Some Kinds of Separation Axioms on Approximation Spaces

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ABSTRACT: In this paper, we study the relationship between the approximate sets induced by equivalence relations and topological spaces. We created a new space known as the topological approximate space by the relation $\tau_\delta = \{Y \subseteq M: \underline{\delta}(Y) = Y\} \cup \phi$. Separation axioms in the induced topological approximate space were also studied and proved it with many theories, properties, and examples.

Keywords: Approximation space , Right neighborhood , Upper approximations , Lower approximations , Separation axiom .

1. INTRODUCTION

Pawlak [1] [2] introduced rough set theory as a solution to issues of detail and ambiguity in information systems. It has proven successful in many areas, and the basic concepts are difficult. Rough sets are defined as functions of a generically. A pair (M, δ) is called approximation space, M is called the universe and δ is known as the equivalency relation on M.This principle supports both upper and lower approximations for subsets of M. Both are based on the equivalence relation on M. The Pawlak approximation group has been generalized in several ways by converting the equivalence relation to any binary relation. Many authors have investigated the relationship between approximate sets and topological space [3 ،4 ،5]. The closure and interior operators of the topology were shown to be represented by the upper and lower approximation operators, which were derived using a transitive and reflexive relationship[6, 7]. Lingyun et al [8].Introduced investigated the topological characteristics of approximation sets and introduced approximate spaces that were generalized using topological techniques. To demonstrate the novel method for regaining missing values for information systems, he provided an example. In this research, we examine the topological characteristics of topological approximate spaces in generic contexts where δ is a equivalence relation on M and M may be infinite. Additionally, we will examine the traditional axioms of connectedness, separation, and compactness in topological approximation spaces before applying them to topological approximate spaces.

2. Preliminaries

Let (M, τ_δ) be a topological approximation space .The relation δ^C is called complement relation of M.
For $n \in M$ .the sets $\delta_N(n) = \{m \in M | n\delta m\}$ is called right neighborhood [9]. (i) For $n \in M$, $(n, n) \in \delta$ is called reflexive, (ii) For all $n, m \in M$, $n\delta m \wedge m\delta n$ is called symmetric, (iii) For all $n, m, z \in M$, $n\delta m \ \wedge \ m\delta z$ then $n\delta z$ is called transitive, $(i), (ii)$ and (iii) is called equivalence relation [10]. β is called base for τ .
I_δ is called Indiscrete approximation space. D_δ is called discrete approximation space.

Definition 2.1: [2] Let (M, δ) is called approximation space.For $Y \subseteq M$.Then,

$\underline{\delta}(Y) = \{x \in M | \delta_N(x) \subseteq Y\}$
$\overline{\delta}(Y) = \{x \in M | \delta_N(x) \cap Y \neq \phi\}$
The operators $\underline{\delta}$, $\overline{\delta}$ are respectively called the lower and Upper approximation operators

Propostion 2.2 : [2] [11] Let (M, δ) be an approximation space.Then:
$(i)\underline{\delta}(Y) \subset Y \subset \overline{\delta}(Y)$
$(ii)\underline{\delta}(M) = \overline{\delta}(M) = M. \overline{\delta}(\phi) = \phi$
$(iii) If\ E \subseteq M$,then $\underline{\delta}(E) = [\overline{\delta}(E^C)]^C$ and $\overline{\delta}(E) = [\underline{\delta}(E^C]^C$
$(iv)\ \ If\ E, N \in M$, $E \subseteq N$ then $\underline{\delta}(E) \subseteq \underline{\delta}(N)\ and\ \overline{\delta}(E) \subseteq \overline{\delta}(N)$

Propostion 2.3 $\tau_\delta = \{Y \subseteq M : \underline{\delta}(Y) = Y\} \cup \phi$ is a topology on M if δ is reflexive.

proof: We only demonstrate that $V_i \in \tau_\delta$ for every $i \in \Lambda$. imply $\cup_i V_i \in \tau_\delta$. Let $V_i \in \tau_\delta$ for every $i \in \Lambda$ and $x \in \cup_i V_i$. There exists $\lambda \in \Lambda$ such that $x \in V_i = \underline{\delta}(V_\lambda)$ For all y such that $x\delta y$, we have
$y \in V_i \subseteq \cup_i V_i$.
This means that
$\cup_i V_i \subseteq \underline{\delta}(\cup_i V_i)$.
that is
$\cup_i V_i = \underline{\delta}(\cup_i V_i)$.
Therefore τ_δ is the topology on M.

Example 2.4 : Let $M = \{a, b, c\}$ and $\delta = \{(a, a), (b, b), (c, c)\}$. is a reflexive relation on δ.
Thus $\delta_N(\{a\}) = \{a\}$, $\delta_N(\{b\}) = \{b\}$, and $\delta_N(\{c\}) = \{c\}$.then
$\underline{\delta}(\{a\}) = \{a\}$, $\underline{\delta}(\{b\}) = \{b\}$, $\underline{\delta}(\{c\}) = \{c\}$
$\underline{\delta}(\{a, b\}) = \{a, b\}$, $\underline{\delta}(\{a, c\}) = \{a, c\}$, $\underline{\delta}(\{b, c\}) = \{b, c\}$, $\underline{\delta}(M) = M$.
$\tau_\delta = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, M, \phi\}$

Corollary 2.5 **:** Let (M, τ_δ) be a T- approximation space .Its shown that if δ is reflexive .Then
$\tau_\delta = \{Y \subseteq M : \underline{\delta}(Y) = Y\} \cup \phi$ is topology on M. Which is called the discrete topology on
(M, δ).

Remark 2.6 : If δ is a reflexive relation on M .The topology by the Lower and upper Approximation operators is the same of the topology by the base.

Theorem 2.7: If δ is transitive and reflexive. Then $\tau_\delta = \{Y \subseteq M : \underline{\delta}(Y) = Y\} \cup \phi$.
$\overline{\delta}$ ($\underline{\delta}$ are respectively). The closure(interior) operators of τ_δ .
proof: since δ is transitive and reflexive. $\underline{\delta}\left(\underline{\delta}(Y)\right) = \underline{\delta}(Y)$ for any $Y \subseteq M$. Consequently,
$\{Y \subseteq M : \underline{\delta}(Y) \cup \phi\} \subseteq \tau_\delta$. $\tau_\delta \subseteq \{Y \subseteq M : \underline{\delta}(Y)\} \cup \phi$ is in Significant.
Let c and i be The closure(interior) operators of τ_δ . since $\underline{\delta}(Y)$ is open and $\underline{\delta}(Y) \subseteq \delta$. we have $\underline{\delta}(Y) \subseteq i(Y)$. On the other hand. For each . $V \subseteq Y$ with $\underline{\delta}(V) = V$. We have
$V = \underline{\delta}(V) \subseteq \underline{\delta}(U)$. hence $i(Y) = \cup V \subseteq Y : \underline{\delta}(V) = V \subseteq \underline{\delta}(Y)$. by the duality. $\overline{\delta}$ and $\underline{\delta}$ are Closure and interior operators of τ_δ.

<u>Propostion 2.8:</u>
Let δ is symmetric on M . for all $Y \subseteq M$. then $\underline{\delta}(Y) = Y$ if $\underline{\delta}(Y^C) = Y^C$.
<u>proof:</u> Assume that $\underline{\delta}(Y) = Y$. It suffices to demonstrate that $Y^C \subseteq \underline{\delta}(Y^C)$. If z $\notin \underline{\delta}(Y^C)$.
then there exists w such that $z\delta w$. but $w \notin Y^C$ .Thus $w \in Y = \underline{\delta}(Y)$. Since δ is symmetric. We have $w\delta z$. hence $z \in Y$. that is z $\notin Y^C$ .we have $Y^C \subseteq \underline{\delta}(Y^C)$.

The contrary can be demonstrated similarly.

Propostion 2.9: [12]
If δ is symmetric and a relation on M, then the topological space (M, τ_δ) has a property that
Y is open $\Leftrightarrow$ Y is closed
proof: since Y is open $\Leftrightarrow Y \in \tau_\delta$.
$\Leftrightarrow \underline{\delta}(Y) = Y$.
$\Leftrightarrow \underline{\delta}(Y^C) = Y^C$.
$\Leftrightarrow \quad Y^C \in \tau_\delta$.
$\Leftrightarrow \quad Y^C$ is open.
$\Leftrightarrow \quad Y \quad is$ closed.

Theorem 2.10: Let (M, τ_δ) be a T- approximation space. Then the family
$\tau_\delta = \{Y \subseteq M : \underline{\delta}(Y) = Y \cup\}\phi$ is a topology on M.

proof: $(1)(i)$ $\phi \in \tau_\delta$ (by definition)
(ii) Since $\underline{\delta}(m) = m$. then $m \in \tau_\delta$.
(2) let $Y, E \in \tau_\delta$. so $\underline{\delta}(Y) = Y$ and $\underline{\delta}(E) = E$.
Then $\underline{\delta}(Y) \cap \underline{\delta}(E) = Y \cap E$.
(3) let $Y, E \in \tau_\delta$. so $\underline{\delta}(Y) = Y$ and $\underline{\delta}(E) = E$.
Then $\underline{\delta}(Y) \cup \underline{\delta}(E) = Y \cup E$.
From $(1), (2)$ *and* (3) we get topology on (M, δ).

3- SEPARATION AXIOMS $T_i^X (i = 0, 1, 2)$ OF APPROXIMATIONS SPACES

Separation axioms are one of the most important branches of topology. The axioms of separation are used in many mathematical fields. In this paper, we study and characterize these axioms, then develop and expand them to an approximate space.We will also describe an approximation spaces by upper and lower approximations. Furthermore, we shall discus separation axioms.

Definition 3.1: [13] A topological space M is said to be a $T_0 -$ space if it satisfy following axiom for any $n, m \in M$, $n \neq m$. There exist an open set Y such that $n \in Y$ but $m \notin Y$.
Definition 3.2: Let (M, τ_δ) be a T- approximation space. Is called T_0^X- approximation space if for all $n, m \in M$. $n \neq m$. There exist an open set $Y \subset M\big(\underline{\delta}(Y) = Y\big) \in \tau_\delta$ such that $(n \in Y \wedge\ m \notin Y$ or $m \in Y\ \wedge n \notin Y)$.

Example 3.3: Let $M = \{a, b, c\}$ and $\delta = \{(a, a), (b, b), (c, c)\}$.
Then $\delta_N(a) = \{a\}$, $\delta_N(\{b\}) = \{b\}$, $\delta_N(\{c\}) = \{c\}$.
$\beta = \{M, \phi, \{a\}, \{b\}, \{c\}\}$.
$\tau_\delta = \{M, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.
$a \neq b, \exists\ \{a\} \in \tau_\delta$ such that $a \in \{a\} \quad \wedge \quad b \notin \{a\}$.
$a \neq c, \exists\ \{a\} \in \tau_\delta$ such that $a \in \{a\} \quad \wedge \quad c \notin \{a\}$.
$b \neq c, \exists\ \{b\} \in \tau_\delta$ such that $b \in \{b\} \quad \wedge \quad c \notin \{b\}$.
Therefore (M, τ_δ) is T_0^X-approximation space.

Proposition 3.4: In the Indiscrete approximation space (M, I_δ) if M is any set containing more than one element,Then Indiscrete approximation space (M, I_δ) is not T_0^X-approximation space. Since M contains more than one element we take $n, m \in M$,$n \neq m$ and $\exists Y \subset M\big(\underline{\delta}(Y) = Y\big)$ containing n but not m or an Y containing m but not n .
Therefore Indiscrete approximation space is not T_0^X-approximation space.

Corollary 3.5: The discrete approximation space (M, D_δ) is T_0^X–approximation space.

proof: Let $n, m \in M$, $n \neq m$. $\exists\ Y \subset M\big(\underline{\delta}\ \{Y\} = Y\big) \in D_\delta$ such that $n \in Y$ and $m \notin Y$.
Therefore discrete approximation space (M, D_δ) is T_0^X–approximation space.

Theorem 3.6: Let (M, τ_δ) be a T- approximation space. Then we have statements are equivalent:
(i) (M, τ_δ) is T_0^X-space .
(ii) For all $n, m \in M$. $n \neq m$ implies $\overline{\delta}(n) \neq \overline{\delta}(m)$

proof: $(i) \Rightarrow (ii)$ Let n, m be any two distinct points of M . We show that $\overline{\delta}(n) \neq \overline{\delta}(m)$. Since (M, δ) is T_0^X-space . $\exists\ Y \subset M\big(\overline{\delta}(Y) = Y\big)$ such that $n \in Y$ and $m \notin Y$ (by Definition).
Hence $m \in Y^C$. So $m \in \overline{\delta}(m)$ as $n \notin \{Y\}^C$.
Therefore $\overline{\delta}(n) \neq \overline{\delta}(m)$.
$(ii) \Rightarrow (i)$ Suppose that for every $n, m \in M$. $n \neq m$ and $\overline{\delta}(n) \neq \overline{\delta}(m)$. Let $v \subset M$ such that $v \subset \overline{\delta}(n)$. hence $v \not\subset \overline{\delta}(m)$. If $n \in \overline{\delta}(m)$ .Then $n \subseteq \overline{\delta}(m)$ which implies that $\overline{\delta}(n) \subseteq \overline{\delta}(m)$ and hence $v \subset \overline{\delta}(m)$ which is a contradiction . thus $n \notin \overline{\delta}(m)$.
Therefore (M, τ_δ) is T_0^X– approximation space.

Example 3.7: Let $M = \{1,2,3\}$ and $\delta = \{(1,1), (2,2), (3,3), (3,1), (1,3)\}$.
Then $\delta_N(1) = \{1,3\}$, $\delta_N(2) = \{2\}$, $\delta_N(3) = \{3,1\}$.
$\tau_\delta = \{M, \phi, \{2\}, \{1,3\}\}$.
Since $\overline{\delta}(\{2\}) \neq \overline{\delta}(\{1,3\})$, $\overline{\delta}(\{M\}) \neq \overline{\delta}(\{1,3\})$, $\overline{\delta}(\{M\}) \neq \overline{\delta}(\{2\})$.
Is T_0^X–approximation space.

Definition 3.8: **[13]** A topological space M is said to be a $T_1 -$ space if it satisfy following axiom for any $n, m \in M$, $n \neq m$. There exist two an open set Y, K such that $n \in Y$ and $m \in K$.

Definition 3.9: Let (M, τ_δ) be a T- approximation space. Is called T_1^X– approximation space if for all $n, m \in M$. $n \neq m$.
$\exists\ Y, K \subset M\big(\underline{\delta}(Y) = Y, \underline{\delta}(K) = K\big) \in \tau_\delta$ such that $(n \in Y \wedge m \notin Y)$ or $(m \in K \wedge n \notin K)$.

Example 3.10: Let $M = \{1,2,3\}$ and $\delta = \{(1,1), (2,2), (3,3)\}$.
Then $\delta_N(1) = \{1\}$, $\delta_N(2) = \{2\}$, $\delta_N(3) = \{3\}$.
$\beta = \{M, \phi, \{1\}, \{2\}, \{3\}\}$.
$\tau_\delta = \{M, \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$.
$1 \neq 2, \exists\ \{1\}\ ,\ \{2\} \in \tau_\delta$ such that $(1 \in \{1\} \quad \wedge \quad 2 \notin \{1\})$ or
$(2 \in \{2\} \quad \wedge \quad 1 \notin \{2\})$.
$1 \neq 3, \exists\ \{1\}\ ,\ \{3\} \in \tau_\delta$ such that $(1 \in \{1\} \quad \wedge \quad 3 \notin \{1\})$ or
$(3 \in \{3\} \quad \wedge \quad 1 \notin \{3\})$.
$2 \neq 3, \exists\ \{2\}\ ,\ \{3\} \in \tau_\delta$ such that $(2 \in \{2\} \quad \wedge \quad 3 \notin \{2\})$ or
$3 \in \{3\} \quad \wedge \quad 2 \notin \{3\})$.
Therefore (M, τ_δ) is T_1^X–approximation space.

Proposition 3.11: The discrete approximation space (M, D_δ) is T_1^X–approximation space.
proof: Let $n, m \in M$, $n \neq m$. $\exists Y, K \subset M\left(\underline{\delta}\{Y\} = Y, \underline{\delta}\{K\} = K\right) \in D_\delta$ (by definition)
We get ($n \in Y$ and $m \notin Y$) or ($m \in K$ and $n \notin K$).
Therefore discrete approximation space (X, D_δ) is T_1^X–approximation space.

Corollary 3.12: Let (M, τ_δ) be a T-approximation space. Every

T_1^X–approximation space is T_0^X–approximation space.

proof: Let $n, m \in M, n \neq m$. since M is T_1^X–approximation space.
$\exists\ Y \subset M(\underline{\delta}(Y) = Y) \in \tau_\delta$ such that $n \in Y$ but $m \notin Y$ and $m \in Y$ but $n \notin Y$.
Therefore (M, τ_δ) is T_0^X–approximation space.

Definition 3.13: [13] A topological space M is said to be a $T_2 -$ space if it satisfy following axiom for any $n, m \in M$, $n \neq m$. There exist two an open set Y, K such that $n \in Y$ and $m \in K$, $Y \cap K = \phi$.

Definition 3.14: Let (M, τ_δ) be a T- approximation space. Is called T_2^X–approximation space if for all $n, m \in M$, $n \neq m$ .There exist two an open set $Y, K \subset M(\underline{\delta}(Y) = Y, \underline{\delta}(K) = K) \in \tau_\delta$ such that $(n \in Y\ and\ m \in K)$. $Y \cap K = \phi$.

Example 3.15: Let $M = \{1,2,3\}$ and $\delta = \{(1,1),(2,2),(3,3)\}$.
Then $\delta_N(1) = \{1\}$, $\delta_N(2) = \{2\}$, $\delta_N(3) = \{3\}$.
$\tau_\delta = \{M, \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$.
$1 \neq 2, \exists\ \{1\}\ ,\ \{2\} \in \tau_\delta$ such that $(1 \in \{1\} \quad \wedge \quad 2 \in \{2\})$.
$\{1\} \cap \{2\} = \phi$.
$1 \neq 3, \exists\ \{1\}\ ,\ \{3\} \in \tau_\delta$ such that $(1 \in \{1\} \quad \wedge \quad 3 \in \{3\})$.
$\{1\} \cap \{3\} = \phi$
$2 \neq 3, \exists\ \{2\}\ ,\ \{3\} \in \tau_\delta$ such that $(2 \in \{2\} \quad \wedge \quad 3 \in \{3\})$.
$\{2\} \cap \{3\} = \phi$
Therefore (X, τ_δ) is T_2^X–approximation space

Proposition 3.16: The The discrete approximation space (M, D_δ) is T_2^X–approximation space .

proof: Let $n, m \in M, n \neq m. \Longrightarrow \{Y\}\ and\ \{K\} \in D_\delta$,$\{\ Y\} \cap \{K\} = \phi$ (by definition)
$(n \in \underline{\delta}\{n\}\ \ and\ \ m \in \underline{\delta}\{m\})$.
Therefore discrete approximation space $(M, D_\delta)\ is\ T_2^X$–approximation space.

Corollary 3.17: Let (M, τ_δ) be a T- approximation space. Every T_2^X– approximation space is T_1^X–approximation space.

proof: Let $n, m \in M, n \neq m$. since M is T_2^X–approximation space.
$\exists\ Y, K \subset M(\underline{\delta}(\{Y\}) = Y,\ \underline{\delta}(\{K\}) = K) \in \tau_\delta$ such that $n \in Y$ but $m \notin Y$ and $m \in K$ but $n \notin K$.
Therefore $(M, \tau_\delta)\ \ is\ T_1^X$–approximation space.

4. CONFLICTS OF INTEREST

In this paper we have some results as shown below

1- Let (M, τ_δ) be a T- approximation space. Then the family $\tau_\delta = \{Y \subseteq M : \underline{\delta}(Y) = Y \cup\}\phi$ is a topology on M.

2- $\tau_\delta = \{Y \subseteq M : \underline{\delta}(Y) = Y\} \cup \phi$ is a topology on M if δ is reflexive.

3- Let δ is symmetric on M . for all $Y \subseteq M$. then $\underline{\delta}(Y) = Y$ if $\underline{\delta}(Y^C) = Y^C$.

4- If δ is transitive and reflexive. Then $\tau_\delta = \{Y \subseteq M : \underline{\delta}(Y) = Y\} \cup \phi$.

$\overline{\delta}$ ($\underline{\delta}$ are respectively). The closure(interior) operators of τ_δ .

5- Let (M, τ_δ) be a T- approximation space. Then we have statements are equivalent:

(i) (M, τ_δ) is T_0^X-space .
(ii) For all $n, m \in M$. $n \neq m$ implies $\overline{\delta}(n) \neq \overline{\delta}(m)$.

6- Let (M, τ_δ) be a T-approximation space. Every T_1^X–approximation space is T_0^X–approximation space.

7- Let (M, τ_δ) be a T- approximation space. Every T_2^X– approximation space is T_1^X–approximation space.

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