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Best Co-Approximation In Weighted Space

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we present best co- approximation in weighted space. The results considered are these of existence of functions of best co- approximation, specifications of co-a proximal and specification co- Chebyshev subspaces.

1. INTRODUCTION

Estimates in approximation theory that (cf [1] , [2], [3], [4]and [5])has recently introduced [6] and [7] As in the case of best approximation, the theory of best co-approximation has been developed to a from obvious in metric space and Banach Characterization, Characterization of co-a proximal space by [8], [9] and [10].

In a class of papers many authors have proved many results on best co approximation in metric space and normed space as [11] and [12]. In this paper introduced some results on existence of functions of best co- approximation specifications of co-a proximal and specification co- Chebyshev subspaces.

Let $X = [-1,1]$, W the set of all weighted functions which as $w: X \to R^+$ and $L_{p,w}(X)$ the space of all unbounded functions, $1 \le p < \infty$ with norm for $f \in L_{p,w}(X)$,

$$\|f\|_{p,w} = \left(\int_{-1}^{1} |f(x)\, w(x)|^p \, dx\right)^{\frac{1}{p}} < \infty.$$

Let A be a subspace of $L_{p,w}(X)$, $f \in L_{p,w}(X)$, find a function a^* in A such that

$$\|f - a^*\|_{p,w} \le \|f - a\|_{p,w} \text{ for every } a \in A$$

We say thata^* is best co approximation of f and the set of best approximation of f by the functions of A is denoted by $\mathcal{P}_\delta$. Clearly

$$\mathcal{P}_\delta(f) = \{\cap_{a\epsilon A} \overline{\mathrm{B}}(f, \|f - a\|_{p,w})\} \cap A$$

Where $\overline{\mathrm{B}}(f, \|f - a\|_{p,w})$ denotes the closed bull in $L_{p,w}(X)$.

As a peer to best co approximation, and the kind of approximation called best co approximation which define by.

If $a^* \in A$, then

$$\|f - a\|_{p,w} \ge \|a^* - a\|_{p,w} \quad \text{for } a \in A.$$

We say that $a^*$is best co-approximation of f and the set of best co-approximation of f by the functions of δ is denoted by $\mathcal{Q}_\delta$. Clearly

$$\mathcal{Q}_\delta(f) = \{\cap_{a\epsilon A} \overline{\mathrm{B}}(f, \|f - a\|_{p,w})\} \cap A$$

The set A is said to be proximal (respectively co-proximal) if $\mathcal{P}_\delta(f)$ (respectively $\mathcal{Q}_\delta(f)$) is non-empty for each $f \in L_{p,w}(X)$, it's said to be chebyshev (respectively chebyshev) $\mathcal{P}_\delta(f)$ (respectively$\mathcal{Q}(f)$) contains exactly one element foe each $f \in L_{p,w}(X)$.

The function $f \in L_{p,w}(X)$ is said to be orthogonal to author function $g \in L_{p,w}(X)$ and denoted by$f \perp g$ if $\|f - g\|_{p,w} \le \|f - \beta g\|_{p,w}$ for any scalar β and f is orthogonal to subspace A of $L_{p,w}(X)$ and denoted by$f \perp A$ if $f \perp a$ for

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$a \in A$.

2. Auxiliary Lemma:

Lemma 2.1: If A is subspace of $L_{p,w}(X)$. Then

$A \subset Q_\delta(f)$ when even the diameter of A is smaller then dist $\|f-A\|_{p,w}$

$\delta(A) < \|f-A\|_{p,w}$, where $\delta(A)$ is the diameter of A.

Lemma 2.2: if A is aconvex subset of strongly locally convex $L_{p,w}(X)$, then $Q_\delta(f)$ is a convex set.

Lemma 2.3: if A is a linear subspace of $L_{p,w}(X)$ and $Q_\delta(0) = \{f \in L_{p,w}(X) : 0 \in Q_\delta(f)\}$. Then

(i) $(Q_\delta(0))^{-1}$ is a closed set containing 0.

(ii) $a_0 \in Q_\delta(f) \Leftrightarrow 0 \in Q_\delta(f - a_0)$ i.e $f - a_0 \in (Q_\delta(0))^{-1}$.

(iii) for $a \in A$ we have $h \in (Q_\delta(a))^{-1} \Leftrightarrow a \in Q_\delta(a+h)$ i.e. $a + h \in (Q_\delta(a))^{-1}$.

Remark 2.4: If $L_{p,w}(X)$ is unbounded functions space and $f_\circ$ fixed function in $L_{p,w}(X)$, then the space

$$\overline{\overline{L_{p,w}(X)}} = \left\{ f:[-1,1] \to \mathbb{R},\ \|f\|_{p,w} = \underbrace{sup}_{\substack{x,y\in[-1,1] \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|} \right\}$$, is complete normed space.

3. Main Results

Theorem 3.1:

Let $f \in L_{p,w}(X)$, $1 \le p < \infty$ and A is linear subspace of unbounded functions space. Then $\mathcal{F}(f)$ is non-empty set for some $f \in L_{p,w}(X)/A$ if and only if $(Q_\delta(f))^{-1}$ (0) is non-singleton.

Proof:- By Lemma 2.3 (ii), then $0 \in (Q_\delta(f))^{-1}$ (0).

Now, suppose $g \in \mathcal{F}(f)$ for some $f \in L_{p,w}(X)/A$. Then By Lemma 2.3 (ii)

$\|f-g\|_{p,w} \neq 0 \in (Q_\delta(f))^{-1}$ (0) and so, $(Q_\delta(f)))^{-1}$ (0) is not a singleton .

Conversely: assume $(Q_\delta(f))^{-1}$ (0) *is* not a singleton.

Then there exists an $(Q_\delta(f) \neq \emptyset) \in (Q_\delta(f))^{-1}$ (0)

And so $0 \in Q_\delta(f)$

i.e. $Q_\delta(f) \neq \emptyset$ for some $f \in L_{p,w}(X)/A$.

■

Theorem 3.2:

Let $L_{p,w}(X)$ be the space of all monatons unbounded function , $1 \le p < \infty$, A is linear subspace of $L_{p,w}(X)$,

$f \in L_{p,w}(X) / A$ and $\lambda \in A$ .Then $\lambda \in \mathcal{F}(f)$, for each $\lambda \in A$, there is an $f_\lambda \in \overline{\overline{L_{p,w}(X)}}$ with the following properties

(i) $|f_\lambda(f) - f_\lambda(\lambda)| \le \|f-\lambda\|_{p,w}$ $\forall f \in L_{p,w}(X) \& \lambda \in A$.

(ii) $f_\lambda(f-\lambda) = 0$.

(iii) $f_\lambda(\lambda_0 - \lambda) = \|\lambda_0 - \lambda\|_{p,w}$.

Proof: Assume that for every $\lambda_\circ \in A$, there exist $f_\lambda \in \overline{\overline{L_{p,w}(X)}}$, satisfies the above conditions (i) ,(ii) and (iii), then

$\|f-\lambda_\circ\|_{p,w} \ge |f_\lambda(f) - f_\lambda(\lambda_\circ)|$ by condition (i).

Since, $|f_\lambda(f) - f_\lambda(\lambda_\circ)| = |f_\lambda(f - \lambda + \lambda - \lambda_\circ)| = |f_\lambda(\lambda - \lambda_\circ)|$ by condition (ii).

$f_\lambda(\lambda - \lambda_\circ) = \|\lambda_0 - \lambda\|_{p,w}$ by condition (iii).

Thus, $\|\lambda_0 - \lambda\|_{p,w} \le \|f-\lambda_\circ\|_{p,w}$ for every $\lambda_\circ \in A$.

Hence, λ is co- approximation of , implies $\lambda \in Q_\delta(f)$.

■

Theorem 3.3:

Let A be a linear subspace of $L_{p,w}(X)$ with $Q_\delta(f) = \emptyset$, for every $f \in L_{p,w}(X)$, there exist no $\in L_{p,w}(X) \ni A \perp \hbar$.

Proof :- Suppose there exists some $\hbar \in L_{p,w}(X)$ such that $A \perp \hbar$, i.e. $a \perp \hbar$ for every $a \in A$. Then

$\|f - \beta\hbar\|_{p,w} \ge \|f-0\|_{p,w}$, for all $f \in L_{p,w}(X)$ and for all scalars β.

This gives

$\|f-0\|_{p,w} \le \|\hbar - f\|_{p,w}$ for all $f \in L_{p,w}(X)$

i.e $0 \in Q_\delta(f)$, thus $Q_\delta(\hbar) \neq \emptyset$

for $\hbar \in L_{p,w}(X)$, this contradiction with hypothesis.

■

Theorem 3.4:

Let M sub space of $L_{p,w}(X)$ and $f \in L_{p,w}(X)$ *then* :

(i) $B = \{g_0 \in M : g_0 \cap_{g\in M} \mathcal{P}_\delta(f)\} \subset Q_\delta(f)$, where

$\|g_0 - f\|_{p,w} = \{\beta f + (1 - \beta) g_0 : \beta \ scalar\}$ is the

linear manifold spanned by g_0 and f .

(ii) For an element $g_0 \in M$ we have $g_0 \in \mathcal{Q}_\delta(f)$

if $A \subset \mathcal{P}_{\delta}{}^{-1}_{\|f-a_0\|}(0) = \{\hbar \in L_{p,w} : 0 \in \mathcal{P}_\delta(\hbar)\}$

Proof :(i) $a_0 \in A$ and $a_0 \in \mathcal{P}_\delta(a)$.

For all $a \in A \Rightarrow a_0 \in A$ and $\|a_0 - a\|_{p,w} \le \|(\beta f + (1-\beta) a_0) - a\|_{p,w}$, for all $a \in A$ and all scalars β .

For all $a \in A$and all scalars β

$\Rightarrow a_0 \in A$ and $\|a_0 - a\|_{p,w} \le \|f - a\|_{p,w}$ for all $g \in M$

i.e. $a_0 \in \mathcal{Q}_\delta(f)$

(ii) $A \subset \mathcal{P}_{\delta}{}^{-1}_{\|f-a_0\|}(0)$, so $0 \in \mathcal{P}_\delta(a)$ for all $a \in A$.

$\Rightarrow \|a_0 - a\|_{p,w} \le \|\beta(f - a_0) - a\|_{p,w}$, for all $a \in A$.

Let $a_1 \in A$. Put $a = a_1 - a_0$ and $\beta = 1$, we obtain

$\|(a_1 - a_0) - 0\|_{p,w} \le \|(f - a_0) - (a_1 - a_0)\|_{p,w} \le \|f - a_1\|_{p,w}$.

So, $a_0 \in \mathcal{Q}_\delta(f)$.

■

Theorem 3.5:

Let A be asb space of linear space $L_{p,w}(X)$. Then following statements: are equivalent.

(i) A is coproximal .

(ii) $L_{p,w}(X) = A + (\mathcal{Q}_\delta(0))^{-1}$

Proof: (i) $\Rightarrow$ (ii) Let $f \in L_{p,w}(X)$ and A coproximal , there is $a_0 \in A$ such that $a_0 \in \mathcal{Q}_\delta(f)$ from lemma 2.3 (ii)

$f - a_0 \in (\mathcal{Q}_\delta(0))^{-1}$.

Since $a_0 \in A$ and $f - a_0 \in \mathcal{Q}_\delta(0)$ implies $f = a_0 + f - a_0 \in A + (\mathcal{Q}_\delta(0))^{-1}$.

Hence $f \in A + (\mathcal{Q}_\delta(0))^{-1}$

i.e. we obtain, $L_{p,w}(X) \subset A + (\mathcal{Q}_\delta(0))^{-1}$, but $A + (\mathcal{Q}_\delta(0))^{-1} \subset L_{p,w}(X)$.

Thus $L_{p,w}(X) = A + (\mathcal{Q}_\delta(0))^{-1}$.

Now we need to show that (ii)$\Rightarrow$(i) , let $f \in L_{p,w}(X) = A + (\mathcal{Q}_\delta(0))^{-1}$.

Then $f = a_0 + a , a_0 \in A$ and $a \in (\mathcal{Q}_\delta(0))^{-1}$.

So, $0 \in \mathcal{Q}_\delta(a) = \mathcal{Q}_\delta(f - a_0)$.

From lemma (ii), $a_0 \in \mathcal{Q}_\delta(a)$.

Hence A is coproximal

■

Theorem 3.6:

Let A be a subspace of linear space $L_{p,w}(X)$. Then following statements are equivalent:

(i) A is coproximal .

(ii) A is closed and for canonical mapping . $M_A : L_{p,w}(X) \to L_{p,w}(X)/A$ such that $M_A(\mathcal{Q}_\delta(0))^{-1} = L_{p,w}(X)/A$.

Proof :-

(i)$\Rightarrow$(ii) Let A be acoproximinal sub space of $L_{p,w}(X)$ we need to prove A is closed and

$M_A(\mathcal{Q}_\delta(0))^{-1} = L_{p,w}(X)/A$

Let $a \in \bar{A}$, $r_0 \in \mathcal{Q}_\delta(a)$.

Then there is a sequence $\{a_n\}$ in A such that $a_n \to a$ and

$\|a - r_0\|_{p,w} \le \|a - k\|_{p,w}$ for $k \in A$

Also $\|a_n - r_0\| \le \|a_n - a\|$ for all n .

We obtain $a_n \to r_0$, but $a_n \to a$

Implies $r_0 = a \in A$

Thus $\bar{A} \subset A \subset \bar{A}$ implies

$A = \bar{A}$

So, A is closed

Now $a_0 \in \mathcal{Q}_\delta(f)$ implies by lemma 2.3 (ii)

$f - a_0 \in \mathcal{Q}_\delta(f)$, then $M_A(f - a_0) = (f - a_0) + A = f + A$

(ii)→(i)

Let $f \in L_{p,w}(x)$

$f + A \in L_{p,w}(X)/A = M_A(\mathcal{Q}_\delta(0))^{-1}$

$\Rightarrow f + A = M_A(h)$, where $h \in (\mathcal{Q}_\delta(0))^{-1}$

$\Rightarrow f + A = h + A$, where $0 \in (\mathcal{Q}_\delta(0))^{-1}$

$\Rightarrow f - h = a_0 \in A$ and $0 \in \mathcal{Q}_\delta(f - a_0)$

from lemma 3.3 (ii)

$\Rightarrow a_0 \in \mathcal{Q}_\delta(f)$

A is coproximinal .

■

Theorem 3.7:

Let A be closed sub space of the space $L_{p,w}(X)$. Then the following .

(i) A is a co-Chebyshev subspace .

(ii) $L_{p,w}(X) = A \oplus (\mathcal{Q}_\delta(0))^{-1}$,where $\oplus$ means that the sum decomposition of each $f \in L_{p,w}(X)$ is a unique .

(iii) A is coproximal and$(\mathcal{Q}_\delta(0))^{-1} - (\mathcal{Q}_\delta(0))^{-1} \cap A = \{0\}$.

(iv) A is coproximinal and the restriction mapping $M_A/(\mathcal{Q}_\delta(0))^{-1}$ is injective.

Proof: (i) →(ii) since A is co-Chebyshev it is coproximinal and by theorem 3.5 $L_{p,w}(X) = A + (\mathcal{Q}_\delta(0))^{-1}$, now we show that the sum decoposition of each $f \in L_{p,w}(X)$ is a unique suppose $f \in L_{p,w}(X)$ and

$f = a_1 + h_1$ *and* $f = a_2 + h_2$

Where $a_1, a_2 \in A$, $h_1, h_2 \in (\mathcal{Q}_\delta(0))^{-1}$

This gives $a_1 - a_2 = h - h_2$ Now $h_1 \in (\mathcal{Q}_\delta(0))^{-1}$

$\Longrightarrow 0 \in \mathcal{Q}_\delta(h_1) \Longrightarrow a_1 \in \mathcal{Q}_\delta(h_1 + a_1)$ by lemma 2.3 (ii) .

i.e. $a_1 \in \mathcal{Q}_\delta(f)$ similarly $a_2 \in \mathcal{Q}_\delta(f)$

since A is co- Chebyshev $a_1 = a_2$ and consequently $h_1 = h_2$

Hence $L_{p,w}(X) = A \oplus (\mathcal{Q}_\delta(0))^{-1}$

(ii)→(iii) $L_{p,w}(X) = A \oplus (\mathcal{Q}_\delta(0))^{-1} \Longrightarrow A$ coproximinal.

By Theorem 3.6 suppose $0 \neq h \in (\mathcal{Q}_\delta(0))^{-1} - (\mathcal{Q}_\delta(0))^{-1} \cap A$

Then . $h = h_1 - h_2$, $h_1 \in (\mathcal{Q}_\delta(0))^{-1}$, $h_2 \in (\mathcal{Q}_\delta(0))^{-1}$

$h_1 \neq h_2$ so, $0 \in \mathcal{Q}_\delta(h_1)$, $0 \in \mathcal{Q}_\delta(h_2)$, now$h_1, h_2 \in (\mathcal{Q}_\delta(0))^{-1}$, $h_1 - h_2 \in A|(0)$ and

$h_1, h_2 \in (\mathcal{Q}_\delta(0))^{-1}$, $h_1 - h_2 \in A|(0)$ and

$h_1 = 0 + h_1 = (h_1 - h_2) + h_2$ a contradication to the uniqueness of the sum decomposition .

Hence $\left[(\mathcal{Q}_\delta(0))^{-1} - (\mathcal{Q}_\delta(0))^{-1} \cap A = \{0\}\right]$.

(iii)→(iv) Suppose $M_A/(\mathcal{Q}_\delta(0))^{-1}$ is not injective , i.e there exists $h_1 - h_2 \in (\mathcal{Q}_\delta(0))^{-1}$

$h_1 \neq h_2$ and $M_A(h_1) = M_A(h_2)$ Then

$0 \neq h_1, h_2 \in (\mathcal{F}(0))^{-1} - (\mathcal{F}(0))^{-1} \cap A$ a contradiction.

(iv)→(i) Suppose $f \in L_{p,w}(X)$ has two distinct best co-aproximation in A say a_1 and a_2 then by observation 2.3 (ii)

$f - a_1$ and $f - a_2 \in (\mathcal{Q}_\delta(0))^{-1}$, $f - a_1 \neq f - a_2$.

But $M_A f - (a_1) = M_A f - (a_2)$ QS

$(f - a_1) - f - a_2) = a_2 - a_1 \in A$,which is a contradiction.

■

Theorem 3.8:

for a closed linear subspace A of $L_{p,w}(X)$ the following statement are equivalent :-

(i) A is co-semi chebysher sub space .

(ii) Each element $f \in L_{p,w}(X)$ has at most one some decomposition as $A + (F(0))^{-1}$

(iii) $\left[(\mathcal{F}(0))^{-1} - (\mathcal{F}(0))^{-1}\right] \cap A = \{0\}$

$M_A|(\mathcal{F}(0))^{-1}$ is injective.

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أفضل تقريب مصاحب في الفضاء الموزون

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الخلاصة:

في هذا البحث تم عرض أفضل تقريب مصاحب في الفضاء الموزون والنتائج التي الحصول عليها هي وجودية الدوال لافضل تقريب مصاحب و وحدانية وتعددية هذه الدوال في فضاءات شبيشيف المصاحبة الجزئية.