University Of AL-Qadisiyah

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

3rd-Order Differential Subordination and Superordination Results for Univalent Analytic Functions Associated with Zeta-Riemann Fractional Differential Operator

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ARTICLE INFO

Article history:
Received: 19 /07/2024
Rrevised form: 29 /07/2024
Accepted : 04 /08/2024
Available online: 30 /09/2024

Keywords:

Differential subordination, Analytic function, Fractional operators, Sandwich theorem, Third-order.

Each keyword to start on a new line

ABSTRACT

Using the results of 3rd order differential subordination, we introduce certain families of admissible functions and discuss some applications of 3rd order differential subordination for normalized analytic functions associated with novel fractional operator namely Zeta-Riemann Fractional differential operator. Some new results on differential subordination and superordination with some sandwich theorems are obtained. Moreover, several particular cases are also noted.

MSC..

https://doi.org/ 10.29304/jqcsm.2024.16.31649

1-Introduction

Let $H(U)$ be a class of analytic functions in the open unit disk $U = \{z: z \in \mathbb{C} \text{ and } |z| < 1\}$. For $n \in N = \{1,2,3,\cdots\}$ and $a \in \mathbb{C}$, let $H[a,n]$ be the subclass of $H(U)$ defined by:

$$H[a,n] = \left\{f: f \in H(U) \text{ and } f(z) = a + \sum_{k=n}^{\infty} a_k z^k \right\}.$$

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Furthermore, suppose that $H_0 = H[0,1]$. Let $A \subset H(U)$ be the class of functions in U which are analytic and have normalized Taylor-Maclaurin series of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \tag{1.1}$$

Let f and g are functions in H(U). We state that f is subordinate to g, (or g is superordinate to f), written

$$f \prec g \ in\ U \text{ or } f(z) \prec g(z), (z \in \mathbb{C}),$$

if there is a Schwarz function $h \in H$, in U such that is analytic with $h(0) = 0$ and $|h(z)| < 1$ $(z \in \mathbb{C})$, such that $f(z) = g(h(z)), (z \in \mathbb{C})$. Moreover, if the function g is univalent in U, then, we have the following equivalence: (see[1, 18, 19]).

$$f(z) \prec g(z) \leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Tayyah and Atshan [26] introduced the following fractional differential operator

$$D^{\alpha}_{(s,k,\omega)} f(z) = \sum_{n=1}^{\infty} \frac{(\omega+1)^{\frac{s(\alpha-1)+1}{k}} \ \Gamma\left(\frac{n}{\omega+1}+1\right)}{k^{-\frac{s(\alpha-1)+1}{k}} \ \Gamma\left(\frac{n}{\omega+1} - \frac{s(\alpha-1)+1}{k}+1\right)} a_n \, z^{(\omega+1)\left(1-\frac{s(\alpha-1)+1}{k}\right)+n-1}, \tag{1.2}$$

$$s \in \mathbb{N}, k > 0, \omega \geq 0, \quad 0 \leq \frac{r(\delta-1)+1}{k} < 1.$$

Bernardi [5] defined the following Hurwitz-Lerch Zeta function:

$$\Phi(z, r, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^r}, \tag{1.3}$$

$r \in \mathbb{C}, a \in \mathbb{C} \backslash \mathbb{Z}_0^-, z \in U, Re(s) > 1$,when $|z| = 1$.

We define the new Hadamard product fractional differential operator.

$$\mathbb{D}_s^{(\alpha,\beta)} f(z) = \left[D^{\alpha}_{(s,1,0)} f(z)\right] * [z^{-(s(\alpha-1)+1)}(\Phi(z, s-1, \alpha+\beta) - \alpha^{-s}]$$

$$= \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n - s(\alpha-1)) \ (n+\alpha+\beta)^{s-1}} a_n z^{n-[s(\alpha-1)+1]}, \tag{1.4}$$

and used to find new results of 3rd-order differential subordination and superordination for univalent analytic functions.

We note that if $s = 1$, then we have Srivastava fractional differential operator in [6] as:

$$\mathbb{D}_1^{(\alpha,\beta)} f(z) = \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n-\alpha+1)} a_n z^{n-\alpha}, \tag{1.5}$$

Example 1. Let $f(z) = z + z^3$, Then by (4) and (5) (see Fig. 1), we have

$\mathbb{D}_1^{\left(\frac{1}{2},0\right)} f(z) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} z^{\frac{1}{2}} + \frac{6}{\Gamma\left(\frac{7}{2}\right)} z^{\frac{5}{2}}$

$\mathbb{D}_2^{\left(\frac{1}{2},0\right)} f(z) = \frac{2}{3} z + \frac{6}{7} z^3 .$

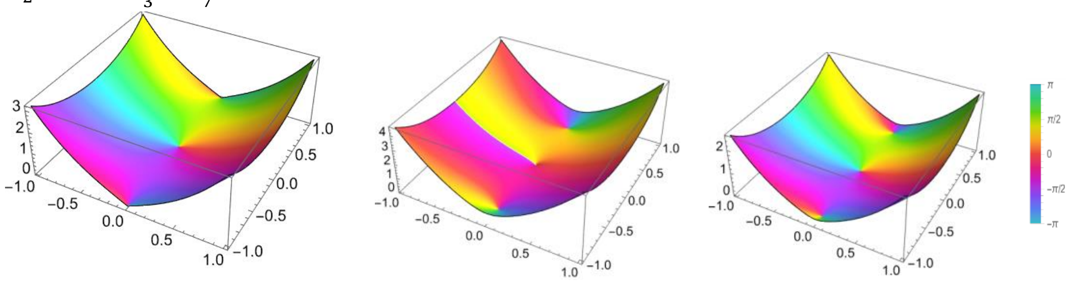

Fig. 1: Complex plot of $f(z)$, $\mathbb{D}_1^{\left(\frac{1}{2},0\right)} f(z)$, $\mathbb{D}_2^{\left(\frac{1}{2},0\right)} f(z)$.

Now, we define a new operator (Zeta-Riemann Fractional differential operator) $\mathfrak{D}_s^{(\alpha,\beta)}: A \longrightarrow A$ by

$$\mathfrak{D}_s^{(\alpha,\beta)} f(z) = (1+\alpha+\beta)^{s-1} \Gamma\big(1-s(\alpha-1)\big)\, z^{s(\alpha-1)+1} \mathbb{D}_s^{(\alpha,\beta)} f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{\Gamma\big(1-s(\alpha-1)\big)\ (1+\alpha+\beta)^{s-1} n!}{\Gamma\big(n-s(\alpha-1)\big)\ (n+\alpha+\beta)^{s-1}} a_n z^n. \tag{1.6}$$

Lemma 1. Let $f \in A$. Then

$$z\left(\mathfrak{D}_s^{(\alpha,\beta)} f(z)\right)' = s(1-\alpha)\mathfrak{D}_s^{\left(\alpha+\frac{1}{s},\beta-\frac{1}{s}\right)} f(z) - [s(1-\alpha)-1]\mathfrak{D}_s^{(\alpha,\beta)} f(z) \tag{1.7}$$

Proof.

$$z\left(\mathfrak{D}_s^{(\alpha,\beta)} f(z)\right)' = z + \sum_{n=2}^{\infty} \frac{\Gamma\big(1-s(\alpha-1)\big)\ (1+\alpha+\beta)^{s-1} n!}{\Gamma\big(n-s(\alpha-1)\big)\ (n+\alpha+\beta)^{s-1}}\, n\, a_n z^n$$

$$= z + \sum_{n=2}^{\infty} \frac{(s-s\alpha)\Gamma\left(1-s\left(\alpha+\frac{1}{s}\right)+s\right)\ (1+\alpha+\beta)^{s-1} n!}{\Gamma\left(n-s\left(\alpha+\frac{1}{s}\right)+s\right)\ (n+\alpha+\beta)^{s-1}} a_n z^n$$

$$+ \sum_{n=2}^{\infty} \frac{(s\alpha-s+1)\Gamma(1-s\alpha+s)\ (1+\alpha+\beta)^{s-1} n!}{\Gamma(n-s\alpha+s)\ (n+\alpha+\beta)^{s-1}} a_n z^n$$

$$= s(1-\alpha)\mathfrak{D}_s^{\left(\alpha+\frac{1}{s},\beta-\frac{1}{s}\right)} f(z) - [s(1-\alpha)-1]\mathfrak{D}_s^{(\alpha,\beta)} f(z).$$

Antonino and Miller [1] (also [27, 28]) have expanded the concept of second-order differential subordination and superordination in U established by Miller and Mocanu [16,18,19] to the third-order case. They derived features of functions p that fulfill the third-order differential subordination:

$$\{\Phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z): z \in U\} \subset \Omega,$$

and also for third-order differential superordination:

$$\Omega \subset \{\Phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z): z \in U\},$$

where Ω is a set in $\mathbb{C}$, p is an analytic function and $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$.

Recently, several authors studied some applications on the concept of second-order differential subordination and superordination and established some sandwich outcomes, like, (see [2,6,15,25]) and third-order for different classes (see [3,27,28]). For some interesting applications related to the differential subordination and superordination in other subjects of mathematics, we may refer to [4,5,6].

The concept of third-order differential subordination was introduced in Ponnusamy and Juneja's work, [27]. Tang et al. introduced recent's study, it is a good example of this (see [27,28]).

The second and third-order terms are used interchangeably. Uneven subordination piqued the interest of many academics in this field. ([6,7,8,9,10,12,13,14,15,17,21,24,29]).

We examine a suitable set of admissible functions associated with the integral operator and construct adequate conditions on the normalized analytic function, known as the sandwich condition, in this paper.

2- Preliminaries

We need the following definitions and lemmas to prove our results.

Definition (2.1) [1]. Let $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$ and suppose that the function h(z) is univalent in U. If the function p(z) is analytic in U and satisfies the following third-order differential subordination:

$$z\left(\mathfrak{D}_s^{(\alpha,\beta)} f(z)\right)' = s(1-\alpha)\mathfrak{D}_s^{\left(\alpha+\frac{1}{s},\beta-\frac{1}{s}\right)} f(z) - [s(1-\alpha)-1]\mathfrak{D}_s^{(\alpha,\beta)} f(z) \tag{2.1}$$

then p(z) is called a solution of the differential subordination (2.1). Furthermore, a given univalent function q(z) is said to as a dominant of the solutions of (2.1), or, more simply, a dominant if $p(z) \prec q(z)$ for all p(z) satisfying (2.1). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants q(z) of (2.1) is said to be the best dominant.

Definition (2.2) [1]. Let $\mathbb{Q}$ be the set of all univalent and analytic functions q on $\overline{U} \backslash E(q)$, where

$$E(q) = \{\xi: \xi \in \partial U : lim_{z\to\xi}\, q(z) = \infty\},$$

and $\min|q'(\xi)| = \rho > 0$ for $\xi \in \partial U \backslash E(q)$. Further, let the subclass of $\mathbb{Q}$ for which $q(0) = a$, be denoted by $\mathbb{Q}(a)$, with $\mathbb{Q}(0) = \mathbb{Q}_0$ and $\mathbb{Q}(1) = \mathbb{Q}_1$.

The method of subordination is applied to an appropriate classes of admissible functions.

The following class of admissible functions was given by Antonino and Miller [1].

Definition (2.3) [1]. Let Ω be a set in $\mathbb{C}$ and $q \in \mathbb{Q}$ and $n \in N\backslash\{1\}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$, which satisfy the following admissibility conditions:

$$\Phi(r, s, t, u; z) \notin \Omega$$

whenever

$$r = q(\xi)\,, \qquad s = k\xi q'(\xi), \qquad Re\left(\frac{t}{s}+1\right) \geq kRe\left(\frac{\xi q''(\xi)}{q'(\xi)}+1\right),$$

and

$$Re\left(\frac{u}{s}\right) \geq k^2 Re\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U, \xi \in \partial U \backslash E(q)$, and $k \geq n$.

Lemma (2.1) [1]. Let $p \in H[a, n]$ with $n \geq 2$, and $q \in \mathbb{Q}(a)$ satisfying the following conditions:

$$Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0 \quad, and \quad \left|\frac{zp'(z)}{q'(\xi)}\right| \leq k,$$

where $z \in U, \xi \in \partial U \setminus E(q)$, and $k \geq n$. If Ω is a set in $\mathbb{C}$, $\Phi \in \Psi_n[\Omega, q]$, and

$$\Phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega,$$

then

$$p(z) \prec q(z), \qquad (z \in U).$$

Definition (2.4) [27]. Let $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$ and assume the function $h(z)$ is analytic in U. If the function $p(z)$ and

$$\Phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z),$$

are univalent in U and satisfies the following third-order differential superordination:

$$h(z) \prec \Phi\left(p(z), zp'^{(z)}, z^2p''^{(z)}, z^3p'''^{(z)}; z\right), \quad (2.2)$$

then $p(z)$ is said to be a solution of the differential superordination. Further an analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q(z) \prec p(z)$ for all $p(z)$ satisfying (2.2). A univalent subordnant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (2.2) is said to be the best subordinant.

Definition (2.5) [27]. Let Ω be a set in $\mathbb{C}$, $q \in H[a, n]$ and $q'(z) \neq 0$. The admissible function class $\Psi'_n[\Omega, q]$ consists of those functions $\Phi : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ that fulfills the following admissibility conditions:

$$\Phi(r, s, t, u; \xi) \in \Omega$$

whenever

$$r = q(z) \quad , \quad s = \frac{zq'(z)}{m} \quad , \quad \mathrm{Re}\left(\frac{t}{s} + 1\right) \leq \frac{1}{m}\mathrm{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$\mathrm{Re}\left(\frac{u}{s}\right) \leq \frac{1}{m^2}\mathrm{Re}\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in U, \xi \in \partial U$, and $m \geq n \geq 2$.

Lemma (2.2) [27]. Let $q \in H[a, n]$ with $\Phi \in \Psi'_n[\Omega, q]$. If

$$\Phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$$

is univalent in U and $p \in \mathbb{Q}(a)$ satisfying the following conditions:

$$\mathrm{Re}\left(\frac{\xi q''(z)}{q'(z)}\right) \geq 0, \qquad \left|\frac{zp'(z)}{q'(z)}\right| \leq m,$$

where $z \in U, \xi \in \partial U$, and $m \geq n \geq 2$, then

$$\Omega \subset \{\Phi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z): z \in U\},$$

indicates that

$$q(z) \prec p(z), \qquad (z \in U).$$

The current paper utilizes the techniques on the third-order differential subordination and superordination outcomes of Antonino and Miller [1], Jeyaraman and Suresh [14] and Tang et al. [26,28], respectively and different conditions (see [9,13,21]). Certain classes of admissible functions are investigated in this current paper, some new results of the third-order differential subordination and superordination for analytic functions in U related to the operator $\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)$ are also mentioned.

3- Results on Third-Order Differential Subordination

Here, we introduce some differential subordination results using the operator, $\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)$.

Definition (3.1). Let Ω be a set in $\mathbb{C}$ and $q \in \mathbb{Q}_0 \cap H_0$. The admissible function class $\mathfrak{I}_j[\Omega, q]$ consists of those functions $\Phi : \mathbb{C}^4 \times U \to \mathbb{C}$ that fulfill the admissibility condition:

$$\Phi(a, b, c, d; z) \notin \Omega,$$

whenever

$$a = q(\xi), \qquad b = \frac{\xi k q'(\xi) + [\delta(1-\alpha) - 1]q(\xi)}{\delta(1-\alpha)},$$

$$\mathrm{Re}\left(\frac{\delta(1-\alpha)[\delta(1-\alpha)c - 2[\delta(1-\alpha)-1]b] + [\delta(1-\alpha)-1]^2 a}{\delta(1-\alpha)b - [\delta(1-\alpha)-1]a}\right) \geq k\,\mathrm{Re}\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right),$$

and

$$\mathrm{Re}\left(\frac{\delta(1-\alpha)^3[d - 3(c-b)] - \delta(1-\alpha)[3[\delta(1-\alpha)-1]a + b] + \left[[\delta(1-\alpha)-1](1-[\delta(1-\alpha)-1]^2)a\right]}{\delta(1-\alpha)b - [\delta(1-\alpha)-1]a}\right) \geq k^2\,\mathrm{Re}\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U, \xi \in \partial U \backslash E(q)$ and $k \in \mathbb{N}\backslash\{1\}$.

Theorem (3.1). Let $\Phi \in \mathfrak{I}_j[\Omega, q]$. If the functions $f \in A$ and $q \in \mathbb{Q}_0 \cap H_0$ satisfies the condition:

$$Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0 \quad , \quad \left|\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta}, \beta-\frac{1}{\delta}\right)} f(z)}{q'(\xi)}\right| \leq k, \tag{3.1}$$

And

$$\left\{\Phi\left(\mathfrak{D}_\delta^{(\alpha,\beta)}f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta}, \beta-\frac{1}{\delta}\right)}f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta}, \beta-\frac{2}{\delta}\right)}f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta}, \beta-\frac{3}{\delta}\right)}f(z); z\right) : z \in U\right\} \subset \Omega \tag{3.2}$$

then

$$\mathfrak{D}_\delta^{(\alpha,\beta)}f(z) \prec q(z), \qquad (z \in U).$$

Proof. Let $p(z)$ be analytic function in U by

$$p(z) = \mathfrak{D}_\delta^{(\alpha,\beta)}f(z). \tag{3.3}$$

From equation (1.7) and differentiating (3.3) with respect to z, we get

$$\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)=\frac{zp'(z)+[\delta(1-\alpha)-1]p(z)}{\delta(1-\alpha)}. \tag{3.4}$$

By a similar argument, yields

$$\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)=\frac{z^2p''(z)+[2\delta(1-\alpha)-1]zp'(z)+[\delta(1-\alpha)-1]^2p(z)}{[\delta(1-\alpha)]^2}, \tag{3.5}$$

and

$$\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)=\frac{z^3p'''(z)+3\delta(1-\alpha)z^2p''(z)+[3[\delta(1-\alpha)]^2-3\delta(1-\alpha)+1]zp'(z)+[\delta(1-\alpha)-1]^3p(z)}{[\delta(1-\alpha)]^3}. \tag{3.6}$$

Define the transformation starting with $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(r,s,t,u)=r, \qquad b(r,s,t,u)=\frac{s+[\delta(1-\alpha)-1]r}{\delta(1-\alpha)},$$

$$c(r,s,t,u)=\frac{t+[2\delta(1-\alpha)-1]s+[\delta(1-\alpha)-1]^2r}{[\delta(1-\alpha)]^2}, \tag{3.7}$$

and

$$d(r,s,t,u)=\frac{u+3\delta(1-\alpha)t+[3[\delta(1-\alpha)]^2-3\delta(1-\alpha)+1]s+[\delta(1-\alpha)-1]^3r}{[\delta(1-\alpha)]^3}. \tag{3.8}$$

Let $\varphi(r,s,t,u)=\Phi(a,b,c,d)=\Phi\left(\begin{matrix} r,\frac{s+[\delta(1-\alpha)-1]r}{\delta(1-\alpha)},\frac{t+[2\delta(1-\alpha)-1]s+[\delta(1-\alpha)-1]^2r}{[\delta(1-\alpha)]^2}, \\ \frac{u+3\delta(1-\alpha)t+[3[\delta(1-\alpha)]^2-3\delta(1-\alpha)+1]s+[\delta(1-\alpha)-1]^3r}{[\delta(1-\alpha)]^3};z\end{matrix}\right)$. (3.9)

The proof will be put to use by Lemma (2.1).Using equations (3.3) to (3.6), and from (3.9), we get

$$\varphi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)=\Phi\left(\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z);z\right) \tag{3.10}$$

Hence, (3.2) leads to

$$\varphi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)\in\Omega,$$

note that

$$\frac{t}{s}+1=\frac{\delta(1-\alpha)(\delta(1-\alpha)c-2[\delta(1-\alpha)-1]b)+[\delta(1-\alpha)-1]^2a}{\delta(1-\alpha)b-[\delta(1-\alpha)-1]a},$$

and

$$\frac{u}{s}=\frac{(\delta(1-\alpha))^3[d-3(c-b)]-\delta(1-\alpha)[3[\delta(1-\alpha)-1]a+b]+\left[[\delta(1-\alpha)-1](1-[\delta(1-\alpha)-1]^2)a\right]}{\delta(1-\alpha)b-[\delta(1-\alpha)-1]a}.$$

As a result, the admissibility condition in Definition (3.1) for $\Phi\in\mathfrak{I}_j[\Omega,q]$ is equivalent to the condition $\varphi\in\Psi_2[\Omega,q]$ as stated in Definition (2.3) with $n=2$. As a result, using (3.1) and Lemma (2.1), we have

$$\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)\prec q(z).$$

The proof of the Theorem (3.1) is complete.

The following outcome is an extension of Theorem (3.1) to the case where the actions of $q(z)$ on ∂U is unknown.

Corollary (3.1). Let $\Omega \subset \mathbb{C}$ and the function q be univalent in U with $q(0) = 1$. Let $\Phi \in \mathfrak{I}_j[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in A$ and $q_\rho \in \mathbb{Q}_0$ satisfies the next conditions:

$$\mathrm{Re}\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geq 0\,, \left|\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{q_\rho'(\xi)}\right| \leq k\,, (z \in U, \xi \in \partial U \backslash E(q_\rho) \text{ and } k \geq 2.)$$

and

$$\Phi\left(\mathfrak{D}_\delta^{(\alpha,\beta)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)} f(z); z\right) \in \Omega,$$

then

$$\mathfrak{D}_\delta^{(\alpha,\beta)} f(z) \prec q_\rho(z), (z \in U).$$

Proof. Using the Theorem (3.1), we obtain

$$\mathfrak{D}_\delta^{(\alpha,\beta)} f(z) \prec q_\rho(z), (z \in U).$$

Corollary asserts the following conclusion (3.1) is now deduced from the subordination characteristic that follows: $q_\rho(z) \prec q(z)$, $(z \in U)$.

If $\Omega \neq \mathbb{C}$ is a domain with only one connection, then $\Omega = h(U)$ for the purpose of conformal mapping $h(z)$ of U onto Ω. The class in this situation is $\mathfrak{I}_j[h(U), q]$ is written as $\mathfrak{I}_j\,[h, q]$.

This is a direct result of the Theorem.(3.1) and Corollary (3.1).

Theorem (3.2). Let $\Phi \in \mathfrak{I}_j[h, q]$. If the function $f \in A$ and $q \in \mathbb{Q}_0 \cap H_0$ satisfy the following conditions:

$$Re\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geq 0\,, \left|\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{q_\rho'(\xi)}\right| \leq k, \quad (3.11)$$

and

$$\Phi\left(\mathfrak{D}_\delta^{(\alpha,\beta)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)} f(z); z\right) \prec h(z). \quad (3.12)$$

then

$$\mathfrak{D}_\delta^{(\alpha,\beta)} f(z) \prec q(z)\,, (z \in U).$$

The following result is a direct result of Corollary. (3.1).

Corollary (3.2). Let $\Omega \subset \mathbb{C}$ and q be univalent in U with $q(0) = 1$. Let $\Phi \in \mathfrak{I}_j[\Omega, q]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in A$ and q_ρ satisfies the following conditions:

$$Re\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geq 0\,, \quad \left|\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{q_\rho'(\xi)}\right| \leq k\,, (z \in U, \xi \in \partial U \backslash E(q_\rho) \text{ and } k \geq 2)$$

and

$$\Phi\left(\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z);z\right)\prec h(z),$$

then

$$\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)\prec q(z),\qquad (z\in U).$$

The best dominant of the differential subordination is seen in the following result (3.12).

Theorem (3.3). Let the function h in U, be univalent, and let $\Phi:\mathbb{C}^4\times U\to\mathbb{C}$ and φ be given by (3.10). Assume the equation of differentiation:

$$\varphi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)=h(z) \tag{3.13}$$

has a solution q(z) with $q(0)=1$, which satisfy condition (3.1). If $f\in A$ satisfies the requirement (3.12) and if

$$\Phi\left(\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z);z\right),$$

is analytic in U, then

$$\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)\prec q(z),\qquad (z\in U)$$

and q(z) is the best dominant.

Proof. From Theorem (3.1), we have q is a dominant of (3.12). Since q satisfies (3.13), it is also a solution of (3.12). Therefore, q will be dominated by all dominants. Hence q is the best dominant. The theorem's proof is complete.

In light of the Definition (3.1), and in the special case $q(z)=Mz$ ($M>0$), the class of admissible functions $\mathfrak{I}_j[\Omega,q]$, denoted by $\mathfrak{I}_j[\Omega,M]$, expresses itself as follows.

Definition (3.2). Let Ω be set in $\mathbb{C}$ and $M>0$. The class of admissible functions $\mathfrak{I}_j[\Omega,M]$ includes those functions $\Phi:\mathbb{C}^4\times U\to\mathbb{C}$ such that

$$\Phi\left(\begin{array}{c}Me^{i\theta},\frac{(k+[\delta(1-\alpha)-1])Me^{i\theta}}{\delta(1-\alpha)},\frac{L+[[2\delta(1-\alpha)-1]k+[\delta(1-\alpha)-1]^2]Me^{i\theta}}{(\delta(1-\alpha))^2},\\ \frac{N+3\delta(1-\alpha)L+[\left(3(\delta(1-\alpha))^2-\delta(1-\alpha)+1\right)k+[\delta(1-\alpha)-1]^3]Me^{i\theta}}{(\delta(1-\alpha))^3};z\end{array}\right)\notin\Omega, \tag{3.14}$$

whenever $z\in U$,

$$Re\left(Le^{-i\theta}\right)\geq(k-1)kM,$$

and

$$Re\left(Ne^{-i\theta}\right)\geq 0,\qquad \forall\theta\in\mathbb{R},k\geq 2.$$

Corollary (3.3). Let $\Phi\in\mathfrak{I}_j[\Omega,M]$. If the function $f\in A$ satisfies:

$$\left|\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)\right|\leq kM,\qquad (z\in U,k\geq 2;M>0)$$

and

$$\Phi\left(\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z);z\right)\in\Omega,$$

then

$$\left|\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)\right|<M.$$

In the special case $\Omega=q(U)=\{w:|w|<M\}$, the class $\mathfrak{J}_j[\Omega,q]$ is simply referred as $\mathfrak{J}_j[M]$. Corollary (3.3) can now be used written as follows .

Corollary (3.4). Let $\Phi\in\mathfrak{J}_j[M]$. If the function $f\in A$ fulfills the following criteria:

$$\left|\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)\right|\le kM,\qquad(z\in U,k\ge 2;M>0)$$

and

$$\left|\left(\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z);z\right)\right|<M,$$

then

$$\left|\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)\right|<M.$$

Corollary (3.5). Let $\ge 2,\ M>\ 0$. If the function $f\in A$ satisfies the following conditions:

$$\left|\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)\right|\le kM,$$

and

$$\left|\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)-\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)\right|\le\frac{M}{\delta(1-\alpha)},$$

then

$$\left|\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)\right|\le M.$$

Proof. Let $\Phi(a,b,c,d;z)=b-a,\ \Omega=h(U)$, where $h(z)=\frac{Mz}{\delta(1-\alpha)}, z\in U, M>0.$

Make use of the Corollary (3.3). We must prove it $\Phi\in\mathfrak{J}_j[\Omega,M]$, in other words, admissibility condition (3.14) is satisfied. This follows readily, since it is seen that

$$|\Phi(a,b,c,d;z)|=\left|\frac{(k-1)}{\delta(1-\alpha)}Me^{i\theta}\right|=\frac{k-1}{\delta(1-\alpha)}M\ge\frac{M}{\delta(1-\alpha)},$$

whenever $z\in U,\theta\in\mathbb{R}$ and $k\ge 2$.

Definition (3.3). Let Ω be a set in $\mathbb{C}$ and $q\in\mathbb{Q}_1\cap H_1$. The class $\mathfrak{J}_{j,1}[\Omega,q]$ of functions that are admissible consists of those functions $\Phi:\mathbb{C}^4\times U\to\mathbb{C}$, which satisfy the following admissibility conditions:

$$\Phi(a,b,c,d;z)\notin\Omega,$$

whenever

$$a = q(\xi), \qquad b = \frac{k\xi q'(\xi) + \delta(1-\alpha)q(\xi)}{\delta(1-\alpha)},$$

$$\mathrm{Re}\left(\frac{\delta(1-\alpha)[a+c-2b]}{b-a}\right) \geq k\mathrm{Re}\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right),$$

and

$$\mathrm{Re}\left(\frac{(\delta(1-\alpha))^2(d-a) - 3\delta(1-\alpha)(\delta(1-\alpha)+1)(c-a) + (b-a)[3(\delta(1-\alpha)+1)^2 - 1]}{b-a}\right) \geq k^2\mathrm{Re}\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in U, \xi \in \partial U \backslash E(q)$ and $k \geq 2$.

Theorem (3.4). Let $\Phi \in \Im_{j,1}[\Omega, q]$. If $f \in A$ be a function and $q \in \mathbb{Q}_1 \cap H_1$ satisfy the next requirements:

$$Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \qquad \left|\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{zq(z)}\right| \leq k, \qquad (3.15)$$

and

$$\left\{\Phi\left(\frac{\mathfrak{D}_\delta^{(\alpha,\beta)} f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)} f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)} f(z)}{z}; z\right) : z \in U\right\} \subset \Omega, \qquad (3.16)$$

then

$$\frac{\mathfrak{D}_\delta^{(\alpha,\beta)} f(z)}{z} \prec q(z), \qquad (z \in U).$$

Proof. The analytic function should be defined p(z) in U by

$$p(z) = \frac{\mathfrak{D}_\delta^{(\alpha,\beta)} f(z)}{z}. \qquad (3.17)$$

Using the equation (1.7) and (3.17), we have

$$\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{z} = \frac{zp'(z) + \delta(1-\alpha)p(z)}{\delta(1-\alpha)}. \qquad (3.18)$$

By a similar argument, we get

$$\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)} f(z)}{z} = \frac{z^2p''(z) + [2\delta(1-\alpha)+1]zp'(z) + (\delta(1-\alpha))^2 p(z)}{(\delta(1-\alpha))^2}, \qquad (3.19)$$

and

$$\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)} f(z)}{z} = \frac{z^3p'''(z) + 3(\delta(1-\alpha)+1)z^2p''(z) + [3\delta(1-\alpha)(\delta(1-\alpha)+1)+1]zp'(z) + (\delta(1-\alpha))^3 p(z)}{(\delta(1-\alpha))^3}. \qquad (3.20)$$

Define the transformation starting with $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(r,s,t,u) = r\,, \qquad b(r,s,t,u) = \frac{s+\delta(1-\alpha)r}{\delta(1-\alpha)},$$

$$c(r,s,t,u) = \frac{t+(2\delta(1-\alpha)+1)s+(\delta(1-\alpha))^2 r}{(\delta(1-\alpha))^2}, \tag{3.21}$$

and

$$d(r,s,t,u) = \frac{u+3(\delta(1-\alpha)+1)t+[3\delta(1-\alpha)(\delta(1-\alpha)+1)+1]s+(\delta(1-\alpha))^3 r}{(\delta(1-\alpha))^3}. \tag{3.22}$$

Let

$$\varphi(r,s,t,u) = \Phi(a,b,c,d;z) = \Phi\left(\begin{array}{c} r, \frac{s+\delta(1-\alpha)r}{\delta(1-\alpha)}, \frac{t+(2\delta(1-\alpha)+1)s+(\delta(1-\alpha))^2 r}{(\delta(1-\alpha))^2}, \\ \frac{u+3(\delta(1-\alpha)+1)t+[3\delta(1-\alpha)(\delta(1-\alpha)+1)+1]s+(\delta(1-\alpha))^3 r}{(\delta(1-\alpha))^3}; z \end{array}\right). \tag{3.23}$$

The proof will make use of Lemma (2.1). Equations are used (3.17) to (3.20), and from (3.23), we have

$$\varphi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) = \Phi\left(\frac{\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{z}; z\right). \tag{3.24}$$

Hence, clearly, (3.16) becomes

$$\varphi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega.$$

Note that

$$\frac{t}{s}+1 = \frac{\delta(1-\alpha)[a+c-2b]}{b-a},$$

and

$$\frac{u}{s} = \frac{(\delta(1-\alpha))^2(d-a)-3\delta(1-\alpha)(\delta(1-\alpha)+1)(c-a)+(b-a)[3(\delta(1-\alpha)+1)^2-1]}{b-a}.$$

As a result, the admissibility condition for $\Phi \in \mathfrak{I}_{j,1}[\Omega, q]$ in Definition (3.3) is the same as the admissibility criterion for $\varphi \in \Psi_2[\Omega, q]$ as stated in the Definition (2.3) with $n = 2$. As a result, using (3.13) and Lemma (2.1), we have

$$\frac{\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)}{z} \prec q(z).$$

Now completes the proof of theorem (3.4).

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this situation, the class $\mathfrak{I}_{j,1}[h(U), q]$ is written as $\mathfrak{I}_{j,1}[\Omega, q]$. This follows immediate consequence of Theorem (3.4), as follows:

Theorem (3.5). Let $\Phi \in \mathfrak{I}_{j,1}[\Omega, q]$. If the functions $f \in A$ and $q \in \mathbb{Q}_1$ satisfy the following conditions:

$$\operatorname{Re}\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geq 0, \qquad \left|\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{z q_\rho'(\xi)}\right| \leq k, \tag{3.25}$$

and

$$\Phi\left(\frac{\mathfrak{D}_\delta^{(\alpha,\beta)} f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)} f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)} f(z)}{z}; z\right) \prec h(z), \tag{3.26}$$

then

$$\frac{\mathfrak{D}_\delta^{(\alpha,\beta)} f(z)}{z} \prec q(z), \qquad (z \in U).$$

In light of the Definition (3.3) and in the special case $q(z) = Mz$, $M > 0$, the class functions that are admissible $\mathfrak{I}_{j,1}[\Omega, q]$, denoted by $\mathfrak{I}_{j,1}[\Omega, M]$ is expressed as follows.

Definition (3.4). Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\mathfrak{I}_{j,1}[\Omega, q]$ consists of those functions $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$ such that:

$$\Phi\left(\begin{array}{c} Me^{i\theta}, \dfrac{k+\delta(1-\alpha)Me^{i\theta}}{\delta(1-\alpha)}, \dfrac{L+([(2\delta(1-\alpha)+1)]k+(\delta(1-\alpha))^2)Me^{i\theta}}{(\delta(1-\alpha))^2}, \\ \dfrac{N+3(\delta(1-\alpha)+1)L+([3(\delta(1-\alpha))(\delta(1-\alpha)+1)+1]k+(\delta(1-\alpha))^3)Me^{i\theta}}{(\delta(1-\alpha))^3}; z \end{array}\right) \notin \Omega, \tag{3.27}$$

whenever

$$z \in U, \qquad \operatorname{Re}\left(Le^{-i\theta}\right) \geq (k-1)kM,$$

and

$$\operatorname{Re}\left(Ne^{-i\theta}\right) \geq 0, \qquad \forall \theta \in \mathbb{R}; k \geq 2.$$

Corollary (3.6). Let $\Phi \in \mathfrak{I}_{j,1}[\Omega, q]$. If the function $f \in A$ satisfies the following conditions:

$$\left|\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{z}\right| \leq kM, \qquad (z \in U, k \geq 2; M > 0),$$

and

$$\Phi\left(\frac{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)}{z}, \frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{z}, \frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{z}, \frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{z}; z\right) \in \Omega,$$

then

$$\left|\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{z}\right| < M.$$

In the special case, when $\Omega = q(U) = \{w: |w| < M\}$, the class $\mathfrak{I}_{j,1}[\Omega, q]$ is simply denoted by $\mathfrak{I}_{j,1}[M]$. Corollary (3.6) can now be expressed as follows:

Corollary (3.7). Let $\Phi \in \mathfrak{I}_{j,1}[\Omega, q]$. If the function $f \in A$ satisfies what follows circumstances:

$$\left|\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{z}\right| \leq kM, \qquad (z \in U, k \geq 2; M > 0),$$

and

$$\left|\Phi\left(\frac{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)}{z}, \frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{z}, \frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{z}, \frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{z}; z\right)\right| < M,$$

then

$$\left|\frac{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)}{z}\right| < M.$$

Definition (3.5). Let $q \in \mathbb{Q}_1 \cap H_1$ and Ω be a set in $\mathbb{C}$. The class $\mathfrak{I}_{j,2}[\Omega, q]$ of admissible functions consists of those functions $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$, which satisfy the following admissibility requirements:

$$\Phi(a, b, c, d; z) \notin \Omega,$$

whenever

$$a = q(\xi), \qquad b = \frac{1}{\delta(1-\alpha)}\left[\frac{k\xi q'(\xi) + \delta(1-\alpha)\big(q(\xi)\big)^2}{q(\xi)}\right],$$

$$\operatorname{Re}\left(\frac{\delta(1-\alpha)[2a^2 + cb - 3ab]}{b-a}\right) \geq k \operatorname{Re}\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right),$$

and

$\mathrm{Re}\left(\left[bc(d-c)\left(\delta(1-\alpha)\right)^2-b\left(\delta(1-\alpha)\right)^2(c-b)(1-b-c+3a)-3\delta(1-\alpha)(c-b)b+2(b-a)+3a\delta(1-\alpha)(b-a)+(b-a)^2\delta(1-\alpha)\left((b-c)\delta(1-\alpha)-3-4a\delta(1-\alpha)\right)+a^2\left(\delta(1-\alpha)\right)^2(b-a)\right](b-a)^{-1}\right)\geq k^2\mathrm{Re}\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$

where $\in U$, $\xi\in\partial U\backslash E(q)$ and $k\geq 2$.

Theorem (3.6). Let $\Phi\in\Im_{j,2}[\Omega,q]$. If the functions $f\in A$ and $q\in\mathbb{Q}_1\cap H_1$ satisfy the following conditions:

$$\mathrm{Re}\left(\frac{\xi q''(\xi)}{q'(\xi)}\right)\geq 0,\qquad \left|\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}\right|\leq k, \tag{3.28}$$

and

$$\left\{\Phi\left(\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{4}{\delta},\beta-\frac{4}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)};z\right):z\in U\right\}\subset\Omega, \tag{3.29}$$

then

$$\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)}\prec q(z),\qquad (z\in U).$$

Proof. The analytic function should be defined $p(z)$ in U by

$$p(z)=\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)}. \tag{3.30}$$

From equation (1.7) and (3.30), we have

$$\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}=\frac{1}{\delta(1-\alpha)}\left[\frac{zp'(z)+\delta(1-\alpha)p^2(z)}{p(z)}\right]=\frac{A}{\delta(1-\alpha)}. \tag{3.31}$$

By a similar argument, we have

(3.32)

$$\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}=\frac{B}{\delta(1-\alpha)},$$

and

$$\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{4}{\delta},\beta-\frac{4}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}=\frac{1}{\delta(1-\alpha)}[B+B^{-1}(C+A^{-1}D-A^{-2}C^2)], \tag{3.33}$$

where

$$B=\frac{zp'(z)}{p(z)}+\delta(1-\alpha)p(z)+\frac{\frac{z^2p''(z)+zp'(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+\delta(1-\alpha)zp'(z)}{\frac{zp'(z)}{p(z)}+\delta(1-\alpha)p(z)},$$

$$C=\frac{z^2p''(z)+zp'(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+\delta(1-\alpha)zp'(z),$$

and

$$D=\frac{z^3p'''(z)+3z^2p''(z)+zp'(z)}{p(z)}-\frac{3z^2\left(p'(z)\right)^2+3z^3p''(z)p'(z)}{(p(z))^2}+2\left(\frac{zp'(z)}{p(z)}\right)^3+\delta(1-\alpha)z^2p''(z)+\delta(1-\alpha)zp'(z).$$

Define the transformation starting with $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(r,s,t,u)=r,\ b(r,s,t,u)=\frac{1}{\delta(1-\alpha)}\left[\frac{s+\delta(1-\alpha)r^2}{\alpha}\right]=\frac{E}{\delta(1-\alpha)},$$

$$c(r,s,t,u)=\frac{1}{\delta(1-\alpha)}\left[\frac{s+\delta(1-\alpha)r^2}{r}+\frac{\frac{t+s}{r}-\left(\frac{s}{r}\right)^2+\delta(1-\alpha)s}{\frac{s}{r}+\delta(1-\alpha)r}\right]=\frac{F}{\delta(1-\alpha)}, \tag{3.34}$$

and

$$d(r,s,t,u)=\frac{1}{\delta(1-\alpha)}[F+F^{-1}(L+HE^{-1}-E^{-2}L^2)], \tag{3.35}$$

where

$$L=\frac{t+s}{r}-\left(\frac{s}{r}\right)^2+\delta(1-\alpha)s,$$

and

$$H=\frac{u+3t+s}{r}-3\left(\frac{s}{r}\right)^2-3\frac{st}{r^2}+2\left(\frac{s}{r}\right)^3+\delta(1-\alpha)(s+t).$$

Let

$$\varphi(r,s,t,u)=\Phi(a,b,c,d)=\Phi\left(r,\frac{E}{\delta(1-\alpha)},\frac{F}{\delta(1-\alpha)},\frac{1}{\delta(1-\alpha)}\left[F+F^{-1}(L+HE^{-1}-E^{-2}L^2)\right]\right). \quad (3.36)$$

The proof will make use of Lemma (2.1). Using the equations (3.30) to (3.33), and from (3.36), we have

$$\varphi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)=$$

$$\Phi\left(\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{4}{\delta},\beta-\frac{4}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)};z\right) \quad (3.37)$$

Hence, clearly (3.29) leads to

$$\varphi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)\in\Omega.$$

We note that

$$\frac{t}{s}+1=\frac{\delta(1-\alpha)[2a^2+cb-3ab]}{b-a},$$

and

$$\frac{u}{s}=\Big[bc(d-c)\big(\delta(1-\alpha)\big)^2-b\big(\delta(1-\alpha)\big)^2(c-b)(1-b-c+3a)-3\delta(1-\alpha)(c-b)b+2(b-a)+3a\delta(1-\alpha)(b-a)+(b-a)^2\delta(1-\alpha)\big((b-c)\delta(1-\alpha)-3-4a\delta(1-\alpha)\big)+a^2\big(\delta(1-\alpha)\big)^2(b-a)\Big](b-a)^{-1}.$$

Thus, the admissibility condition for $\Phi\in\mathfrak{I}_{j,2}[\Omega,q]$ in Definition (3.5) is the same as the criteria of admissibility for $\varphi\in\Psi_2[\Omega,q]$ as stated in the Definition (2.3) with $n=2$. As a result, using (3.30) and Lemma (2.1), we have

$$\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)}\prec q(z),$$

this completes the proof of the Theorem (3.6).

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this situation, the class $\mathfrak{I}_{j,2}[h(U),q]$ is written as $\mathfrak{I}_{j,2}[\Omega,q]$. This follows immediate consequence of Theorem (3.6) is stated below without proof.

Theorem (3.7). Let $\Phi\in\mathfrak{I}_{j,2}[\Omega,q]$. If the functions $f\in A$ and $q\in\mathbb{Q}_1$ satisfy the following conditions (3.29) and

$$\Phi\left(\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{4}{\delta},\beta-\frac{4}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)};z\right)\prec h(z),$$

then

$$\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{\mathfrak{D}_{\delta}^{(\alpha,\beta)} f(z)} \prec q(z), \qquad (z \in U).$$

4- Results on Third-Order Differential Superordination

Definition (4.1). Let Ω be a set in $\mathbb{C}$ and $q \in \mathbb{Q}_0 \cap H_0$ with $q'(z) \neq 0$. The class of admissible functions $\mathfrak{I}'_j[\Omega, q]$ consists of those functions $\Phi: \mathbb{C}^4 \times U \to \mathbb{C}$, that satisfy the following admissibility conditions:

$$\Phi(a, b, c, d; \xi) \in \Omega,$$

whenever

$$a = q(z), \qquad b = \frac{zq'(z) + m\delta(1-\alpha)q(z)}{m\delta(1-\alpha)},$$

$$\mathrm{Re}\left(\frac{\delta(1-\alpha)[\delta(1-\alpha)c - 2[\delta(1-\alpha)-1]b] + [\delta(1-\alpha)-1]^2 a]}{\delta(1-\alpha)b - [\delta(1-\alpha)-1]a}\right) \leq \frac{1}{m}\mathrm{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$\mathrm{Re}\left(\frac{(\delta(1-\alpha))^3[d - 3(c-b)] - \delta(1-\alpha)[3[\delta(1-\alpha)-1]a + b] + \left[[\delta(1-\alpha)-1]a(1 - [\delta(1-\alpha)-1]^2)\right]}{\delta(1-\alpha)b - [\delta(1-\alpha)-1]a}\right) \leq \frac{1}{m^2}\mathrm{Re}\left(\frac{z^2 q'''(z)}{q'(z)}\right),$$

where $z \in U, \xi \in \partial U$ and $m \geq 2$.

Theorem (4.1). Let $\Phi \in \mathfrak{I}'_j[\Omega, q]$. If the functions $f \in A$, with $\mathfrak{D}_{\delta}^{(\alpha,\beta)} f(z) \in \mathbb{Q}_0$ and if $q \in H_0$ with $q'(z) \neq 0$, satisfying the following conditions:

$$Re\left(\frac{\xi q''(z)}{q'(z)}\right) \geq 0, \qquad \left|\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{q'(z)}\right| \leq m, \quad (4.1)$$

and the function

$$\Phi\left(\mathfrak{D}_{\delta}^{(\alpha,\beta)} f(z), \mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z), \mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)} f(z), \mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)} f(z); z\right),$$

is univalent in U, then

$$\Omega \subset \left\{\Phi(\mathfrak{D}_{\delta}^{(\alpha,\beta)} f(z), \mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z), \mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)} f(z), \mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)} f(z); z \in U\right\}, (4.2)$$

implies that

$$q(z) \prec \mathfrak{D}_{\delta}^{(\alpha,\beta)} f(z), \qquad (z \in U).$$

Proof. Let the function $p(z)$ be defined by (3.3) and φ by (3.8). Since $\Phi \in \mathfrak{I}'_j[\Omega, q]$. From (3.10) and (4.2), we have

$$\Omega \subset \{\Phi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z): z \in U\}.$$

From (3.7) and (3.8), We can observe that the admissibility condition for $\Phi \in \mathfrak{I}'_j[\Omega, q]$ in Definition (4.1) is the same as the admissibility criterion for $\varphi \in \Psi'_n[\Omega, q]$ as stated in the Definition (2.5) with $n = 2$. Hence $\varphi \in \Psi'_2[\Omega, q]$ as well as (4.2) and Lemma (2.2), we have

$$q(z) \prec \mathfrak{D}_{\delta}^{(\alpha,\beta)} f(z), \qquad (z \in U).$$

This completes the proof of the Theorem (4.1).

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of onto Ω. In this case, the class $\mathfrak{J}_j{}'[h(U), q]$ is written as $\mathfrak{J}_j{}'[h, q]$. This follows an immediate repercussion of Theorem (4.1) is stated below.

Theorem (4.2). Let $\Phi \in \mathfrak{J}_j{}'[h, q]$ and h be analytic in U. If th function $f \in A$, and $\mathfrak{D}_\delta^{(\alpha,\beta)} f(z) \in \mathbb{Q}_0$ and $q \in H_0$ with $q'(z) \neq 0$, satisfying the following conditions (4.1) and the function

$$\Phi\left(\mathfrak{D}_\delta^{(\alpha,\beta)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)} f(z); z\right),$$

is univalent in U, then

$$h(z) \prec \Phi\left(\mathfrak{D}_\delta^{(\alpha,\beta)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)} f(z); z\right), \qquad (4.3)$$

implies that

$$q(z) \prec \mathfrak{D}_\delta^{(\alpha,\beta)} f(z), \qquad (z \in U).$$

Theorem (4.1) and (4.2) may only be utilized to get third-order differential superordination of the forms' subordination (4.2) or (4.3).

The following theorem gives the existence of the best subordinant of (4.3) for suitable Φ.

Theorem (4.3). Let h be univalent function in U and $\Phi: \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ and φ be given by (3.9). Assume the differential equation:

$$\varphi(q(z), zq'(z), z^2 q''(z), z^3 q'''(z); z) = h(z), \quad (4.4)$$

has a solution $q(z) \in \mathbb{Q}_0$. If the functions $f \in A$, and $\mathfrak{D}_\delta^{(\alpha,\beta)} f(z) \in \mathbb{Q}_0$ and if $q \in H_0$ with $q'(z) \neq 0$, which satisfy the following criteria (4.1) and the function

$$\Phi\left(\mathfrak{D}_\delta^{(\alpha,\beta)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)} f(z); z\right),$$

is analytic in U, then

$$h(z) \prec \Phi\left(\mathfrak{D}_\delta^{(\alpha,\beta)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)} f(z), \mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)} f(z); z\right),$$

implies that

$$q(z) \prec \mathfrak{D}_\delta^{(\alpha,\beta)} f(z), \qquad (z \in U)$$

and q(z) is the best subordinant.

Proof. From Theorem (4.1) and (4.2), we see that q is a subordinant of (4.3). Since q satisfies (4.4), it is also a solution of (4.3) and therefore, q will be subordinant by all subordinants. Hence q is the best subordinant. The proof of Theorem (4.3) is complete.

Definition (4.2). Let Ω be a set in $\mathbb{C}$ and $q \in H_1$ with $q'(z) \neq 0$. The class of admissible functions $\mathfrak{J}_{j,1}'[\Omega, q]$ includes those functions $\Phi: \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$, that satisfy the following admission requirements:

$$\Phi(a, b, c, d; \xi) \in \Omega,$$

whenever

$$a = q(z), \qquad b = \frac{zq'(z) + m\,\delta(1-\alpha)q(z)}{m\delta(1-\alpha)},$$

$$Re\left(\frac{\delta(1-\alpha)[c+a-2b]}{b-a}\right) \le \frac{1}{m} Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$Re\left(\frac{(\delta(1-\alpha))^2(d-a) - 3(\delta(1-\alpha))(\delta(1-\alpha)+1)(c-a) + (b-a)[3(\delta(1-\alpha)+1)^2 - 1]}{b-a}\right) \le \frac{1}{m^2} Re\left(\frac{z^2 q'''(z)}{q'(z)}\right),$$

where $z \in U, \xi \in \partial U$ and $m \ge 2$.

Theorem (4.4). Let $\Phi \in \mathfrak{J}'_{j,1}[\Omega, q]$. If the function $f \in A$ and $\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{z} \in \mathbb{Q}_1$, and if $q \in H_1$ with $q'(z) \ne 0$, satisfying the following conditions:

$$Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \ge 0, \qquad \left|\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{zq'(z)}\right| \le m, \quad (4.5)$$

and the function

$$\Phi\left(\frac{\mathfrak{D}_\delta^{(\alpha,\beta)} f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)} f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)} f(z)}{z}; z\right),$$

is univalent in U, then

$$\Omega \subset \left\{\Phi\left(\frac{\mathfrak{D}_\delta^{(\alpha,\beta)} f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)} f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)} f(z)}{z}, \frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)} f(z)}{z}; z\right) : z \in U\right\}, \quad (4.6)$$

implies that

$$q(z) \prec \frac{\mathfrak{D}_\delta^{(\alpha,\beta)} f(z)}{z}, \qquad (z \in U).$$

Proof. Let the function $p(z)$ be defined by (3.17) and φ by (3.23). Since $\Phi \in \mathfrak{J}'_{j,1}[\Omega, q]$, from (3.24) and (4.6) yield

$$\Omega \subset \{\varphi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in U\}$$

From the equations (3.21) and (3.22), it is clear that admissibility is a requirement $\Phi \in \mathfrak{J}'_{j,1}[\Omega, q]$ in Definition (4.1) is equivalent to the admissibility condition for φ as stated in Definition (2.3) with $n = 2$. Hence $\varphi \in \Psi'_2[\Omega, q]$ and by using (4.6) and Lemma (2.2), we have

$$q(z) \prec \frac{\mathfrak{D}_\delta^{(\alpha,\beta)} f(z)}{z}, \qquad (z \in U).$$

The proof of Theorem (4.4) is complete.

If $\Omega \ne \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\mathfrak{J}'_{j,1}[h(U), q]$ is written as $\mathfrak{J}'_{j,1}[h, q]$. This is a direct consequence of the Theorem. (4.4).

Theorem (4.5). Let $\Phi \in \Im'_{j,1}[h,q]$ and h be analytic in U. If the functions $f \in A$, with $q \in H_1$ and $q'(z) \neq 0$, satisfying the following conditions (4.5) and the function

$$\Phi\left(\frac{\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)}{z},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{z},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{z},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{z};z\right),$$

is univalent in U, then

$$h(z) \prec \Phi\left(\frac{\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)}{z},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{z},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{z},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{z};z\right),$$

implies that

$$q(z) \prec \frac{\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)}{z}, \qquad (z \in U).$$

Definition (4.3). Let Ω be a set in $\mathbb{C}$ and $q \in H_1$ with $q'(z) \neq 0$. The class $\Im'_{j,2}[\Omega,q]$ of admissible functions $\Im'_{j,2}[\Omega,q]$ consists of those functions $\Phi: \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$, that satisfy the following admissibility conditions:

$$\Phi(a,b,c,d;\xi) \in \Omega,$$

whenever

$$a = q(z), \qquad b = \frac{1}{\delta(1-\alpha)}\left[\frac{zq'(z) + m\,\delta(1-\alpha)\big(q(z)\big)^2}{m\,q(z)}\right],$$

$$Re\left(\frac{\delta(1-\alpha)[cb+2a^2-3ab]}{b-a}\right) \leq \frac{1}{m}\,Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$\mathrm{Re}\left(\left[bc(d-c)\big(\delta(1-\alpha)\big)^2 - b\big(\delta(1-\alpha)\big)^2(c-b)(1-b-c+3a) - 3\delta(1-\alpha)(c-b)b + 2(b-a) + 3a\delta(1-\alpha)(b-a) + (b-a)^2\delta(1-\alpha)\big((b-c)\delta(1-\alpha) - 3 - 4a\delta(1-\alpha)\big) + a^2\big(\delta(1-\alpha)\big)^2(b-a)\right](b-a)^{-1}\right) \leq \frac{1}{m^2}\,\mathrm{Re}\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in U$, $\xi \in \partial U \backslash E(q)$ and $m \geq 2$.

Theorem (4.6). Let $\Phi \in \Im'_{j,2}[\Omega,q]$. If the function $f \in A$ and $\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)} \in \mathbb{Q}_1$ and if $q \in H_1$ with $q'(z) \neq 0$, satisfying the following conditions:

$$Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \qquad \left|\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}\right| \leq m, \qquad (4.7)$$

and the function

$$\Phi\left(\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{(\alpha,\beta)}f(z)},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)},\frac{\mathfrak{D}_\delta^{\left(\alpha+\frac{4}{\delta},\beta-\frac{4}{\delta}\right)}f(z)}{\mathfrak{D}_\delta^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)};z\right),$$

is univalent in U, then

$$\Omega\subset\left\{\Phi\left(\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{4}{\delta},\beta-\frac{4}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)};z\right):z\in U\right\},\quad(4.8)$$

implies that

$$q(z)\prec\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)},\qquad(z\in U).$$

Proof. Let the function p(z) be defined by (3.30) and φ by (3.36). Since $\Phi\in\mathfrak{I}'_{j,2}[\Omega,q]$, we find from (3.37) and (4.8) that

$$\Omega\subset\{\varphi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z):z\in U\}.$$

From equations (3.34) and (3.35), we can see that the requirement for admissibility is $\Phi\in\mathfrak{I}'_{j,2}[\Omega,q]$ in Definition (4.3) is the same as the admissibility condition for φ as given in Definition (2.5), when $n=2$. Hence $\varphi\in\Psi'_2[\Omega,q]$ and by using (4.7) and Lemma (2.2), we have

$$q(z)\prec\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)},\qquad(z\in U).$$

The proof of Theorem (4.6) is complete.

Theorem (4.7). Let $\Phi\in\mathfrak{I}'_{j,2}[\Omega,q]$. If the function $f\in A$ and $\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)}\in\mathbb{Q}_1$, and if $q\in H_1$ with $q'(z)\neq0$, satisfying the following conditions (4.7) and the function

$$\Phi\left(\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{4}{\delta},\beta-\frac{4}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)};z\right),$$

is univalent in U, then

$$h(z)\prec\Phi\left(\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{4}{\delta},\beta-\frac{4}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)};z\right),$$

implies that

$$q(z)\prec\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)},\qquad(z\in U).$$

5- Sandwich Results

By combining Theorem (3.2) and (4.2), we obtain the following sandwich-type theorem.

Theorem (5.1). Let h_1 and q_1 be analytic functions in U. Also let h_2 be univalent function in U and $q_2\in\mathbb{Q}_0$ with $q_1(0)=q_2(0)=1$ and $\Phi\in\mathfrak{I}_j[h_2,q_2]\cap\mathfrak{I}'_j[h_1,q_1]$. If the function $f\in A$ with $\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)\in\mathbb{Q}_0\cap H_0$ and the function

$$\Phi\left(\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z);z\right),$$

is univalent in U, and if the conditions (3.1) and (4.1) are satisfied, then

$$h_1(z)\prec\Phi\left(\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z),\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z);z\right)\prec h_2(z)$$

implies that

$$q_1(z)\prec\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)\prec q_2(z),\qquad(z\in U).\qquad(5.1)$$

Combining Theorems (3.5) and (4.5), The following sandwich-type theorem is obtained.

Theorem (5.2). Let h_1 and q_1 be analytic functions in U, and let h_2 be univalent function in U and $q_2\in\mathbb{Q}_1$ with $q_1(0)=q_2(0)=1$ and $\Phi\in\mathfrak{J}_{j,1}[h_2,q_2]\cap\mathfrak{J}'_{j,1}[h_1,q_1]$. If the function $f\in A$ with $\frac{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)}{z}\in\mathbb{Q}_1\cap H_1$ and the function

$$\Phi\left(\frac{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)}{z},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{z},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{z},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{z};z\right),$$

is univalent in U, and the conditions (3.15) and (4.5) are contented, then

$$h_1(z)\prec\Phi\left(\frac{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)}{z},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{z},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{z},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{z};z\right)\prec h_2(z),$$

implies that

$$q_1(z)\prec\frac{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)}{z}\prec q_2(z),\qquad(z\in U).\,(5.2)$$

Theorem (5.3). Let h_1 and q_1 be analytic functions in U, and let h_2 be univalent function in U and $q_2\in\mathbb{Q}_1$ with $q_1(0)=q_2(0)=1$ and $\Phi\in\mathfrak{J}_{j,2}[h_2,q_2]\cap\mathfrak{J}'_{j,2}[h_1,q_1]$. If the function $f\in A$ with $\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)}\in\mathbb{Q}_1\cap H_1$ and the function

$$\Phi\left(\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{4}{\delta},\beta-\frac{4}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)};z\right),$$

is univalent in U, and the conditions (3.28) and (4.7) are contented, then

$$h_1(z)\prec\Phi\left(\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{2}{\delta},\beta-\frac{2}{\delta}\right)}f(z)},\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{4}{\delta},\beta-\frac{4}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{3}{\delta},\beta-\frac{3}{\delta}\right)}f(z)};z\right)\prec h_2(z),$$

implies that

$$q_1(z)\prec\frac{\mathfrak{D}_{\delta}^{\left(\alpha+\frac{1}{\delta},\beta-\frac{1}{\delta}\right)}f(z)}{\mathfrak{D}_{\delta}^{(\alpha,\beta)}f(z)}\prec q_2(z),\qquad(z\in U).\qquad(5.3)$$

Conclusion. In summary, the structure of 3rd order differential subordination and superordination has been generalized to normalized analytic functions via new fractional differential operator. The provided theorems generalize pre-existing results and introduce new classes of admissible functions under specific conditions. These theorems and corollaries provide more thought-provoking discoveries into the dual topics differential subordination and fractional calculus , presentation several essential applications in $\mathbb{C}$. subsequent investigations may explore the applicability of these results in broader contexts, further enriching the theoretical landscape.

Reference

1. J. A. Antonino, S. S. Miller, Third-order differential inequalities and subordination in the complex plane, Complex Var. Elliptic Equ., 56(2011), 439-454.
2. W. G. Atshan and A. A. R. Ali, On some sandwich theorems of analytic functions involving Noor- Sãlãgean operator, Advances in Mathematics: Scientific Journal, 9(10) (2020), 8455-8467.
3. W. G. Atshan, A. H. Battor and A. F. Abaas, On third-order differential subordination results for univalent analytic functions involving an operator, Journal of Physics: Conference Series, 1664 (2020) 012044, 1-19.
4. P. Agarwal, A. A. Hyder and M. Zakarya, Well-Posedness of Stochastic Modified Kawahara Equation. Adv. Differ. Equ., 1(2020), 1–10.
5. G. AlNemer, M. Zakarya, H. A. A. El-Hamid, M. R. Kenawy and H. M. Rezk, Dynamic Hardy-Type Inequalities with Non-Conjugate Parameters. Alex. Eng. J., 59(2020), 4523–4532.
6. W. G. Atshan and A. A. R. Ali, On sandwich theorems results for certain univalent functions defined by generalized operators, Iraqi Journal of Science, 62(7) (2021), pp: 2376-2383.
7. W. G. Atshan and S. R. Kulkarni, On application of differential subordination for certain subclass of meromorphically p-valent functions with positive coefficients defined by linear operator, Journal of Inequalities in Pure and Applied Mathematics, 10(2) (2009), Article 53, 11pp.
8. W. G. Atshan, A. H. Battor and A. F. Abaas, Some sandwich theorems for meromorphic univalent functions defined by new integral operator, Journal of Interdisciplinary Mathematics, 24(3) (2021), 579-591.
9. W. G. Atshan, R. A. Hiress and S. Altinkaya, On third-order differential subordination and superordination properties of analytic functions defined by a generalized operator, Symmetry, 14 (2) (2022), 418, 1-17.
10. A.A. Attiya, T.M. Seoudy and A. Albaid, Third-Order Differential Subordination for Meromorphic Functions Associated with Generalized Mittag-Leffler Function. Fractal Fract. 2023, 7, 175.
11. S. D. Bernardi, " Convex and Starlike Univalent Functions", Transaction of American Mathematical Society, 1969, 135, 429-449.
12. S. R. Bakheet, W. G. Atshan, Third-order sandwich results for analytic univalent functions defined by integral operator. Adv. Mech., 10(2022), 1178-1197.
13. A. M. Darweesh, W. G. Atshan, A. H. Battor and A. A. Lupas, Third-order differential subordination results for analytic functions associated with a certain differential operator, Symmetry, 14(1) (2022), 99, 1-15.
14. M. P. Jeyaraman and T. K. Suresh, Third-order differential subordination of analytic functions, Acta Univ. Apulensis Math. Inform., 35 (2013), 187–202.
15. Saliu A, Jabeen K, Ravichandran V. Differential subordination for certain strongly starlike functions. Rendiconti del Circolo Matematico di Palermo Series 2. 2024 Feb;73(1):1-8.
16. S. S. Miller and P. T. Mocanu and M. O. Reade, Subordination-preserving integral operators. Trans. Amer. Math. Soc., 283(1984), 605–615.
17. M. Sharma, NK. Jain and S. Kumar , Starlikeness of analytic functions using special functions and subordination. Boletín de la Sociedad Matemática Mexicana. 2024 Jul;30(2):55.
18. S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, Complex Var. Elliptic Equ., 48 (2003), 815-826.
19. S. S. Miller and P. T. Mocanu, Differential Subordination: Theory and Applications; Series on Monographs and Textbooks in Pure and Applied Mathematics; Marcel Dekker Incorporated: New York, NY, USA, 2000; Volume 225.
20. S. Ponnusamy and O. P. Juneja, Third-order differential inequalities in the complex plane, in Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, (1992).

21. D. Ra˘ducanu, Third order differential subordinations for analytic functions associated with generalized Mittag-Leffler functions, Mediterr. J. Math., 14 (4) (2017), Article ID 167, 1–18.
22. H. M. Rezk, G. AlNemer, H. A. Abd El-Hamid, A. H. Abdel-Aty, K. S. Nisar and M. Zakarya, Hilbert-Type Inequalities for Time Scale Nabla Calculus. Adv. Differ. Equ., 1(2020), 1–21.
23. H. M. Srivastava, S. Owa, " Univalent Functions, Fractional Calculus, and Their Applications", Wiley: New York, NY, USA, 1989.
24. T. M. Seoudy, Some applications of third-order differential subordination for analytic functions involving k-Ruscheweyh derivative operator. Afrika Matematika. 2023 Jun;34(2):29.
25. Q. A. Shakir W. G. Atshan. "On Sandwich Results of Univalent Functions Defined by Generalized Abbas-Atshan Operator." Journal of Al-Qadisiyah for Computer Science and Mathematics 15.4 (2023): 11-20.
26. A.S. Tayyah and W. G. Atshan, New Results on r, k, μ-Riemann–Liouville Fractional Operators in Complex Domain with Applications, Fractal and Fractional, 8(3)(2024), 165,1-17.
27. H. Tang, H. M. Srivastava, E. Deniz and S. Li, Third-order differential superordination involving the generalized Bessel functions, Bull. Malays. Math. Sci. Soc., 38 (2015), 1669–1688.
28. H. Tang, H. M. Srivastava, S. Li and L. Ma, Third order differential subordination and superordination results for meromorphically multivalent functions associated with the Liu-Srivastava operator, Abstr. App. Anal. , 2014 (2014), Article ID 792175, 1-11.
29. H. Tang and E. Deniz, Third-order differential subordination results for analytic functions involving the generalized Bessel functions, Acta Math. Sci. Ser. B Engl. Ed.,34 (2014),1707-1719.