Pseudo-Weakly-N-Quasi- Injective Modules

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Abstract

Let R be a commutative ring with unity and M be a unitary R-module. An R-module M is said to be N-injective where N is an R-module if $f(N) \subseteq M$ for each $f \in Hom(N, E(M))$, where E(M) is the injective hull of M. And M is called weakly-N-injective if for each $f \in Hom(N, E(M))$ there exists a submodule X of E(M) such that $f(N) \subseteq X \cong M$. In this paper we give generalizations for the concepts N-injective and weakly- N-injective modules we call them pseudo-N-quasi-injective and pseudo-weakly-N-quasi-injective modules respectively. We call an R-module M pseudo-N-quasi-injective modules if $f(N) \subseteq M$ for each monomorphism $f: N \to \overline{M}$ where $\overline{M}$ is quasi injective hull of M. And we call M is Pseudo-weakly-N-quasi-injective module if for each monomorphism $f: N \to \overline{M}$, there exists a submodule X of $\overline{M}$ such that $f(N) \subseteq X \cong M$. Our main goal in this work is to study the basic properties of these concepts, and give examples, characterizations of pseudo-weakly-N-quasi-injective and study the relation of these concepts with other modules.

Introduction

Let R be a commutative ring with identity and M , N be two R-modules. M is said be N-injective if $f(N) \subseteq M$ for each $f \in Hom(N, E(M))$, where E(M) is the injective hull of M. This concept was introduced first by Azumaya [3]. Weakly-N-injective module was introduced originally by [10] as a generalization of N-injective module. Since then the study of this concept has been extensively in[9],[11]. We introduce in this paper generalizations of both N-injective module and weakly-N-injective module. We call an R-module M pseudo-N-quasi-injective modules if $f(N) \subseteq M$ for each monomorphism $f: N \to \overline{M}$ where $\overline{M}$ is quasi injective hull of M .And we call An R-module M is Pseudo-weakly-N-quasi-injective module if for each monomorphism $f: N \to \overline{M}$, there exists a submodule X of $\overline{M}$ such that $f(N) \subseteq X \cong M$.

Also we introduce a new concept named pseudo-invertible submodule to prove that if M is a torsion free R-module and N is pseudo-invertible submodule of M such that M is pseudo-weakly-M/N-quasi-injective. Then N is pseudo-invertible submodule of $\overline{M}$.

1- <u>Pseudo-N-quasi-injective modules</u>

In this section we introduce the definition of pseudo-N-quasi-injective module as a generalization of N-injective module and gives some basic properties, examples of this concept.

<u>Definition 1.1</u>

Let M and N be two R-modules. M is called pseudo-N-quasi-injective modules if $f(N) \subseteq M$ for each monomorphism $f: N \to \overline{M}$, where $\overline{M}$ is a quasi-injective hull of M

<u>Examples and remarks 1.2</u>

1. Every N-injective module is pseudo-N-quasi-injective module, but the converse is not true , as the following example shows: Z_3 as a Z- module is pseudo-N-quasi-injective module , but not Z-injective, for if $f: Z \to E(Z_3) = Z_{3^\infty}$ defined by $f(n) = \frac{n}{3^3} + Z$ for all $n \in Z$. It is clear that $f(Z) = Z_9$ which is not embeds in Z_3.
2. Z_n as a Z-module is pseudo-Z-quasi-injective module.
3. Every injective module is pseudo-N-quasi-injective modules for any R-module N.
4. Every quasi-injective module is pseudo-N-quasi-injective module for any R-module N.
5. Every pseudo-injective module is pseudo-N-quasi-injective module for any R-module N.
6. Every semi-simple R-module is pseudo-N-quasi-injective module for any R-module N.
7. M is pseudo-injective if and only if M is pseudo- $\overline{M}$ -quasi-injective module.
8. Z as a Z-module is pseudo-Z-quasi-injective module.
9. If M is pseudo-N-quasi-injective module and L is a submodule of N , it not necessary that M is pseudo-L-quasi-injective module, as the following example shows:

From(8) Z as a Z-module is pseudo-Z-quasi-injective module . but Z is not pseudo-3Z-quasi-injective module, for if $g: 3Z \to \overline{Z} = Q$ is monomorphism define by $g(3n) =$3n/7 for all n in Z, but $g(3Z)$ is not embeds in Z.

Before we give the next proposition, we need to recall the following definitions.

A non-zero submodule K of an R-module M is said to be an essential submodule of M if $L \cap K \neq 0$ for every non-zero submodule L of M. And we said that M is an essential extension of K [8].

A non-zero R-module M is called a rational extension of an R-submodule N of M, if for all $m_1, m_2 \in M, m_2 \neq 0$ there exists an element $r \in R$ such that $r\ m_1 \in N, rm_2 \neq 0$ [6].

Proposition 1.3

If M is pseudo-N-quasi-injective module, then every essential extension of M is pseudo-N-quasi-injective module.

Proof

Let H be essential extension R-module of M, and let $f: N \to \overline{H}$ be a monomorphism. Since M is an essential submodule of H, then $\overline{M} = \overline{H}$,and hence $f: N \to \overline{M}$ is a monomorphism. But M is pseudo-N-quasi-injective module, hence $f(N) \subseteq M$. Therefore $f(N) \subseteq H$. The following corollary is immediate consequence of Proposition 1.3.

Corollary 1.4

If M is pseudo-N-quasi-injective module, then $\overline{M}$ is pseudo-N-quasi-injective module. Since every rational extension is an essential extension [7], we have the following corollary

Corollary 1.5

If M is pseudo-N-quasi-injective module, then every rational extension of M is pseudo-N-quasi-injective module.

Proposition 1.6

Let M, N and H be an R-modules. If M is pseudo-N-quasi-injective module and M is pseudo-H-quasi-injective module, then M is pseudo- $N \oplus H$-quasi-injective module.

Proof

Let $f: N \oplus H \to \overline{M}$ be a monomorphism and let $j_1: N \to N \oplus H, j_2: H \to N \oplus H$ be the injection homomorphism, then $f \circ j_1: N \to \overline{M}$ and $f \circ j_2: H \to \overline{M}$ are monomorphisms. But M is both pseudo-N-quasi-injective and pseudo-H-quasi-injective module. Therefore $f \circ j_1(N) \subseteq M$ and $f \circ j_2(H) \subseteq M$. So $f(N \oplus 0) \subseteq M$ and $f(0 \oplus H) \subseteq M$. Therefore $f(N \oplus 0) + f(0 \oplus H) \subseteq M$ .But $f(N \oplus H) \subseteq f(N \oplus 0) + f(0 \oplus H) \subseteq M$. Therefore M is pseudo-$N \oplus H$-quasi-injective module.

Corollary 1.7

If M is an R-module and $N_1, N_2, \dots, N_n$ be an R-modules, such that M is pseudo-N_i-quasi-injective module for each i=1,2,…,n. then M is pseudo-$\oplus_{i=1}^{n} N_i$-quasi-injective module.

Note:

Let M be an R-module, and N is a submodule of M, then it is not necessary that M is pseudo-N-quasi-injective module as it shown in the following example.

Example1.8

The Z-module Z is pseudo-Z-quasi-injective. Let N=3Z be a submodule of the Z-module Z. We claim that Z is not pseudo-3Z-quasi-injective module. For if $f: 3Z \to \bar{Z} = Q$, defined by $f(3n)$ =3n/5 for each $n \in Z$, f is a monomorphism, $f(3Z) \subseteq Q$, but $f(3Z) \not\subseteq Z$. Therefore Z is not pseudo-3Z-quasi-injective module.

However under certain condition on a submodule of M, we could treat the above case. First we need to recall the following definitions. A submodule K of an R-module M is called pseudo-stable submodule if $f(K) \subseteq K$ for each monomorphism $f: K \to M$. M is called a fully pseudo stable if each submodule of M is pseudo stable [1].

Proposition 1.9

Let M be an R-module, and N is a submodule of M. If N is pseudo-stable submodule of $\overline{M}$, then M is pseudo-N-quasi-injective.

Proof

Let $g: N \to \overline{M}$ be a monomorphism. Since N is pseudo-stable submodule of $\overline{M}$, then $g(N) \subseteq N$. Therefore $g(N) \subseteq M$, and hence M is pseudo-N-quasi-injective.Θ

Corollary 1.10

Let N be a submodule of an R-module M. If $\overline{M}$, is fully-pseudo stable R-module, then M is pseudo-N-quasi-injective. In particular M is pseudo-M-quasi-injective.

Corollary 1.11

Let M be fully pseudo-stable module over Notherian ring, and N is submodule of M then M is pseudo-N-quasi-injective.

Proof

Since M is fully pseudo stable R-module over Notherian ring, then E(M) is fully pseudo stable injective envelop of M by [1.Th.2.15 ch.2]. Since $\overline{M}$, is a submodule of E(M), then $\overline{M}$, is a pseudo- stable R-module [1]. Hence the proof followed by corollary 1.10.

Recall that an R-module M is terse if every distinct submodules of it are not isomorphic [12].

It is well known that terse module and fully pseudo-stable module are equivalent [1.Prop.2.11 ch2], we have another consequence of Prop.1.9.

Corollary 1.12

Let N be a submodule of an R-module M. if $\overline{M}$ is terse module, then M is pseudo-N-quasi-injective R-module. In particular M is pseudo-M-quasi-injective R-module.

Recall that an R-module M is Q-module if every submodule of M is quasi-injective [2].

Corollary 1.13

Let M be uniform Q-module over Notherian ring. If N is a submodule of M, then M is pseudo-N-quasi-injective module.

Proof

Since M is uniform Q-module, then M is fully pseudo stable R-module by [1.Th1.4 ch3]. Now M is fully pseudo-stable submodule over Notherian ring. Hence the proof followed by cor.1.12.

Note

The class of pseudo-N-quasi-injective modules is not closed under submodules. For example Q as Z-module is pseudo-Q-quasi-injective, and Z is not pseudo-Q-quasi-injective submodule.

But it turns out that the class of pseudo-N-quasi-injective modules is closed under direct summands. The following proposition shows the case.

Proposition 1.14

A direct summand of pseudo-N-quasi-injective module is also pseudo-N-quasi-injective for any R-module N.

Proof

Suppose that $M = K \oplus H$ be pseudo-N-quasi-injective R-module. Let $f: N \to \bar{K}$ and $g: N \to \bar{H}$ be monomorphisms. Define $h: N \to \bar{K} \oplus \bar{H}$ by $h(n) = (f(n), g(n))$ for all n in N. Clearly, h is well- defined monomorphism. Let $i: \bar{K} \oplus \bar{H} \to \bar{M}$ be the inclusion monomorphism, then $i \circ h: N \to \bar{M}$ is a monomorphism. Since M is pseudo-N-quasi-injective, then $i \circ h(N) = h(N) \subseteq M$. but $h(N) = (f(N), g(N))$.

Therefore $h(N) = (f(N), g(N)) \subseteq M = K \oplus H$, which implies that $f(N) \subseteq K$ and $g(N) \subseteq H$. Hence each of K and H are pseudo-N-quasi-injective.

2- Pseudo-Weakly-N-Quasi-injective modules

In this section we introduce the concept of Pseudo-Weakly-N-Quasi-injective module as a generalization of the concept of weakly-N-injective module, and study the basic properties and give examples, characterizations of this concept, and study the relations of Pseudo-Weakly-N-quasi-injective modules with other know modules.

Definition 2.1

Let M and N be two R-modules. M is called Pseudo-weakly-N-quasi-injective module if for each monomorphism $f: N \to \bar{M}$, there exists a submodule X of $\bar{M}$ such that $f(N) \subseteq X \cong M$.

Examples and Remarks2.2

1. Every weakly-N-injective R-module is pseudo-weakly-N-quasi-injective module, but the converse is not true as the following example shows:

Z_2 as a Z-module is pseudo- weakly-Z-quasi-injective module; however Z_2 is not weakly-Z- injective. For if $f: Z \to E(Z_2) = Z_{2^\infty}$ defined by $f(n) = n/2^3 + Z$ is a homomorphism for all $n \in Z$. it is clear that $f(Z) = Z_8$ which is not embedded in Z_2

2. Every quasi-injective R-Module is pseudo-weakly-N-quasi-injective module for each R-module N.
3. Every pseudo-injective R-module is pseudo-weakly-N-quasi-injective module for each R-module N.
4. Z as a Z-module is not pseudo-weakly-Q-quasi-injective module.
5. Every pseudo-N-quasi-injective R-module is pseudo-weakly-N-quasi-injective R-module. But the converse is not true.

The Z-module Z is pseudo-weakly-2Z-quasi-injective, since if $f: 2Z \to \bar{Z} = Q$, defined by $f(2n) = 2n/3$ for each $n \in Z$ is a monomorphism. We take $X = (2/3)$ the submodule of Q generated by $2/3$. We get $f(2Z) = (2/3) \cong Z$. However Z is not pseudo-2Z- quasi-injective since $f(2Z) \nsubseteq Z$.

The following propositions give some properties of pseudo-weakly-N-quasi-injective modules. Before we give the next proposition we recall the following definition.

An R-submodule H of an R-module M is called fully invariant if $f(H) \subseteq H$ for all $f \in End_R(M)$ [12].

Proposition 2.3

Let N_1 and N_2 be two submodules of an R-module M, such that $N_1 \subseteq N_2$ and N_1 is fully- invariant submodule of $\bar{M}$. If M is pseudo-weakly-N_2 -quasi-injective R-module, then M is pseudo-weakly-N_1-quasi-injective module.

Proof

Let $f: N_1 \to \bar{M}$ be a monomorphism, and consider the following diagram

$$\begin{tikzcd} N_1 \arrow[r, "g_1"] \arrow[d, "f"'] & N_2 \arrow[r, "g_2"] & \bar{M} \arrow[dll, "h"] \\ \bar{M} & & \end{tikzcd}$$

Where g_1, g_2 are the inclusion homomorphisms, since $\bar{M}$ is pseudo-injective R-module, so there exist an R-homomorphism $h: \bar{M} \to \bar{M}$ such that $h \circ g_2 \circ g_1 = f$. But M is pseudo-weakly-N_2-quasi-injective R-module and $g_2: N_2 \to \bar{M}$ is a monomorphism, so there exist a submodule X of $\bar{M}$ such that $g_2(N_2) = N_2 \subseteq X \cong M$. Then $f(N_1) = h \circ g_2 \circ g_1(N_2) = h(N_1) \subseteq N_1$ (since N_1 is a fully-invariant submodule of $\bar{M}$). Therefore $f(N_1) \subseteq N_2 \subseteq X \cong M$. That is $f(N_1) \subseteq X \cong M$. Therefore M is pseudo-weakly-N_1-quasi-injective R-module. Θ

Corollary 2.4

Let N_1 and N_2 are two submodule of an R-module M such that $N_1 \subseteq N_2$ and N_1 is fully invariant submodule of $\bar{M}$. If M is pseudo-weakly-N_2-quasi-injective R-module, then M is pseudo-weakly-$N_1 \cap N_2$-quasi-injective R-module.

Since the intersection of finite collection of fully-invariant submodules of an R-module M is again fully-invariant submodule [4] we have the following corollary as a consequence of proposition2.3.

Corollary 2.5

Let $N_1, N_2, \ldots, N_n$ be finite family of submodules of an R-module M such that N_i is fully-invariant submodule of $\bar{M}$ for all i=1,2,…,n. If M is pseudo-weakly-N_i-quasi-injective R-module for all i=1,2,…,n, then M is pseudo-weakly- $\cap_{i=1}^{n} N_i$ -quasi-injective R-module.

Remark 2.6

A direct summand of pseudo-weakly-N-quasi-injective R-module is not pseudo-weakly-N-quasi-injective R-module for any R-module N, as the following example shows.

Let $M = Z \oplus Q$ and $N = Q$ and $R = Z$ we claim that $Z \oplus Q$ is pseudo-weakly-Q-quasi-injective:

Let $f: Q \to \overline{Z \oplus Q} = Q \oplus Q$ be a monomorphism. Therefore $f(Q) \cong 0 \oplus K_1$ or $f(Q) \cong K_2 \oplus 0$ where K_1 and K_2 are submodules of Q.

Case one: if $f(Q) \cong 0 \oplus K_1$ then $f(Q) \subseteq 0 \oplus Q \subseteq Z \oplus Q \subseteq Q \oplus Q$. Let $X \cong M = Z \oplus Q$. then $f(Q) \subseteq X \cong M = Z \oplus Q$.

Case two: similarly, if $f(Q) \cong K_2 \oplus 0$. Therefore $Z \oplus Q$ is pseudo-weakly-Q-quasi-injective Z-module. But it is clear that Z is not pseudo-weakly-Q-quasi-injective. Θ

Proposition 2.7

Let M be an R-module, and N be a submodule of $\overline{M}$. If M is pseudo-weakly-N-quasi-injective R-module, then M is pseudo-weakly-K-quasi-injective R-module for each submodule K of N.

Proof

Let $f: K \to \overline{M}$ be a monomorphism. Consider the following diagram:

$$\begin{tikzcd} K \arrow[r, "i_1"] \arrow[d, "f"'] & N \arrow[r, "i_2"] & \overline{M} \arrow[dll, "g"] \\ \overline{M} & & \end{tikzcd}$$

Where $i_1: K \to N$ *and* $i_2: N \to \overline{M}$ are the inclusion homomorphism.

Since $\overline{M}$ is pseudo-injective R-module, then there exists a homomorphism $g: \overline{M} \to \overline{M}$ such that $g \circ i_2 \circ i_1 = f$, that is $g = f$ so g is a monomorphism also. Let $h = g|_N: N \to \overline{M}$. h is a monomorphism, also, since M is pseudo-weakly-N-quasi-injective R-module so, there exist a submodule X of $\overline{M}$ such that $h(N) \subseteq X \cong M$. But $f(K) \subseteq X \cong M$. Hence M is pseudo-weakly-K-quasi-injective R-module. Θ

Corollary 2.8

Let M be an R-module, and N, L are submodules of $\overline{M}$. If M is pseudo-weakly-N-quasi-injective R-module, then M is pseudo-weakly-$N \cap L$-quasi-injective R-module.

In the next proposition we show that the class of pseudo-weakly-N-quasi-injective R-module is closed under essential extension.

Proposition 2.9

Let M and N be two R-modules such that K is an essential extension of M. If M is pseudo-weakly-N-quasi-injective R-module, then K is also pseudo-weakly-N-quasi-injective R-module.

Proof

Let $f: N \to \overline{K}$ be a monomorphism. Since K is an essential extension of M Then $\overline{M} = \overline{K}$. Hence $f: N \to \overline{M}$ is a monomorphism. But M is pseudo-weakly-N-quasi-injective, therefore there exists a submodule X of $\overline{M}$ such that $f(N) \subseteq X \cong M$. Let $g: X \to M$ be an isomorphism. Since X is a submodule of $\overline{M} = \overline{K}$, then X is submodule of $\overline{K}$. That is $f_1: X \to \overline{K}$ is the inclusion homomorphism. Consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f_1"'] & M \arrow[r, "f_2"] & K \arrow[r, "f_3"] \arrow[dll, "h_2"'] & \overline{K} \arrow[dlll, "h_1"] \\ \overline{K} & & & \end{tikzcd}$$

Where $f_2: M \to K$ and $f_3: K \to \overline{K}$ are inclusion homomorphisms.

Since $\overline{K}$ is pseudo-injective R-module, then there exists a homomorphism $h_1: \overline{K} \to \overline{K}$ such that $h_1 \circ f_3 \circ g = f_1$. We claim that $Ker h_1 = \{0\}$. Let $0 \neq k_1 \in \overline{K}$ and $h_1(k_1) = 0$. Since M is an essential in K and K is an essential in $\overline{K}$ then M is an essential in $\overline{K}$[5]. Therefore there exists $0 \neq r \in R$ such that $0 \neq rk_1 \in M$, and since g is an epimorphism, there exists $x \in X$ such that $g(x) = rk_1$.

Now $x = h_1 \circ f_3 \circ f_2 \circ g(x) = h_1(rk_1) = rh_1(k_1) = 0$. Hence $rk_1 = 0$ a contradiction, therefore $k_1 = 0$. and hence h_1 is a monomorphism. Let $h_2 = h_1|K: K \to \overline{K}$ be a monomorphism. Then $h_2 \circ f_2 \circ g = f_1$ and hence $f_1(X) = X \subseteq h_2(K) \cong K$. Therefore $f(N) \subseteq X \subseteq h_2(K)$ which mean that K is pseudo-weakly-N-quasi-injective. Θ

The next theorem gives an interesting characterization of pseudo-weakly-N-quasi-injective modules.

Theorem2.10

Let M and N be two R-modules. Then M is pseudo-weakly-N-quasi-injective if and only if for every monomorphism $f: N \to \overline{M}$ there exists an R-monomorphism $h: N \to M$ and R- monomorphism $g: M \to \overline{M}$ such that $g \circ h = f$.

Proof

Assume that M is pseudo-weakly-N-quasi-injective module. Let $f: N \to \overline{M}$ be a monomorphism. Then there exists a submodule X of $\overline{M}$ such that $f(N) \subseteq X \cong M$, this implies that $f: N \to X$ is a monomorphism. Let $\alpha: X \to M$ be an isomorphism we take $h = \alpha \circ f$. So $h: N \to M$ is a monomorphism. Let $g = inc \circ \alpha^{-1}$, where $inc: X \to \overline{M}$ is the inclusion homomorphism. Then $g: M \to \overline{M}$ is a monomorphism. Now $g \circ h = (inc \circ \alpha^{-1}) \circ (\alpha \circ f) = inc \circ f = f$. Which prove the only if part .

To prove the if part: let $f: N \to \overline{M}$ be a monomorphism , then by our hypothesis there exists a monomorphism $h: N \to M$ and a monomorphism $g: M \to \overline{M}$ such that $f = g \circ h$. We take $X = g(M)$. Then X is a submodule of $\overline{M}$ and $X \cong M$. Moreover $f(N) = g(h(N)) \subseteq g(M) = X \cong M$.

Therefore M is pseudo-weakly-N-quasi-injective module Θ.

The next proposition explains the behavior of pseudo-weakly-N-quasi-injective module under isomorphism.

Proposition 2.11
Let M, N and H be an R-modules . if M is pseudo-weakly-N-quasi-injective and $N \cong H$ then M is pseudo-weakly-H-quasi-injective module.

Proof
Let $f: H \to \bar{M}$ be a monomorphism and $g: N \to H$ be an isomorphism. Then $f \circ g: N \to \bar{M}$ is a monomorphism. But M is pseudo-weakly-N-quasi-injective, so there exists a submodule X of $\bar{M}$such that $f \circ g(N) \subseteq X \cong M$. but $f \circ g(N) = f\big(g(N)\big) = f(H) \subseteq X \cong M$. Hence $f(H) \subseteq X \cong M$and so M is pseudo-weakly-H-quasi-injective. Θ

Proposition 2.12
Let M_1 , M_2 and N be R- modules, such that $M_1 \cong M_2$. If M_1 is pseudo-weakly-N-quasi-injective, then M_2 is pseudo-weakly-N-quasi-injective.

Proof
Let $f: N \to \bar{M}_2$be a monomorphism. Since $M_1 \cong M_2$ then $\bar{M}_1 = \bar{M}_2$. Hence $f: N \to \overline{M_1}$ is a monomorphism. But $M_1$is pseudo-weakly-N-quasi-injective R-module, therefore there exists a submodule X of $\bar{M}_1$ such that $f(H) \subseteq X \cong M$. That is g(X) is a submodule of $\bar{M}_2$ under an isomorphism $g: \bar{M}_1 \to \bar{M}_2$. So $X \cong g(X)$. Hence $f(N) \subseteq g(X) \cong X \cong M_1 \cong M_2$ that is $f(N) \subseteq g(X) \cong M_2$ which implies that M_2 is pseudo-weakly-N-quasi-injective. Θ
Before we give the following proposition, we introduce the following definition.

Definition 2.13
A submodule N of an R-module M is called pseudo-invertible if $Hom(M/N, M) = 0$ for each monomorphism $f: M/N \to M$.
Recall that an R-module M is torsion free if $T(M) = \{m \in M: rm = 0 \; for\; some\; r \in R\} = 0$ [7].

Proposition 2.14
Let M be a torsion free R-module and N be pseudo-invertible submodule of M such that M is pseudo-weakly-M/N-quasi-injective. Then N is pseudo-invertible submodule of $\bar{M}$.

Proof
Assume that N is not pseudo-invertible submodule of $\bar{M}$. That is there exists a non-zero monomorphism $f: \bar{M}/N \to \bar{M}$. Therefore there exists $\bar{m} = m + N \in \bar{M}/N$ with $m \in \bar{M} \; and \; m \notin N$ such that $0 \neq f(m+N) = y \; for\; some\; y \in \bar{M}$. Let $i: M/N \to \bar{M}/N$ be the inclusion homomorphism. Then $f \circ i$: $M/N \to \bar{M}$ is a monomorphism. But M is pseudo-weakly-M/N-quasi-injective, so there exists a submodule X of $\bar{M}$ such that $f \circ i(M/N) \subseteq X \cong M$. Let $g: X \to M$ be an isomorphism, then $g \circ f \circ i$: $M/N \to M$ is a monomorphism. But since N is pseudo-invertible submodule of M, therefore $g \circ f \circ i = 0$. $Thus\; f \circ i = 0$ and hence $f(M/N) = 0$. But $f(m+N) \neq 0$, and M is an essential submodule of $\bar{M}$, therefore, there exists $0 \neq r \in R$, such that $rm \in M$. Hence $rm + N \in M/N$ and $f(rm+N) = 0 = rf(m) + N = ry$. But $\bar{M}$ is torsion free, so that $r = 0$, which is a contradiction. Therefore N is pseudo-invertible submodule of $\bar{M}$Θ

Proposition2.15
Let M, $M'$and N be R-modules, such that $\overline{M \oplus M'} = \bar{M} \oplus \overline{M'}$. If M and $M'$are pseudo weakly-N-quasi-injective, then $M \oplus M'$is pseudo weakly-N-quasi-injective module.

Proof
Let $f: N \to \overline{M \oplus M'}$ be a monomorphism , then $f: N \to \bar{M} \oplus \overline{M'}$be a monomorphism. But $f = (f_1, f_2)$ where $f_1: N \to \bar{M} \; and \; f_2: N \to \overline{M'}$are monomorphism. Since M is pseudo weakly-N-quasi-injective, then there exists a submodule X of $\bar{M}$ such that $f_1(N) \subseteq X \cong M$, and since $M'$is pseudo weakly-N-quasi-injective, then there exist a submodule Y of $\overline{M'}$such that $f_2(N) \subseteq Y \cong M'$. on other hand$f_1(N) = f_1(N) \oplus 0 \subseteq X \oplus Y \cong M \oplus M'$ and $f(N) = (f_1(N), f_2(N)) \subseteq X \oplus Y \cong M \oplus M'$, which implies that $M \oplus M'$ is pseudo weakly-N-quasi-injective.Θ
In the following theorem we shall characterize pseudo weakly-R-quasi-injective R-module M.

Theorem 2.16
Let M be an R-module. Then M is pseudo weakly-R-quasi-injective if and only if for each element x of $\bar{M}$ with $ann_R(x) = 0$, there exists a submodule X of $\bar{M}$such that $x \in X \cong M$.

Proof
Assume that M is pseudo weakly-R-quasi-injective R-module. Let $x \in \bar{M}$ such that $ann_R(x) = 0$. Define $f: R \to \bar{M}$ by $f(r) = rx \; for\; all\; r \in R$. Clearly $f$is well-define monomorphism. But M is pseudo weakly-R-quasi-injective R-module. Thus there exists a submodule X of $\bar{M}$ such that $f(R) \subseteq X \cong M$. But $x = 1 \cdot x \in f(R)$. Hence $x \in X \cong M$.
Conversely:
Suppose that for each element $x \in \bar{M}$ with $ann_R(x) = 0$ there exists $x \in X \cong M$. We have to show that M is pseudo weakly-R-quasi-injective. Let $f: R \to \bar{M}$ be a monomorphism. Since $1 \in R$, then $f(1) \in \bar{M}$, let $x = f(1)$, therefore $ann_R(x) = ann_R(f(1)) = f(ann_R(1)) = f(0) = 0$. Hence there exists a submodule X of $\bar{M}$ such

that $x \in X \cong M$. To show that $f(R) \subseteq X$. Let $b \in f(R)$, then $b = f(r)\ for\ some\ r\ in\ R$. Thus $b = rf(1) = rx \in X$. Therefore $f(R) \subseteq X$ and hence M is pseudo weakly-R-quasi-injective R-module. Θ

The next theorem gives a characterization of pseudo weakly-R-quasi-injective R-module R.

Theorem2.17

Let R be a ring. Then R is pseudo weakly-R-quasi-injective R-module if and only if each element $a \in \bar{R}$ with $ann_R(a) = 0$, there exists an element $b \in \bar{R}$ such that $a \in Rb\ and\ ann_R(b) = 0$

Proof

Suppose that R is pseudo weakly-R-quasi-injective R-module. Let $a \in \bar{R}$ such that $ann_R(a) = 0$. Define $f: R \to \bar{R}$ by $f(r) = ra\ for\ all\ r\ in\ R$. It can be easily shown that f is well-defined R-monomorphism. Since R is pseudo weakly-R-quasi-injective R-module, then there exists a submodule X of $\bar{R}$ such that $f(R) \subseteq X \cong R$. Clearly $f(R) = Ra$. Thus $Ra \subseteq X$ which implies that $a = 1.a \in X$. let $g: R \to X$ be an isomorphism. So there exists an element $c \in R$ such that $a = g(c)$. Hence $a = g(c.1) = cg(1) = cb \in Rb$.therefore $a \in Rb$. It is left to show that $ann_R(b) = 0$. Let $r \in ann_R(b)$ .then r.b=0 and hence $0 = rg(1) = g(r.1) = g(r)$, iplise that r=0. Hence $ann_R(b) = 0$.

Conversely:

Let $f: R \to \bar{R}$ be a monomorphism. Then $f(1) \in \bar{R}$. Let $a = f(1)$, then $ann(a)_R = ann(f(1))_R = f(ann(1)_R) = f(0) = 0$. Therefore there exists an elements $b \in \bar{R}$ such that $a \in Rb$ and $ann(b)_R = 0$. We take X=Rb, implies that $X \subseteq \bar{R}$. But $Rb \cong R/ann(b)_R \cong R$. Moreover $f(R) = \{f(r): r \in R\} = \{rf(1): r \in R\} = Ra \subseteq Rb$. Therefore $f(R) \subseteq X \cong R$. Which proved that R is pseudo weakly-R-quasi-injective R-module .Θ

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مقاسات شبه اغمارية – N – ضعيفة كاذبة

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المستخلص

لتكن R حلقة ابدالية بمحايد وM مقاسا أحاديا على R. يقال أن M مقاس اغماري –N، حيث N مقاس علىR إذا كان $f(N) \subseteq M$ لكل $f \in Hom(N,E(M))$ حيث ان E(M) الغلاف الاغماري للمقاس M. ويقال ان M مقاس اغماري –N– ضعيف إذا كان لكل $f \in Hom(N,E(M))$ يوجد مقاس جزئي X من E(M) بحيث ان $f(N) \subseteq X \cong M$. في هذا البحث اعطينا تعميمات لمفاهيم المقاس الاغماري-N و المقاس الاغماري-N ضعيف و اسمينهما بالمقاس شبه الاغماري—N كاذب و المقاس شبه الاغماري-N– ضعيف كاذب على التوالي. يقال ان المقاس M شبه اغماري – N- كاذب اذا كان $f(N) \subseteq M$ لكل تشاكل متباين $f: N \to \bar{M}$ حيث ان $\bar{M}$ هو الغلاف الشبه اغماري للمقاس M. ويقال للمقاس M بانه المقاس شبه الاغماري-N– ضعيف كاذب اذا كان لكل تشاكل متباين $f: N \to \bar{M}$ يوجد مقاس جزئي X من المقاس$\bar{M}$ بحيث ان $f(N) \subseteq X \cong M$.

هدفنا الرئيسي في هذا البحث دراسة الخواص الأساسية لهذين المفهومين وإعطاء امثلة ومكافئات لمفهوم ا لمقاس شبه الاغماري-N– ضعيف كاذب و دراسة علاقة هذه المفاهيم بالمقاسات الأخرى.