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Shifted modified chebyshev direct method for solving quadratic optimal control problem

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Abstract

The area of Chebyshev polynomial functions plays an important role in mathematic with applications in computer science and engineering. New shifted modified Chebyshev polynomials are considered in the present article. An explicit expression of the definition for shifted modified Chebyshev polynomials with some important relations and interesting properties is first derived. Then new expression formulation for constructing shifted modified Chebyshev operation matrix of derivative is given. Such polynomials are used as basis functions to propose and analyze direct numerical technique to optimal control problem having a quadratic performance index, based on shifted modified Chebyshev polynomials together with its operation matrix of derivative, such a problem is reduced into an optimization technique which can be solved easily using quadratic programming algorithm. For confirming the validity and accuracy of the presented method some numerical examples are included a long with a comparison between the obtained results against the exact one.

Introduction:

Optimal control problem is very important role in many application areas including economies, engineering and finance. They are a branch of optimization theory which concerned with minimizing post or maximizing a payout pertaining. There is an important and very useful method for solving optimal control problems. This method is labelled as direct method in which the optimal solution is obtained by direct minimization of the cost function subject to constraint. Therefor, three techniques by employing the direct method: state parameterization reducing the problem to a nonlinear optimization problem [1-11].

Chebyshev polynomials control and state variables are expanded by Chebyshev polynomials and Chebyshev spectral procedure was introduced by [12]. In 2019 A dane presented a technique using Chebyshev polynomials for solving linear quadratic optimal control system [13]. Other method based on Chebyshev polynomials can be found in [14-16].

In the present article, the sections are organized as follows: section 2 gives new bases functions called shifted modified Chebyshev polynomial with an explicit expression formula. Operational matrix of derivative for such polynomials is constructed in section 3 while an efficient approach together with the obtained operation matrix using state vector parameterization is illustrated in section 4. Section 5 gives numerical examples to illustrate the reliability of the suggested method. Finally, some concluded remarks are discussed in section 6.

1. Shifted Modified Chebyshev Polynomials:

The recurrence relation of shifted Chebyshev polynomials is

$$Ms_{m+1}(t) = (2t - 1)Ms_m - Ms_{m-1}, \qquad \forall\, n \geq 1 \tag{1}$$

with the initial conditions.

$Ms_0(t) = 2$ and $Ms_1(t) = 2t - 1$

The Pascal array generates shifted modified chebyshev polynomials is

$$\begin{array}{lllllll} 2 & & & & & & \\ -1 & 2 & & & & & \\ -1 & -4 & 4 & & & & \\ 2 & 0 & -12 & 8 & & & \\ -1 & 8 & 8 & -32 & 16 & & \\ -1 & -10 & 20 & 40 & -80 & 32 & \\ 2 & 0 & -48 & 32 & 144 & -192 & 64 \end{array}$$

The explicit expression of shifted modified Chebyshev polynomials can be defined as

$$Ms_m(t) = \sum_{k=0}^{[\frac{m}{2}]} (-1)^k \frac{m}{m-k} \binom{m-k}{k} (2t-1)^{m-2k}, \; m \in N \tag{2}$$

2. Operation Matrix of Differentiation for Shifted Modified Chebyshev Polynomials:

Theorem 1

For all $m \geq 1$, the first derivative of shifted Chebyshev polynomials interims of their shifted modified Chebyshev polynomials themselves is given as

$$\dot{M}s_m(t) = D\, M\, s\, (t) \tag{3}$$

where the matrix D is the operational matrix derivative defined by

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & \cdots 0 \\ 3 & 0 & 6 & 0 & 0 & 0 & \cdots 0 \\ 0 & 8 & 0 & 8 & 0 & 0 & \cdots 0 \\ 5 & 0 & 10 & 0 & 10 & 0 & \cdots 0 \\ 0 & 12 & 0 & 12 & 0 & 12 & \cdots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 2m & 0 & 2m & 0 & 2m & \cdots 2m \end{bmatrix}$$

For even m, while the matrix D is defined for odd m as

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & \cdots 0 \\ 3 & 0 & 6 & 0 & 0 & 0 & \cdots 0 \\ 0 & 8 & 0 & 8 & 0 & 0 & \cdots 0 \\ 5 & 0 & 10 & 0 & 10 & 0 & \cdots 0 \\ 0 & 12 & 0 & 12 & 0 & 12 & \cdots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ m & 0 & 2m & 0 & m & 0 & \ldots . m \end{bmatrix}$$

From the obtained matrix of derivative one conclude that

$$\dot{M}s\,(t) = 2m \sum_{\substack{i=1 \\ i=odd}}^{m-1} Ms_i\,(t) \qquad \text{for even } m \tag{4}$$

and

$$\dot{M}s(t) = 2m \sum_{\substack{i=2 \\ i\ even}}^{m-1} Ms_i\,(t) + mMs_0(t) \quad \text{for odd } m \tag{5}$$

Theorem 2

Suppose that the function $x(t)$ can be expressed as a linear combination of shifted modified Chebyshev polynomials as

$$x(t) = \sum_{i=0}^{m} S_i\, Ms_i\,(\mathrm{t}) \tag{6}$$

Then the derivative of $x(t)$ with respect to t is given by

$$\dot{x}\,(\mathrm{t}) = \sum_{i=0}^{m-1} t_i\, Ms_i(t) \tag{7}$$

$$\varphi t_{m-2} = 2(m-1)S_{m-1}$$

$$t_{r-1} = t_{r+1} + 2rS_r, = (m-2)\,,\ldots\ldots, 2, 1$$

3. The Suggested Design Approach:

Suppose that the problem of a finding a control variable for a system

$$\dot{x}(t) = f(t\,, x(t)\,, u(t)\,) \tag{8}$$

where $u(t) : [0\,,1] \to R^n$ and $x(t) : [0\,,1] \to R^n$ and f is a real valued continuously differentiable function the aim is to find a pricewise continuous u^* and the related variable x^* which optimize in a sense the performance index

$$\mathrm{J} = \int_0^1 L\big(\,t\,, x(t), u(t)\big)dt \tag{9}$$

Subject to the conditions together with Eq. 8

$$X(0) = x_0 \tag{10}$$

where $x_0 \in R^n$

A new parameterization using shifted modified Chebyshev polynomials

Step 1

Express u(t) as a function of $t\,, x(t)\,, \dot{x}\,(t)$

$$u(t) = \varphi(t\,, x(t)\,, \dot{x}\,(t)) \tag{11}$$

Step 2

Expand $x(t)$ as a linear combination of shifted Chebyshev series
$x_m(t) = \sum_{k=0}^{m} a_k\, Ms_k\,(\mathrm{t}),\ m = 1,2, \dots ..$

or $x_m(t) = A^T \mathrm{M}s(t)$ (12)

Step 3

Rewrite Eq. 11 using Eq. 12
$u_m(t) = \varphi\ (t\,, \sum_{k=0}^{m} a_k\, Ms_k(t)\,, \sum_{k=0}^{m} a_k\, \dot{M}s_k(t)$

$u_m(t) = (\,t, A^T M\, s(t), A^T DM\, s(t)\,)$ (13)

where $A^T = [a_{0\,,}\, a_1, \dots, a_m]^T$

$Ms(t) = [Ms_0\,, Ms_1, \dots, Ms_m]$

Step 4

Substitute the approximation given in Eq. 12 and Eq. 13 into Eq. 9
$\mathrm{J} = \int_0^1 L\,(\,t\,,\ A^T Ms\,(t), \varphi(\mathrm{t}),\ A^t M\, s(t)\,, A^T DMs(t)\,)\,d\,t$ (14)

Thus, the problem is reduced to quadratic function of the unknown vector A^T
Step 5

Replace the initial condition in Eq. 10 to be equality constraint as
$x_n(0) = \sum_{k=0}^{m} a_k\, Ms_k(0) = x_0$ (15)

The converted new problem will be
$\min_{a \in R^{n+1}} \{A^T\, HA\}$ (16)

Subject to $F\, a = b$ (17)

Note that Eq. 15 leads to the linear constraint, Eq. 17

4. Numerical Examples:

The efficiency of the presented method is illustrated by considering two numerical examples. The analytical solution is given for each example to allow validation of the method by comparing the obtained results with the corresponding exact solution.

Example 1

The aim is to obtain the optimal control which minimize

$J = \frac{1}{2}\int_0^1 (\,x^2 + u^2\,)\,d\,t$ (18)

when $u(t) = \dot{x}(t) + x(t)$ (19)

$x(0) = 1$ (20)

J_{exact} = 0.1929092978

This example contains state variable $x(t)$ and control variable $u(t)$. Here $x(t)$ approximated by Shifted modified Chebyshev polynomials of unknown parameters, then $u(t)$ can be obtained from Eq. 19. By substituting the conditions from Eq. 20 in $x(t)$ and substituting $x(t)$ and $u(t)$ in Eq. 18 to find optimal value J^*

In this approach, the state variable $x(t)$ and control variable $u(t)$ are approximate by

$$x(t) = \sum_{i=0}^{3} a_i \, Ms_i(t) \qquad (21)$$

where $Ms_i(t)$ is shifted modified Chebyshev polynomials, the control variable is found from Eq. 19 using the shifted modified Chebyshev operational matrix of derivative, yields

$$u(t) = (a_0 + a_1)Ms_0(t) + (a_1 + 4a_2)Ms_1(t) + a_2 Ms_2(t) \qquad (22)$$

The expression for J is obtained by substitution Eq. 21 and Eq. 22 into Eq. 18,

$$J = [a_0 \; a_1 \; a_2]\begin{bmatrix} 8 & 4 & -6.6666 \\ 4 & 4.6666 & -0.2001 \\ -6.6666 & -0.2001 & 11.0662 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \qquad (23)$$

The following additional equation can be obtained using Eq. (20)

$$2a_0 - a_1 - a_2 = 1 \qquad (24)$$

The reduced problem is to minimize Eq. 23 subjects to Eq. 24. Standard quadratic programming method is used to obtain the optimal values a_0 a_1 a_2 as follows

$a_0 = 0.3877, a_1 = -3.583, a_2 = 0.1337$

Table 1 lists the optimal functional values, J for different values of n together with the exact error.

Table 1 optimal functional values for Example 1

N	The optimal value	Exact Error $\lvert J_{exact} - J_{opp} \rvert$
2	0.1943	-0.0014
3	0.1929	0.0000
4	0.1929	0.0000

A good approximation is obtained when $n = 4$ with the optimal values are

$a_0 = 0.3978, a_1 = -0.4161, a_2 = 0.1490, a_3 = -0.0286, a_4 = 0.0055$

Figure 1 plots the obtained approximate solution $x(t), u(t)$ for $n = 4$ against the analytic one given by

$$x(t) = \frac{2\sqrt{2}-3}{-e^{2\sqrt{2}}+2\sqrt{2}-3} e^{\sqrt{2}t} + \left(1 - \frac{2\sqrt{2}-3}{-e^{2\sqrt{2}}+2\sqrt{2}-3}\right) e^{-\sqrt{2}t}$$

and $u(t) = \frac{2\sqrt{2}-3}{-e^{2\sqrt{2}}+2\sqrt{2}-3}(\sqrt{2}+1)e^{\sqrt{2}t} - \left(1 - \frac{2\sqrt{2}-3}{-e^{2\sqrt{2}}+2\sqrt{2}-3}\right)(\sqrt{2}-1)e^{-\sqrt{2}t}$

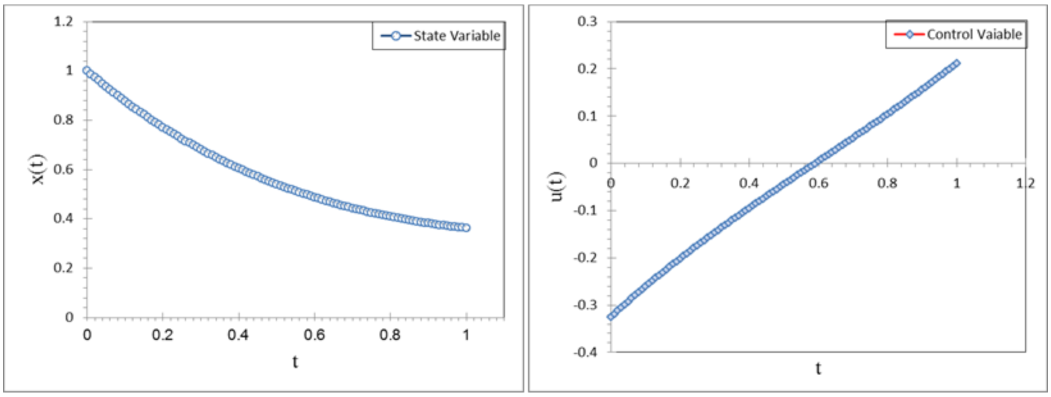

Fig. 1: the approximate solution x(t),u(t) for n = 4 and the exact solution of example 1

Example 2

$J = \int_0^1 (x^2 + u^2) d\, t$ (25)

Subject to $\dot{x} = u$ (26)

$x(0) = 1$ (27)

In this example there is state variable $x(t)$ and control variable $u(t)$. Here $x(t)$ approximated by Shifted modified Chebyshev polynomials with unknown parameters and $u(t)$ can be obtained by Eq. 26. Then by substituting the condition from Eq. 27 in $x(t)$ and substituting $x(t)$ and $u(t)$ in Eq. 25 to find optimal value J^*.

Now the state variable $x(t)$ and control variable $u(t)$ are approximate by

$x(t) = \sum_{i=0}^{3} a_i\, M_i^* \text{ (t)}$

We have numerical results obtained by table 2

Table 2: Numerical results for example 2

N	The optimal value	Exact Error $\lvert J_{exact} - J_{opp} \rvert$
2	0.7618	-0.0003
3	0.7616	-0.0001
4	0.7616	-0.0001

Figure 2 plots the obtained approximate solution $x(t), u(t)$ for $n = 4$ against the analytic one given by

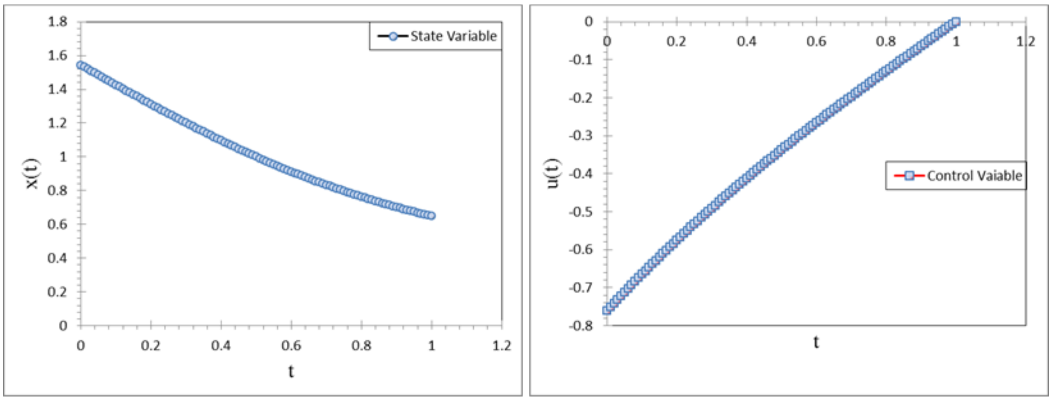

Fig. 2: the approximate solution $x(t), u(t)$ for $n = 4$ and the exact solution of example 2

Discussion

New and efficient algorithm for solving special class of optimal control problem called quadratic optimal control problem is presented. The approximated solution is based on state parameterization using new basis function, named shifted modified Chebyshev polynomials. First, an optimization problem $(n+1) \times (n+1)$ dimension space is reduced to one dimensional optimization problem. The obtained reduced problem can be solved easily. Illustrative examples are included to show the accuracy of the proposed method.

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طريقة مباشرة لشيبيشف المحسنة المزاحة لحل مسائل السيطرة المثلى التربيعية

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متعددات حدود شيبيشف المطورة المزاحة، مسألة السيطرة المثلى، مصفوفة العمليات، أمثلة عددية، الدوال المتعامدة

الخلاصة:

تلعب مساحة دوال متعددات حدود شيبيشف دوراً مهماً للبحوث في الرياضيات مع التطبيقات في علوم الحاسبات والهندسة. في هذا البحث، تم دراسة دوال شيبيشف المحسنة المزاحة الجديدة. تم اشتقاق صيغة صريحة لتعريف متعددات حدود شيبيشف المحسنة المزاحة ثم كونت مصفوفة العمليات الجديدة لمشتقات متعددات الحدود المقدمة استخدمت كدوال اساسية لاقتراح وتحليل اسلوب عددي مباشر لحل مسائل السيطرة المثلى التي تمتلك معامل اداء تربيعي. بالاعتماد على متعددات حدود شيبيشف المحسنة مع مصفوفة العمليات للمشتقة، ثم اختزال مسالة السيطرة المثلى الى مسالة امثليه وممكن حلها بسهولة باستخدام خوارزمية البرمجة التربيعية. لتأكيد قابلية ودقة الطريقة المقترحة. تم حل بعض الامثلة العددية مع المقارنة بين النتائج التي حصلنا عليها مع الحل الحقيقي.