 5, then $S_N = \{\{\omega_2, \omega_n\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}, \{\omega_3, \omega_5\}, \dots \{\omega_1, \omega_{n-1}\}\}$, so $\beta = \{\emptyset, \{\omega_2, \omega_n\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}, \{\omega_3, \omega_5\}, \dots \{\omega_1, \omega_{n-1}\}, \{\omega_2\}, \{\omega_3\}, \dots, \{\omega_{n-1}\}, \{\omega_n\}, \{\omega_1\}\}$
Now, we have basis that includes all singleton set of a space. Thus, T_A will be a discrete topology. Hence, C_n is GATTS. Look at FIGURE .4

Example 2.15: C_4 is GATTS.
$S_N = \{\{\omega_2, \omega_4\}, \{\omega_1, \omega_3\}\}$.
$\beta = \{\emptyset, \{\omega_2, \omega_4\}, \{\omega_1, \omega_3\}\}$
$T_A = \{\emptyset, V(G), \{\omega_2, \omega_4\}, \{\omega_1, \omega_3\}\}$
Clear that every open in T_A is T-open set, so C_4 is GATTS. Look at FIGURE .5

Proposition 2.16: Every star graph S_n is GATTS.
Proof: $S_N = \{\{\omega_2, \omega_3, \omega_4, \dots, \omega_n\}, \{\omega_1\}\}$.
$\beta = \{\emptyset, \{\omega_2, \omega_3, \omega_4, \dots, \omega_n\}, \{\omega_1\}\}$.
$T_A = \{\emptyset, V(G), \{\omega_2, \omega_3, \omega_4, \dots, \omega_n\}, \{\omega_1\}\}$
Clear, $(V(G), T_A)$ is definitely disconnected space, so by Proposition 2.10(iii).
S_n is GATTS. Look at FIGURE .6

Example 2.17: A star S_5 is GATTS.
We have, $S_N = \{\{\omega_2, \omega_3, \omega_4, \omega_5\}, \{\omega_1\}\}$. Then a basis $\beta = \{\emptyset, \{\omega_2, \omega_3, \omega_4, \omega_5\}, \{\omega_1\}\}$ and $T_A = \{\emptyset, V(G), \{\omega_2, \omega_3, \omega_4, \omega_5\}, \{\omega_1\}\}$
Clear that every open in T_A is T-open set, so S_5 is GATTS. Look at FIGURE .7

Proposition 2.18: Every weal graph W_n is GATTS.
Proof: Let $V(G) = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_4, \dots, \omega_{n-2}, \omega_{n-1}, \omega_n\}$ and so, $S_N = \{\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_4, \dots, \omega_{n-2}, \omega_{n-1}\}, \{\omega_n, \omega_{n-1}, \omega_2\}, \{\omega_n, \omega_1, \omega_3\}, \{\omega_n, \omega_2, \omega_4\}, \{\omega_n, \omega_3, \omega_5\}, \dots, \{\omega_n, \omega_1, \omega_{n-2}\}\}$ then we have $\beta = \{\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_4, \dots, \omega_{n-2}, \omega_{n-1}\}, \{\omega_n, \omega_{n-1}, \omega_2\}, \{\omega_n, \omega_1, \omega_3\}, \{\omega_n, \omega_2, \omega_4\}, \{\omega_n, \omega_3, \omega_5\}, \dots, \{\omega_n, \omega_1, \omega_{n-2}\}, \{\omega_{n-1}, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}, \{\omega_3, \omega_5\}, \dots, \{\omega_1, \omega_{n-2}\}, \{\omega_n\}, \{\omega_n, \omega_2\}, \{\omega_n, \omega_3\}, \{\omega_n, \omega_4\}, \dots, \{\omega_n, \omega_{n-2}\}\}$.
It is clear from the base that $(V(G), T_A)$ is GT_1-space, so by Proposition 2.10
W_n is GATTS. Look at FIGURE .8

Example 2.19: A weal W_5 is GATTS.
Since, $S_N = \{\{\omega_1, \omega_3, \omega_5\}, \{\omega_2, \omega_3, \omega_4, \omega_5\}, \{\omega_1, \omega_2, \omega_4\}$, then we have $\beta = \{\{\omega_1\}, \{\omega_2, \omega_4\}, \{\omega_3, \omega_5\}, \{\omega_1, \omega_2, \omega_4\}, \{\omega_2, \omega_3, \omega_4, \omega_5\}, \{\omega_1, \omega_3, \omega_5\}\}$, and $T_A = \{\emptyset, V(G), \{\omega_2, \omega_3, \omega_4, \omega_5\}, \{\omega_1, \omega_3, \omega_5\}, \{\omega_1\}, \{\omega_2, \omega_4\}, \{\omega_3, \omega_5\}, \{\omega_1, \omega_2, \omega_4\}\}$
Clear, $(V(G), T_A)$ is definitely disconnected space, so by Proposition 2.10(iii).
W_5 is GATTS. Look at FIGURE .9

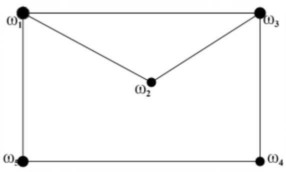
FIGURE .1

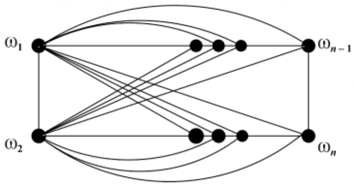
FIGURE .2

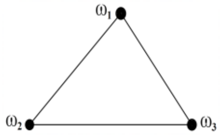
FIGURE .3

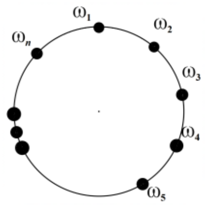
FIGURE .4

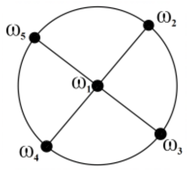
FIGURE .9

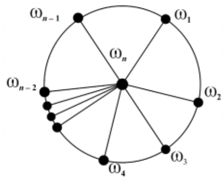
FIGURE .8

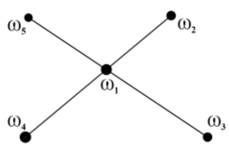
FIGURE .7

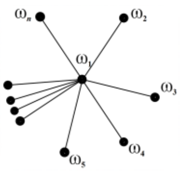
FIGURE .6

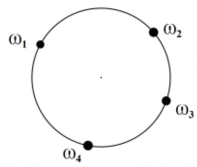
FIGURE .5

Definition 2.20: Let $(V(G), T_A)$ be GATS and $(\acute{V}(H), \acute{E}(H))$ be a subgraph of $(V(G), E(G))$. Then the class $\beta_H = \{\acute{V}(H) \cap B : B \in \beta, \beta$ is basis of $T_A\}$ is called a basis of a relative topology (denoted by $\acute{T}_A$) on $\acute{V}(H)$.

Proposition 2.21: Let $(V(G), T_A)$ be GATS such that any element of subbase of T_A is T-open and $(\acute{V}(H), \acute{E}(H))$ is a subgraph of $(V(G), E(G))$. If $\acute{V}(H)$ is a closed set in $V(G)$, then a pair of a relative topology $\acute{T}_A$ and $\acute{V}(H)$ is GATTS.
Proof: Since $\acute{V}(H)$ is a closed set in $V(G)$, then $\acute{V}(H)$ is T-open set (Look at FIGURE .10 and FIGURE .11). We have the elements of a basis β are T-open sets, and so, the elements of a basis β_H are T-open sets. But, every element in $\acute{T}_A$ is a union of elements of β_H. Hence, $(\acute{V}(H), \acute{T}_A)$ is GATTS.

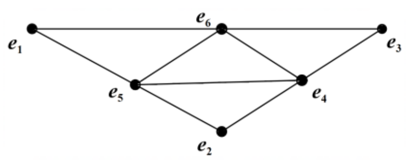

FIGURE .10

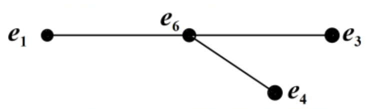

FIGURE .12

Now, $S_N = \{\{\omega_6, \omega_5\}, \{\omega_4, \omega_5\}, \{\omega_6, \omega_4\}, \{\omega_6, \omega_2, \omega_3, \omega_5\}, \{\omega_4, \omega_6, \omega_2, \omega_1\}, \{\omega_4, \omega_1, \omega_3, \omega_5\}\}$, thus
$\beta = \{\{\omega_6, \omega_5\}, \{\omega_4, \omega_5\}, \{\omega_6, \omega_4\}, \{\omega_6, \omega_2, \omega_3, \omega_5\}, \{\omega_4, \omega_6, \omega_2, \omega_1\}, \{\omega_4, \omega_1, \omega_3, \omega_5\}, \{\omega_5\}, \{\omega_6\}, \{\omega_4\}, \{\omega_6, \omega_2\}, \{\omega_3, \omega_5\}, \{\omega_1, \omega_4\}\}$.
$T_A = \{\emptyset, V(G), \{\omega_6, \omega_5\}, \{\omega_4, \omega_5\}, \{\omega_6, \omega_4\}, \{\omega_6, \omega_2\}, \{\omega_3, \omega_5\}, \{\omega_1, \omega_4\}, , \{\omega_5\}, \{\omega_6\}, \{\omega_4\}, , ,,,$
$\{\omega_6, \omega_3, \omega_5\}, \{\omega_6, \omega_4, \omega_5\}, \{\omega_1, \omega_5, \omega_4\}, \{\omega_4, \omega_5, \omega_1\}, \{\omega_4, \omega_2, \omega_6\}, \{\omega_4, \omega_6, \omega_1\},$
$\{\omega_4, \omega_5, \omega_6\}, \{\omega_2, \omega_5, \omega_6\}, \{\omega_3, \omega_5, \omega_6\}, \{\omega_4, \omega_5, \omega_3\}, \{\omega_4, \omega_5, \omega_3\}, \{\omega_6, \omega_2, \omega_3, \omega_5\},$
$\{\omega_4, \omega_6, \omega_2, \omega_1\}, \{\omega_4, \omega_1, \omega_3, \omega_5\}, , \{\omega_4, \omega_6, \omega_5, \omega_2\}, \{\omega_4, \omega_6, \omega_5, \omega_3\},$
$\{\omega_4, \omega_2, \omega_6, \omega_3, \omega_5\}, \{\omega_4, \omega_2, \omega_6, \omega_1, \omega_5\}, \{\omega_4, \omega_1, \omega_6, \omega_3, \omega_5\}, , , \}$
Let $\acute{V}(H) = \{\omega_4, \omega_3, \omega_6, \omega_1\}$

$\beta_H = \{\{\omega_6\}, \{\omega_3\}, \{\omega_4\}, \{\omega_6, \omega_4\}, \{\omega_6, \omega_3\}, \{\omega_3, \omega_4\}, \{\omega_1, \omega_4\}\}$ $\acute{T}_A =$
$\{\emptyset, \acute{V}(H), \{\omega_6\}, \{\omega_3\}, \{\omega_4\}, \{\omega_6, \omega_4\}, \{\omega_6, \omega_3\}, \{\omega_3, \omega_4\}, \{\omega_1, \omega_4\}, \{\omega_6, \omega_4, \omega_3\}, \{\omega_6, \omega_4, \omega_1\}, \{\omega_3, \omega_4, \omega_1\}\}$
Clear that $\acute{T}_A$ is GT_1-space and so, $(\acute{V}(H), \acute{T}_A)$ is GATTS.

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